Name ________________

Student Number ________________

Instructor: S. Qiao

SFWR ENG 3X03/COMP SCI 4X03

Day Class
Duration of examination: two hours
McMaster University Final Examination

December 2012

This examination paper includes **6** pages and **5** questions. You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancy to the attention of your invigilator.

SPECIAL INSTRUCTIONS: This paper must be returned with your answers. Open book and notes, no electronics, use of McMaster standard (Casio-FX991) calculator is allowed.

1. (4 marks) The following segment of (Matlab) code computes $\sqrt{x^2 + y^2}$:

```
n = sqrt(x*x + y*y);
```

Applying the scaling technique, modify the above code so that it avoids unnecessary overflow or underflow.

```
m = max(abs(x), abs(y));
x = x/m;
y = y/m;
n = sqrt(x*x + y*y);
n = m*n;
```

Continued on Page 2

2. The cubic spline function $s(x)$ on the interval $[x_i, x_{i+1}]$ is defined by

$$s(x) = wy_{i+1} + \bar{w}y_i + h_i^2[(w^3 - w)\sigma_{i+1} + (\bar{w}^3 - \bar{w})\sigma_i], \quad i = 1, 2, ..., n-1,$$

where $h_i = x_{i+1} - x_i$,

$$w = (x - x_i)/h_i, \quad \bar{w} = 1 - w,$$

and, σ_i and σ_{i+1} are unknown parameters. Set up an equation corresponding to the continuity of $s'(x)$ at x_i, $1 < i < n$. In other words, $s'_-(x_i) = s'_+(x_i)$.

(a) (4 marks) $s'_+(x_i)$: on interval $[x_i, x_{i+1}]$

$$w' = 1/h_i, \qquad \bar{w}' = -1/h_i, \qquad w(x_i) = 0, \quad \bar{w}(x_i) = 1$$

$$s'_+(x_i) = y_{i+1}/h_i - y_i/h_i + h_i^2[(3w^2(x_i)/h_i - 1/h_i)\sigma_{i+1} + (-3\bar{w}^2(x_i)/h_i + 1/h_i)\sigma_i]$$

$$= (y_{i+1} - y_i)/h_i + h_i[-\sigma_{i+1} - 2\sigma_i]$$

$$= \Delta_i - h_i(\sigma_{i+1} - 2\sigma_i)$$

(b) (4 marks) $s'_-(x_i)$: on $[x_{i-1}, x_i]$

$$w' = 1/h_{i-1}, \quad \bar{w}' = -1/h_{i-1}, \quad w(x_i) = 1, \quad \bar{w}(x_i) = 0$$

$$s'_-(x_i) = y_i/h_{i-1} - y_{i-1}/h_{i-1} + h_{i-1}^2[(3w^2(x_i)/h_{i-1} - 1/h_{i-1})\sigma_i + (-3\bar{w}^2(x_i)/h_{i-1} + 1/h_{i-1})\sigma_{i-1}]$$

$$= (y_i - y_{i-1})/h_{i-1} + h_{i-1}[2\sigma_i + \sigma_{i-1}]$$

$$= \Delta_{i-1} + h_{i-1}(2\sigma_i + \sigma_{i-1})$$

(c) (2 marks) The equation:

$$\Delta_i - h_i(\sigma_{i+1} - 2\sigma_i) = \Delta_{i-1} + h_{i-1}(2\sigma_i + \sigma_{i-1})$$

$$h_{i-1}\sigma_{i-1} + 2(h_{i-1} + h_i)\sigma_i + h_i\sigma_i = \Delta_i - \Delta_{i-1}$$

Continued on page 3

3. The error in the Simpson's rule is given by

$$I - S = -\frac{1}{2880}\sum_{i=1}^{n} h_i^5 f^{iv}(y_i) + \cdots$$

where $h_i = x_{i+1} - x_i$ and $y_i = (x_i + x_{i+1})/2$. Doubling the number of panels in the Simpson's rule can be expected to reduce the error in the composite trapezoidal rule by roughly the factor of $1/16$. That is, if I is the exact integral, S_1 is the one-panel result, and S_2 is the two-panel result, then $I - S_2 \approx (I - S_1)/16$.

(a) (5 marks) Derive an estimation for $I - S_2$ using S_1 and S_2. That is, find the factor α in $I - S_2 \approx \alpha(S_2 - S_1)$.

$$I - S_2 \approx \frac{1}{16} I - \frac{1}{16} S_1$$

$$\frac{15}{16}(I - S_2) \approx \frac{1}{16}(S_2 - S_1)$$

$$I - S_2 \approx \frac{1}{15}(S_2 - S_1)$$

(b) (6 marks) Apply the Simpson's rule to

$$\int_0^{\pi} (\sin(x))dx$$

with one panel $[0, \pi]$ for S_1 and two panels $[0, \pi/2]$ and $[\pi/2, \pi]$ for S_2

$S_1 = \frac{\pi}{6}(\sin 0 + 4\sin(\frac{\pi}{2}) + \sin\pi)$

$= \frac{4\pi}{6} \approx 2.09$

$S_2 = \frac{\pi}{12}(\sin 0 + 4\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 4\sin\frac{3\pi}{4} + \sin\pi)$

$= \frac{\pi}{12}(2\sqrt{2} + 2 + 2\sqrt{2}) \approx 2.00$

Continued on Page 4

4. Consider the second-order differential equation

$$y'' = -3y' - y, \quad y(0) = 0 \quad \text{and} \quad y'(0) = 3,$$

(a) (5 marks) Express this second-order ODE as an equivalent system of two first-order ODEs, including the initial conditions for the system.

$$u = \begin{bmatrix} y \\ y' \end{bmatrix}$$

$$u' = \begin{bmatrix} u_2 \\ -3u_2 - u_1 \end{bmatrix} \qquad u(0) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

(b) (5 marks) Perform one step of the forward Euler's method for this ODE system using a stepsize of $h = 0.1$.

$$u_+ = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + h \begin{bmatrix} u_2 \\ -3u_2 - u_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 0.3 \\ -0.9 \end{bmatrix}$$

$$= \begin{bmatrix} 0.3 \\ 2.1 \end{bmatrix}$$

Continued on Page 5

(c) (5 marks) Perform one step of the backward Euler's method for this ODE system using a stepsize of $h = 0.1$.

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + h \begin{bmatrix} u_2 \\ -3u_2 - u_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \end{bmatrix} + 0.1 \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Solve

$$\begin{bmatrix} 1 & -0.1 \\ 0.1 & 1.3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0.229 \\ 2.29 \end{bmatrix}$$

Continued on Page 6

5. (10 marks) Carry out one iteration of Newton's method for finding a zero of the function:

$$f(x) = x - \cos x,$$

with starting point $x_0 = 1.0$.

$x_0 = 1.0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1.0 - \frac{1.0 - \cos(1.0)}{1 + \sin(1.0)}$$

$$= 1.0 - \frac{0.460}{1.84}$$

$$= 0.75$$

END!