Natural Cubic Interpolation

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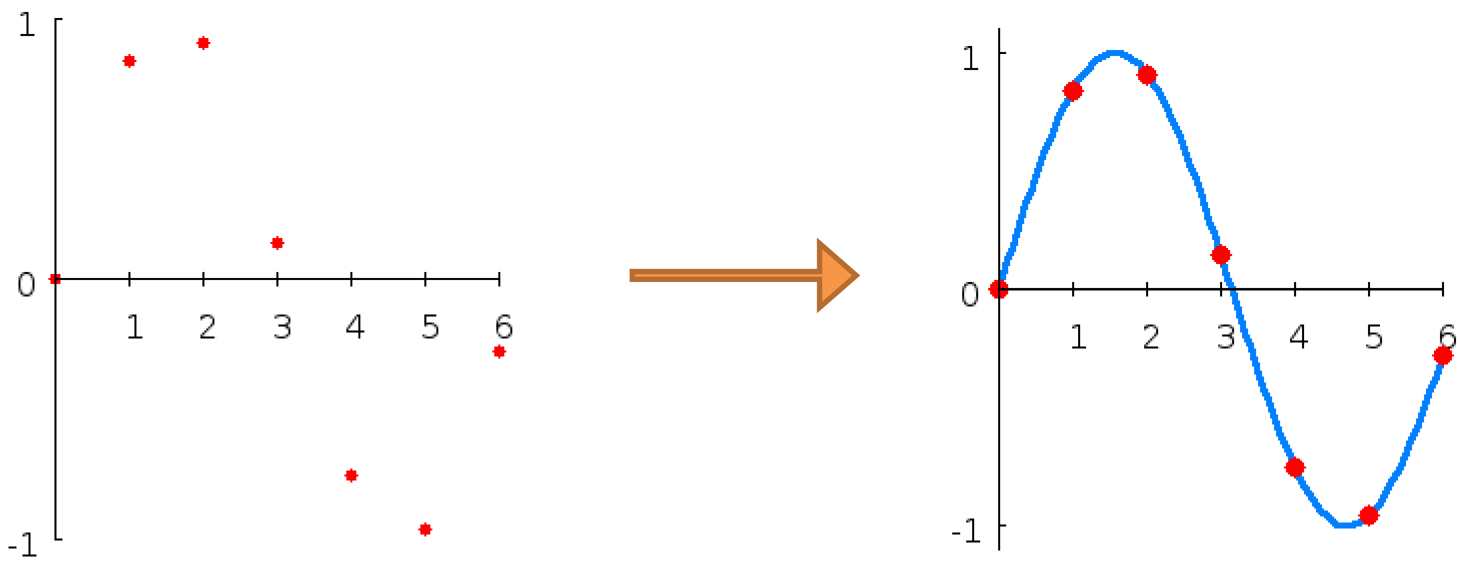
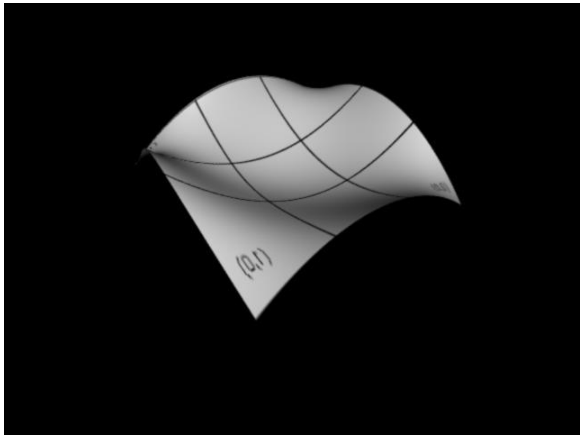
10/24/2012

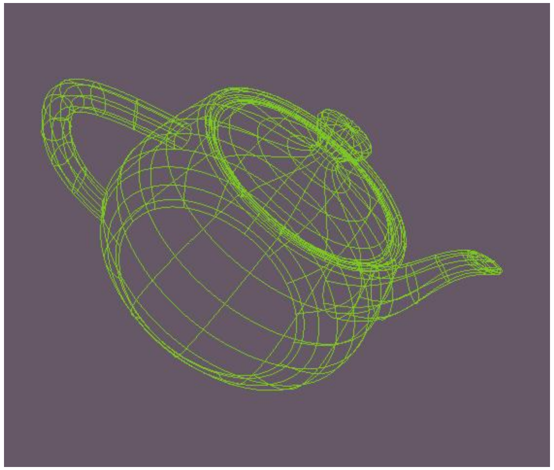
Interpolation

- Construct a function crosses known points
- Predict the value of unknown points



Interpolation in modeling





Interpolation

- Polynomial Interpolation
  - Same polynomial for all points
  - Vandermonde Matrix, ill-conditioned

- Lagrange Form
  - Hard to evaluate

- Piecewise Interpolation
  - Different polynomials for each interval

Lagrange form

Given k+1 points

$$(x_0, y_0), \ldots, (x_j, y_j), \ldots, (x_k, y_k)$$

Define:

$$L(x) := \sum_{j=0}^{k} y_j \ell_j(x)$$

where

$$\ell_j(x) := \prod_{\substack{0 \le m \le k \\ m \neq j}} \frac{x - x_m}{x_j - x_m} = \frac{(x - x_0)}{(x_j - x_0)} \cdots \frac{(x - x_{j-1})}{(x_j - x_{j-1})} \frac{(x - x_{j+1})}{(x_j - x_{j+1})} \cdots \frac{(x - x_k)}{(x_j - x_k)}.$$

Lagrange form

$$\ell_j(x_i) = \delta_{ji} = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases}$$

$$L(x_i) = \sum_{j=0}^{k} y_j \ell_j(x_i) = \sum_{j=0}^{k} y_j \delta_{ji} = y_i.$$

Lagrange form

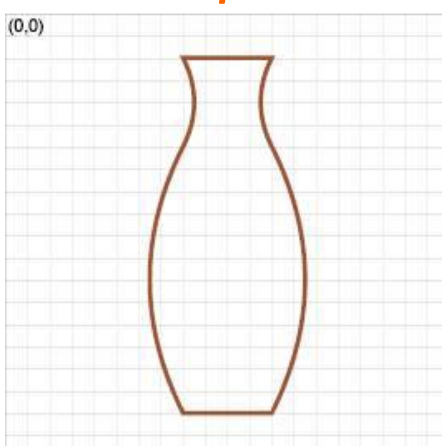
- Example: interpolate f(x) = x^2, for x = 1,2,3

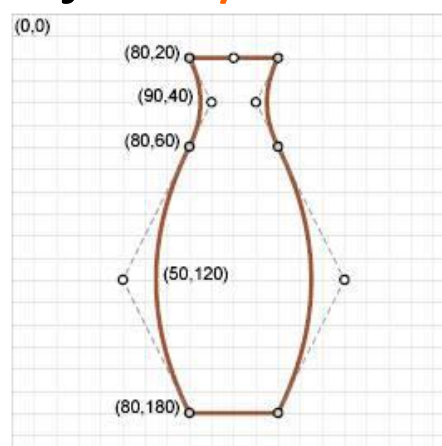
$$
\begin{aligned}
x_0 &= 1 \qquad f(x_0) = 1 \\
x_1 &= 2 \qquad f(x_1) = 4 \\
x_2 &= 3 \qquad f(x_2) = 9.
\end{aligned}
$$

$$
\begin{aligned}
L(x) &= 1 \cdot \frac{x-2}{1-2} \cdot \frac{x-3}{1-3} + 4 \cdot \frac{x-1}{2-1} \cdot \frac{x-3}{2-3} + 9 \cdot \frac{x-1}{3-1} \cdot \frac{x-2}{3-2} \\
&= x^2.
\end{aligned}
$$

How to represent models

- Specify every point along a model?
  - Hard to get precise results
  - Too much data, too hard to work with generally
- Specify a model by a small number of "control points"
  - Known as a *spline curve* or just *spline*

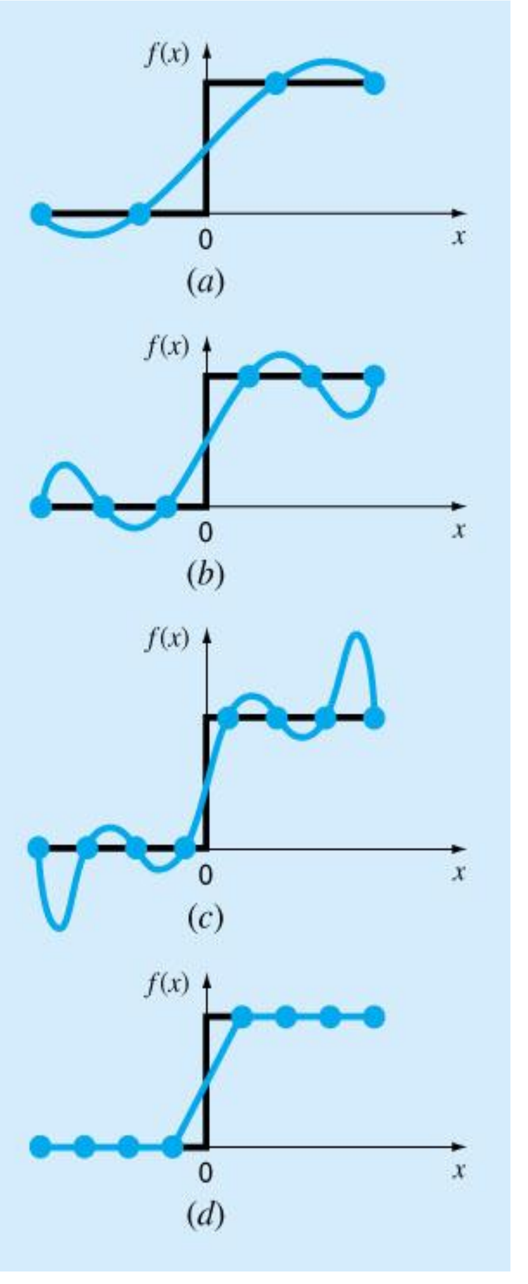




Spline Interpolation

- For some cases, polynomials can lead to erroneous results because of round off error and overshoot.

- Alternative approach is to apply lower-order polynomials to subsets of data points. Such connecting polynomials are called *spline functions*.



Spline Interpolation Definition

- Given n+1 distinct **knots** x_i such that:

$$x_0 < x_1 < \ldots < x_{n-1} < x_n,$$

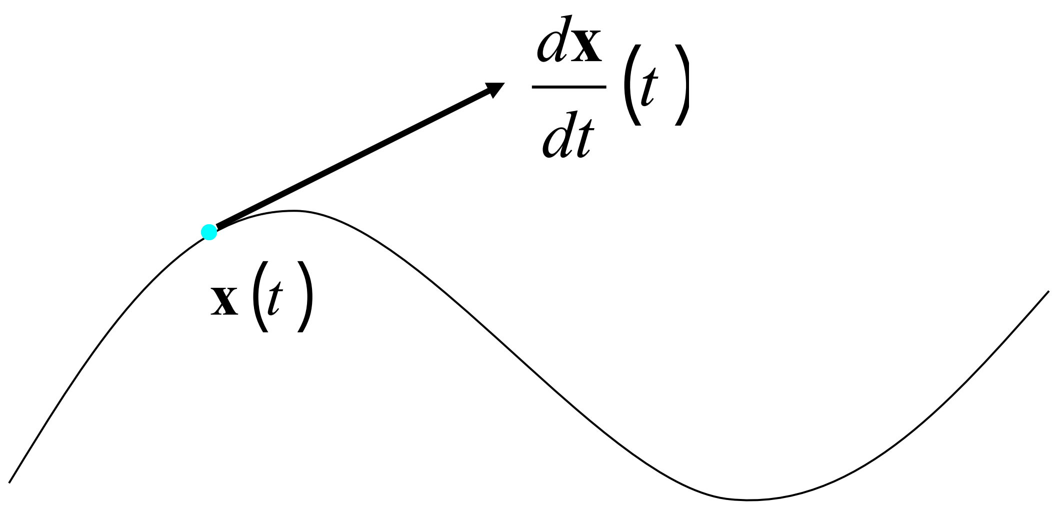
with n+1 **knot values** $\boldsymbol{y_i}$ find a spline function

$$S(x) := \begin{cases} S_0(x) & x \in [x_0, x_1] \\ S_1(x) & x \in [x_1, x_2] \\ \vdots & \vdots \\ S_{n-1}(x) & x \in [x_{n-1}, x_n] \end{cases}$$

with each $S_i(x)$ a polynomial of degree at most n.

Tangent

- The derivative of a curve represents the tangent vector to the curve at some point



(a) Linear spline

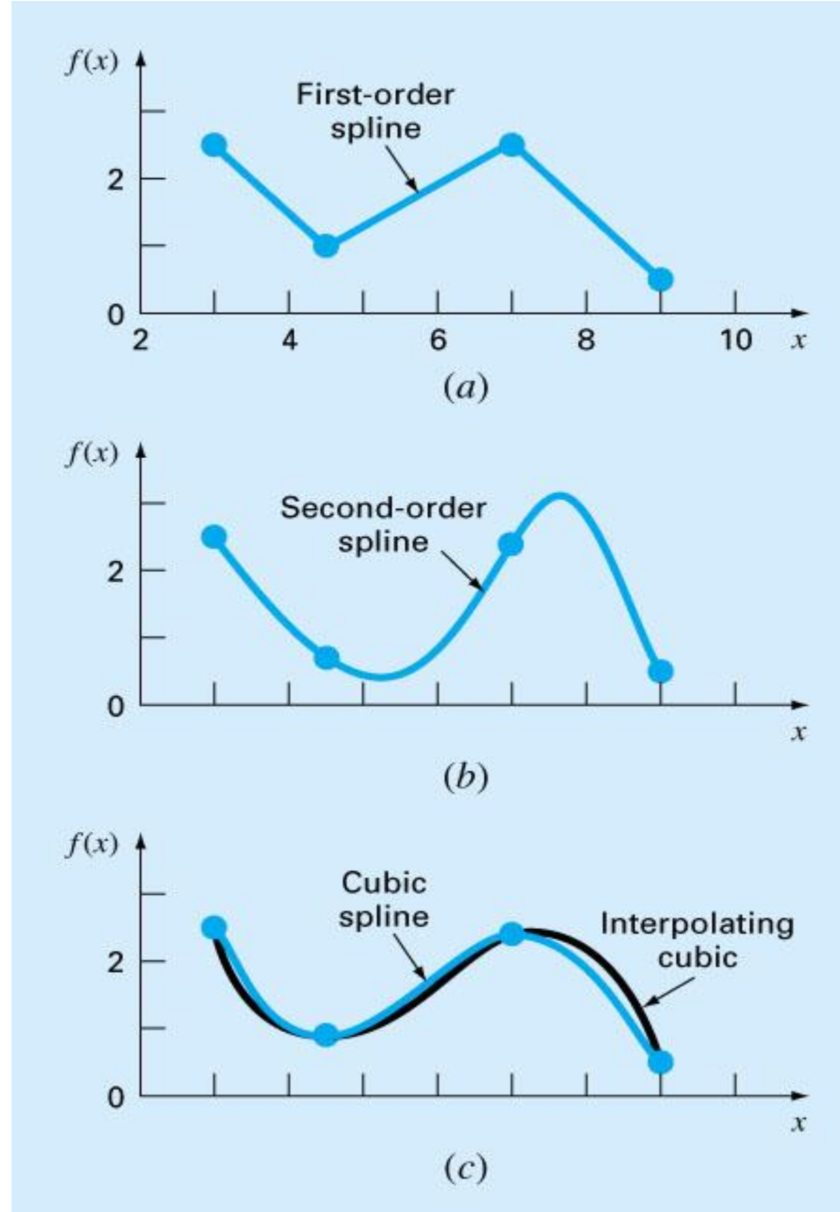
- Derivatives are not continuous
- Not smooth

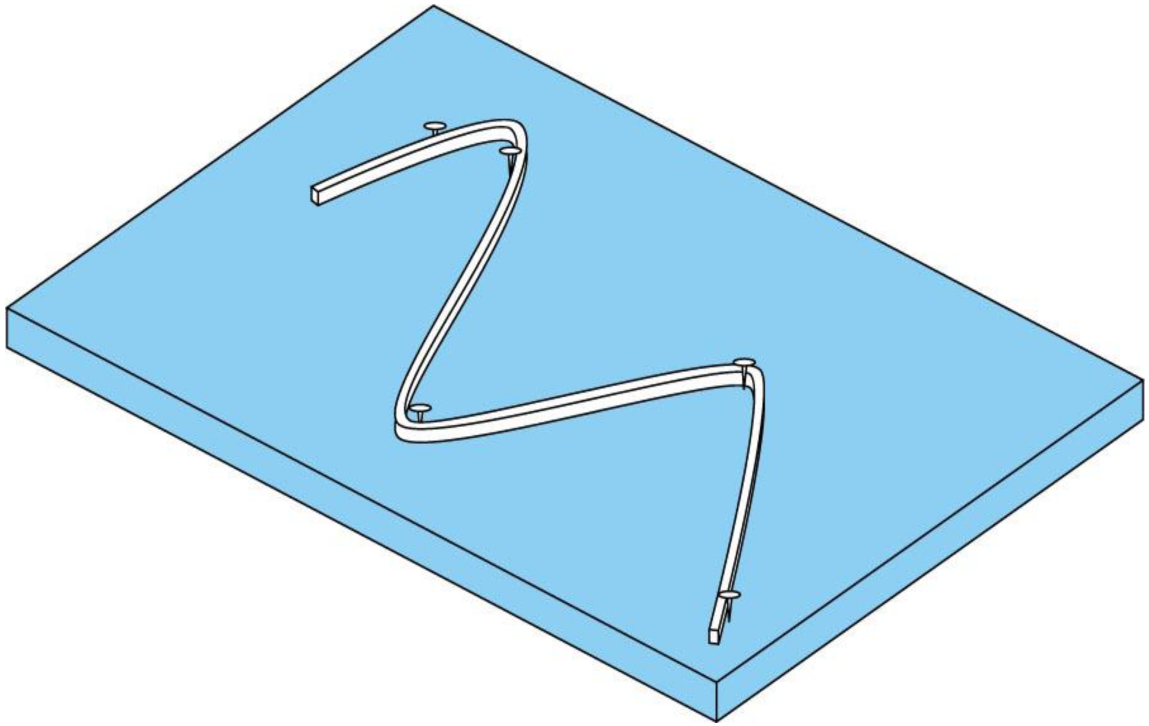
(b) Quadratic spline

- Continuous 1st derivatives

(c) Cubic spline

- Continuous 1st & 2nd derivatives
- Smoother

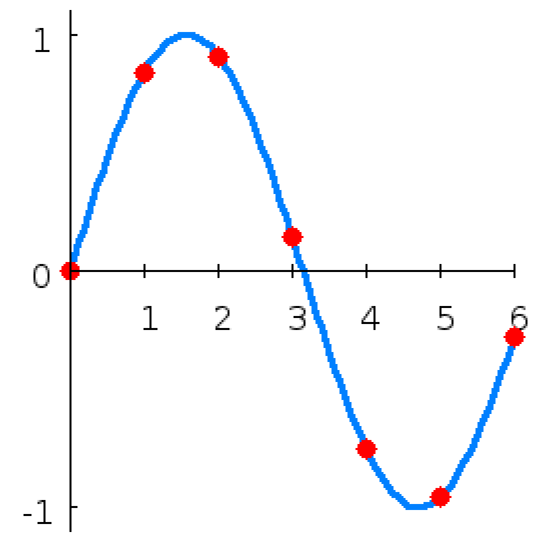




- Why cubic?
  - Good enough for some cases
  - The degree is not too high to be easily solved

Natural Cubic Spline Interpolation

- $S_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$ (Given n points)
  - 4 Coefficients with n-1 subintervals = 4n-4 equations
  - There are 4n-6 conditions
    - Interpolation conditions
    - Continuity conditions
    - Natural Conditions
  - $S''(x_0) = 0$
  - $S''(x_n) = 0$
  - **$O(n^3)$**

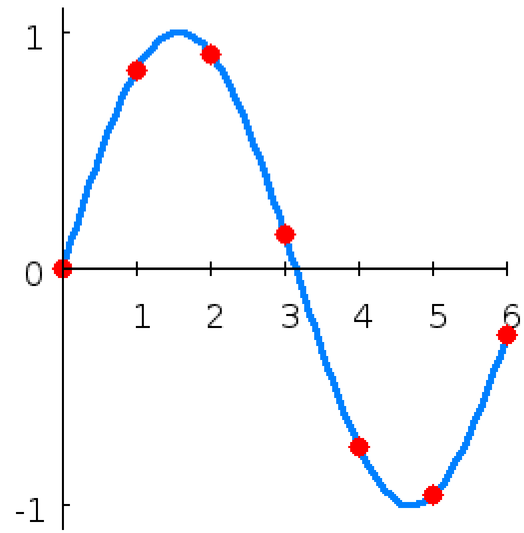


Natural Cubic Spline Interpolation

- A clever method
  – Construct S(x)
  - ➢ Lagrange Form thought

  – Solve tridiagonal matrix
  - ➢ Using decompt & solvet (2-1)

  – Evaluate of S(z)
  - ➢ Locate z in some interval (using binary search)
  - ➢ Using Horner's rule to evaluate



Thanks