Adaptive algorithms for computing the principal Takagi vector of a complex symmetric matrix

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\textbf{A B S T R A C T}

In this paper, we present a unified framework for deriving and analyzing adaptive algorithms for computing the principal Takagi vector of a complex symmetric matrix. Eight systems of complex-valued ordinary differential equations (complex-valued ODEs) are derived and their convergence behavior is analyzed. We prove that the solutions of the complex-valued ODEs are asymptotically stable. The systems can be implemented on neural networks. Finally, we show experimental results to support our analyses.

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1. Introduction

A complex symmetric matrix can be diagonalized by a unitary matrix. Specifically, let $A = B + iC$ be a complex symmetric matrix, where $B, C \in \mathbb{R}^{n \times n}$ are symmetric, then there exist a unitary matrix $Q \in \mathbb{C}^{n \times n}$ and a positive semidefinite diagonal matrix $\Sigma \in \mathbb{R}^{n \times n}$ such that (see, e.g., [36])

$$A = Q \Sigma Q^T, \quad \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n). \quad (1.1)$$

This factorization of a complex symmetric matrix is called the Autonne-Takagi factorization, or the Takagi factorization in short, originally proposed by Autonne [2] and Takagi [29]. The columns of $Q$ are called the Takagi vectors of $A$ and the diagonal elements of $\Sigma$ are its Takagi values. We denote $\Sigma$ and $Q$ as the Takagi value matrix and the associated Takagi vector matrix of $A$, respectively. In particular, if $\sigma_1$ is the largest Takagi value, then we call $q_1$ the principal Takagi vector associated with $\sigma_1$.

The Takagi factorization reveals the symmetry of a complex symmetric matrix. One advantage of the factorization is that it can save storage and computation by about half. The Takagi factorization of a complex symmetric matrix has applications in the Grunsky inequalities [27], computation of the near-best uniform polynomial or rational approximation of a high degree polynomial on a disk [30], the complex independent component analysis [11], and nuclear magnetic resonance [3].

Throughout this paper, unless stated otherwise, $I, J$, and $N$ denote the index upper bounds, lower case letters $x, u, \ldots$ for scalars, bold lower case letters $\mathbf{x}, \mathbf{u}, \ldots$ for vectors, bold capital letters $\mathbf{A}, \mathbf{B}, \ldots$ for matrices of order $I$, and calligraphic letters $\mathcal{A}, \mathcal{B}, \ldots$ for matrices of order $2I$. This notation is consistently used for entries. For example, the entry with row index $i$ and column index $j$ of a matrix $A$, i.e., $(A)_{ij}$, is denoted by $a_{ij}$ (also $(A)_{i} = x_i$). We denote $\overline{A}$, $\mathbf{A}^\star$, and $\overline{\mathbf{A}}^\star$ the complex conjugate, the transpose and the complex conjugated transpose of $\mathbf{A}$, respectively. We use $|a|$ to denote the modulus of a complex number $a$. The real and imaginary parts of $z \in \mathbb{C}$ are denoted by $\Re(z)$ and $\Im(z)$, respectively.

There are several ways of computing the Takagi factorization. For example, the complex problem can be transformed into a real...
problem of double size. Since \(Q\) is unitary and \(Q^* = (Q^T)^T\), thus \(A(QQ^T) = QX Q^T\). Let \((\sigma, q)\) be a Takagi value-vector pair and \(q = x + iy\), where \(x, y \in \mathbb{R}^l\), then \(\tilde{A} = \sigma \tilde{q}\), that is,

\[
\begin{bmatrix}
B & C \\
C & -B
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \sigma
\begin{bmatrix}
x \\
y
\end{bmatrix}
\text{and}
\begin{bmatrix}
B & C \\
C & -B
\end{bmatrix}
\begin{bmatrix}
-x \\
x
\end{bmatrix}
= -\sigma
\begin{bmatrix}
-x \\
x
\end{bmatrix}.
\]

Denote

\[
A = \begin{bmatrix}
B & C \\
C & -B
\end{bmatrix}
\]

and note that \(A \in \mathbb{R}^{2l \times 2l}\) is symmetric. Thus, the Takagi factorization of \(A\) can be obtained from the eigenvalue decomposition of the real symmetric \(A\) of double size.

Suppose that \((\sigma, q)\) is a Takagi pair of \(A\). Let \(v = \tilde{q}\) and \(u = \tilde{q}\), then

\[
Av = \sigma u, \quad \tilde{Au} = A^* u = \sigma v,
\]

which implies that \((\sigma, u, v)\) is a singular pair of \(A\). Hence we can also compute the Takagi factorization of a complex symmetric matrix to obtain its singular value decomposition. The interested readers can be referred to \[12,14,15,28\] and their references for numerical computation of the singular value decomposition of matrices. However, as shown in \[36\], for a complex symmetric matrix \(A\), we have

(a) if \((\sigma, q)\) is a Takagi pair of \(A\), then \((\sigma, q, \tilde{q})\) is its singular pair;
(b) if \((\sigma, u, v)\) is a singular pair of \(A\), we can not directly obtain the Takagi vector associated with \(\sigma\) from \(u\) and \(v\).

Finally, similar to the computation of the SVD (see, e.g., \[12\], Bunse-Gerstner and Gragg \[7\]) that a standard algorithm for computing the Takagi factorization of a complex symmetric matrix consists of two stages: First, a complex symmetric matrix is reduced to a complex symmetric tridiagonal matrix; Second, the Takagi factorization of the complex symmetric tridiagonal matrix from the first stage is computed. For the first stage, Qiao et al. \[29\] derived a block Lanczos method for tridiagonalizing complex symmetric matrices. There are two methods for implementing the second stage: the divide-and-conquer method \[36\] and a twisted factorization method \[37\]. There are other numerical algorithms for computing the Takagi factorization of a complex symmetric matrix \[7,36,37\]. The existing algorithms are designed for the complete Takagi factorization.

In some applications, however, only the principal Takagi vector is required. Especially in the situation where we want to investigate the behavior of each component of the principal Takagi vector of a parameterized complex symmetric matrix. Moreover, we will shortly show that an algorithm for computing the principal Takagi vector of a complex symmetric matrix can be used to compute the complete Takagi factorization. So, in this paper, we focus on the computation of the principal Takagi vector of a complex symmetric matrix.

Let \(A \in \mathbb{C}^{l \times l}\) be symmetric, whose Takagi values are positive and distinct, that is, \(\sigma_1 > \sigma_2 > \cdots > \sigma_N > 0\) and \(q_i\) the principal Takagi vector. Rewriting (1.1) as \(A = \sum_{i=1}^{l} \sigma_i q_i q_i^T\), we get

\[
A - \sigma_1 q_1 q_1^T = \sum_{i=2}^{l} \sigma_i q_i q_i^T,
\]

a symmetric matrix whose principal Takagi vector is \(q_1\). Repeating the process, we can successively compute \(q_1, q_2, \ldots, q_i\). From (1.1), the Takagi values \(\sigma_i\) can be obtained by \(\sigma_i = q_i^T A q_i\), \(i = 1, 2, \ldots, l\). The following algorithm computes \(Q = [q_1, \ldots, q_l]\) and \(\Sigma = \text{diag} (\sigma_1, \ldots, \sigma_l)\) in the complete Takagi factorization (1.1) by calling a procedure of computing the principal Takagi vector of a complex symmetric matrix.

**Input:** Symmetric \(A \in \mathbb{C}^{l \times l}\) with positive and distinct Takagi values.

**Output:** The Takagi vectors \(q_i\) and Takagi values \(\sigma_i\) with \(i = 1, 2, \ldots, l\).

\[
A_i = A.
\]

for \(i = 1, 2, \ldots, l\) do

Compute \(q_i\): the principal Takagi vector of \(A_i\),

\[
\sigma_i = q_i^T A_i q_i,
\]

\[
A_{i+1} = A_i - \sigma_i q_i q_i^T.
\]

end for

This paper presents a unified approach to complex-valued neural networks for computing the principal Takagi vector of a complex symmetric matrix with positive and distinct Takagi values.

### 2. Basics of real-valued functions of complex variables

For a function \(f : \mathbb{C} \to \mathbb{C}\), its complex derivative at \(x \in \mathbb{C}\), if it exists, is defined as the limit:

\[
f'(x) := \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.
\]

We know that \(f\) is differentiable in the complex sense, if and only if the Cauchy–Riemann conditions hold. However, in many practical applications, functions are not differentiable in the complex sense. In this paper, we consider optimization problems where the objective functions are real valued with complex variables. The objective functions are not complex differentiable, unless they are constant functions \[5\]. In order to deal with the problem, we provide an alternative formulation which is based on the real derivatives but similar to the complex derivative. The purpose of this section is to introduce some basics of the Wirtinger derivative of a real valued function with complex variables, including the gradient and the Hessian matrix.

Suppose that \(f : \mathbb{C} \to \mathbb{R}\) is a real-valued function with complex variables. Let \(z = x + iy \in \mathbb{C}\) and \(x, y \in \mathbb{R}^l\). We introduce a calculus of the differential operators, developed in principal by Wirtinger, often called the Wirtinger calculus. We refer to \[6,16,26,31\] for the underlying framework of the complex derivatives.

**Definition 2.1.** Let \(z = x + iy \in \mathbb{C}^l\), where \(x, y \in \mathbb{R}^l\). The cogradient operator \(\frac{\partial}{\partial z}\) and the conjugate cogradient operator \(\frac{\partial}{\partial \bar{z}}\) are defined by

\[
\frac{\partial}{\partial z} = \frac{1}{2} \begin{bmatrix}
\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \\
\vdots \\
\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}
\end{bmatrix}
\]

and

\[
\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \begin{bmatrix}
\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \\
\vdots \\
\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}
\end{bmatrix}.
\]

For example, define \(f(z) = z^* z\) for all \(z = x + iy \in \mathbb{C}^l\), then

\[
\frac{\partial}{\partial z} = x - iy, \quad \frac{\partial}{\partial \bar{z}} = x + iy.
\]

Note that the Cauchy–Riemann conditions for \(f(z)\) to be analytic at \(z\) can be expressed compactly, using the cogradient as \(\frac{\partial f}{\partial z} = 0\), i.e., \(f\) is a function of \(z\) only. Analogously, \(f\) is said to be analytic at \(z\) if and only if \(\frac{\partial f}{\partial \bar{z}} = 0\).

Let \(\frac{\partial}{\partial x} = [\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_l}]^T\) and \(\frac{\partial}{\partial y} = [\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_l}]^T\), it follows immediately from the above definition that

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}.
\]

The Hessian matrix is defined by

\[
H_1(z) := \begin{bmatrix}
H_{xx}(z) & H_{xz}(z) \\
H_{zx}(z) & H_{zz}(z)
\end{bmatrix}
\]

or

\[
H_2(z) := \begin{bmatrix}
H_{xx}(z) & H_{xy}(z) \\
H_{yx}(z) & H_{yy}(z)
\end{bmatrix}.
\]
where, for $k, l = 1, 2, \ldots, l$,

$$
\begin{align*}
(H_{2z})_{i l}(z) &= \frac{\partial}{\partial z_i} \left( \frac{\partial f(z)}{\partial z_k} \right), \\
(H_{2z})_{i l}(z) &= \frac{\partial}{\partial z_i} \left( \frac{\partial f(z)}{\partial z_k} \right).
\end{align*}
$$

Note that $\mathcal{H}_1(z) \in \mathbb{C}^{2l \times 2l}$ is symmetric, whereas $\mathcal{H}_2(z) \in \mathbb{C}^{2l \times 2l}$ is Hermitian.

3. A framework

In this section, we describe a framework of deriving and analyzing neural network systems for computing the principal Takagi vector of a complex symmetric matrix. We refer to [1,4,13,20,33,35,38,39] for solving a complex-valued nonlinear convex programming problem by a complex-valued neural network model.

Wang et al. [34] proposed a dynamics of the complex-valued neural network model for computing the principal Takagi vector of a complex symmetric matrix $A$:

$$
\frac{dZ(t)}{dt} = f(Z; A), \quad \text{where } f(Z; A) = (z^*Z - Z^*Az + z^*Az) / 2.
$$

(3.1)

where $z = [z_1, z_2, \ldots, z_l]^T \in \mathbb{C}^l$ represents the state of the network. It is proved in [34] that the equilibrium point $1$ of (3.1) corresponds to the principal Takagi vector of $A$. A simple discrete-time iterative algorithm corresponding to the neural network described in (3.1) can be written as

$$
\begin{align*}
Z_{k+1} &= z_k + \eta_k f(Z_k; A) \\
\text{with } f(z_k; A) &= (z_k^*Z_k) - z_k^*Az_k + z_k^*Az_k.
\end{align*}
$$

(3.2)

where $z_k \in \mathbb{C}^l$ is a given initial value and $\{\eta_k\}$ is a decreasing gain sequence. Following [17], we make the following assumption on $\{\eta_k\}$.

**Assumption 3.1.** The gain sequence $\{\eta_k \geq 0\}$ is decreasing such that $\sum_{k=0}^{\infty} \eta_k = \infty$, $\sum_{k=0}^{\infty} \eta_k^r < \infty$ for some $r > 1$ and $\lim_{k \to \infty} (\eta_k - \eta_k^{r+1}) < \infty$.

Based on the relationship between the Takagi pairs of $A$ and the eigenpairs of $A$, we can apply the theory of stochastic approximation to prove the convergence of $z_k$ in (3.2) to the principal Takagi vector of $A$. Note that the stable stationary solution of (3.1) is a convergence point of (3.2). Hence we either solve (3.1) or analyze its stable stationary point.

In the following, from the framework, we derive and analyze eight different adaptive algorithms for computing the principal Takagi vector, as a realization and development of (3.2). The function $f(z_k; A)$ for various adaptive algorithms is given by:

- **Type I** $A_k - z_k^*Az_k + z_k^*Az_k$.
- **Type II** $\frac{1}{z_k} \left( A_k - z_k^*Az_k + z_k^*Az_k \right)$.
- **Type III** $A_k - z_k^*Az_k + z_k^*Az_k = z_k^*Az_k$.
- **Type IV** $z_k^*Az_k - z_k^*Az_k + z_k^*Az_k$.
- **Type V** $z_k^*Az_k - z_k^*Az_k + z_k^*Az_k$.
- **Type VI** $A_k - \mu_k z_k^*Az_k - z_k$.
- **Type VII** $A_k - z_k^*Az_k + z_k^*Az_k - z_k$.
- **Type VIII** $A_k - z_k^*Az_k + z_k^*Az_k - \mu_k z_k^*Az_k - z_k$.

Type I is based on nonlinear programming, Types II, III, and IV are based on the generalized Rayleigh quotient criterion, Type V is based on the information theory criterion, Type VI is based on the penalty function, and Types VII and VIII are based on the augmented Lagrangian criterion.

Note that we cannot ensure that $z^*A^2z + z^*Az$ is always positive, where $z$ is the limit of the sequence $\{z_k, k = 1, 2, \ldots\}$, which is derived from the above eight adaptive algorithms. If $z^*A^2z + z^*Az < 0$, when considering Type I and setting $u = e^{i\pi/2}z$, we have

$$
\frac{A^2u}{2} = u'A^2u + u'A^2u > 0.
$$

Although there exist many numerical algorithms to computing the Takagi factorization of complex symmetric matrices (see, e.g., [7,36,37]), it is more effective to use eight adaptive algorithms for computing the Takagi vectors of complex symmetric matrices in the following situations:

(a) when we only want to derive the Takagi vector complex symmetric matrices, associated to the largest Takagi value;

(b) when we want to draw the graph about each component of the Takagi vector of parameterized complex symmetric matrices, associated to the largest Takagi value.

3.1. Type I

For Type I, the adaptive algorithm is derived from the following nonlinear programming

$$
\min_{z \in \mathbb{C}^l} \left\{-z^*Az + z^*Az \right\}, \quad \text{subject to } z^*z = 1.
$$

When $f(z_k; A)$ is Type I, we have

$$
\frac{dZ(t)}{dt} = A - z(t)^*Az + z(t)^*Az / 2.
$$

(3.4)

where $z(t) \in \mathbb{C}^l$ is the continuous time counterpart of $z_k$ with $t \geq 0$ representing time. The following theorem gives the stable equilibrium points and the convergence properties of (3.4).

**Theorem 3.1.** Under the conditions of Assumption 3.1, given a symmetric matrix $A \in \mathbb{C}^{l \times l}$ with $\sigma_1 > \sigma_2 \geq \cdots \geq \sigma_l \geq 0$, let $z(t) = \sum_{i=1}^l a_i(t)q_i$ be a solution of the complex-valued ODE (3.4) expressed in terms of the set $\{q_1, q_2, \ldots, q_l\}$ of all the unitary Takagi vectors of $A$, then for any initial condition $z(0) = z_0 \in \mathbb{C}^l$ with $z_0^*q_i \neq 0$ and the real and imaginary parts of $a_i(0)$ being nonzero, the coefficients $a_i(t)$ $(i = 2, 3, \ldots, l)$ of the solution $z(t)$ are given by

$$
\frac{d}{dt} a_i(t) = (a_i(t) + \sigma_i a_i(t)) e^{-(\sigma_i + \eta_i) t}.
$$

(3.5)
Furthermore, we also have $\Re(a_i(t)) \to -1$ and $\Im(a_i(t)) \to 0$ as $t \to \infty$.

The points $\pm q_i$ are (uniformly) asymptotically stable. The domain of contraction of $q_i$ is $D(q_i) = \{z \in \mathbb{C} : \Re(z^* q_i) > 0 \}$ and that of $-q_i$ is $D(q_i) = \{z \in \mathbb{C} : \Re(z^* q_i) < 0 \}$.

**Proof.** Substituting $z(t)$ in (3.4) with $z(t) = \sum_{i=1}^{l} a_i(t) q_i$ and pre-multiplying the both side with $q_i^*$, we obtain

$$\frac{d a_i(t)}{dt} = \sigma_i \Re(a_i(t)) - q_i \sum_{j=1}^{l} \sigma_j \Re((a_j(t))^2 - \Im(a_j(t))^2),$$

$$i = 1, 2, \ldots, l. \quad (3.6)$$

Define $b \in \mathbb{C}^{l-1}$ such that

$$\Re(b_i) = \frac{\Re(a_i)}{\Re(a_1)}, \quad \Im(b_i) = \frac{\Im(a_i)}{\Re(a_1)}, \quad i = 2, \ldots, l,$$

It then follows from (3.6) that

$$\frac{d \Re(b_i(t))}{dt} = -\Re(b_i(t)) \sigma_1 - \sigma_i$$

and

$$\frac{d \Im(b_i(t))}{dt} = -\Im(b_i(t)) \sigma_1 + \sigma_i, \quad i = 2, \ldots, l,$$

which implies (3.5). From (3.5), we have $\Re(a_i(t)) \to 0$ and $\Im(a_i(t)) \to 0$ as $t \to \infty$, for $i = 2, \ldots, l$. Thus, we have $\Re(a_1(t)) \to x(t)$ and $\Im(a_1(t)) \to y(t)$ as $t \to \infty$, where $x(t)$ and $y(t)$ satisfies the following ordinary differential equations:

$$\frac{dx(t)}{dt} = \sigma_1 x(t) - \sigma_1 (x(t)^2 - y(t)^2)x(t),$$

$$\frac{dy(t)}{dt} = -\sigma_1 y(t) - \sigma_1 (x(t)^2 - y(t)^2)y(t). \quad (3.7)$$

When letting $u(t) = x(t)^2$ and $v(t) = y(t)^2$, we have

$$\frac{du(t)}{dt} = -\sigma_1 u(t) - \sigma_1 (u(t) - v(t))u(t)^2,$$

$$\frac{dv(t)}{dt} = -\sigma_1 v(t) - \sigma_1 (u(t) - v(t))v(t)^2,$$

which implies that

$$u(t) = \frac{e^{2\sigma_1 t} u(0) - 2\sigma_1 v(t) t + 1}{e^{2\sigma_1 t} u(0) - 2\sigma_1 v(t) t + 1}.$$

with $u(0) = x(0)^2 > 0$ and $v(0) = y(0)^2 > 0$, where $x(0)$ and $y(0)$ are the initial values of (3.7). Thus, $u(t) \to 1$ and $v(t) \to 0$ as $t \to \infty$, which lead to $\Re(a_1(t)) \to -1$ and $\Im(a_1(t)) \to 0$ as $t \to \infty$. Thus the sign of $\Re(a_i(t))$, $i = 2, \ldots, l$, is determined by the sign of the initial $\Re(a_1(t)) = \Re(z(t^* q_1))$. From (3.5), $\Re(a_i(t)) \to 0$ and $\Im(a_i(t)) \to 0$, as $t \to \infty$, for $i = 2, 3, \ldots, l$. Thus $z(t) \to \pm q_1$ as $t \to \infty$. □

**Remark 3.1.** Similar to the proof of Theorem 3.1, we also have

$$\Re(a_i(t)) = \Re(a_i(0)) e^{\sigma_1 - \sigma_i t}, \quad \Im(a_i(t)) = \Im(a_i(0)) e^{\sigma_1 - \sigma_i t},$$

$$i = 2, 3, \ldots, l.$$

It is easy to verify that both $\Re(a_i(t))$ and $\Im(a_i(t))$ are the high order infinitesimal of $\Re(a_1(t))$, which also hold for the other adaptive algorithms in this paper.

The convergence of (3.3) is now a direct consequences of the above theorem and Theorem 1 of Ljung [17].

3.2. Types II, III and IV

All the three types are derived from the objective function:

$$J(z_k; A) = \frac{-z_k^* A z_k + z_k^* \bar{A} z_k}{2z_k^* z_k}.$$

(3.8)

called the generalized Rayleigh quotient of $A$. Our models are generalizations of the previous work by Cirrincione et al. [10], Luo et al. [18] and Oja [22], where the real case is considered. The gradient of (3.8) with respect to $z_k$ is

$$\nabla_{z_k} J(z_k; A) = \frac{1}{z_k^* z_k} \left( A z_k - z_k^* A z_k + z_k^* \bar{A} z_k \right).$$

Hence, the adaptive gradient descent algorithm for Type II is

$$z_{k+1} = z_k - \eta_k \nabla_{z_k} J(z_k; A) = z_k - \eta_k \frac{1}{z_k^* z_k} \left( A z_k - z_k^* A z_k + z_k^* \bar{A} z_k \right).$$

(3.9)

and the complex-valued ODE associated with (3.9) is

$$\frac{dz(t)}{dt} = \frac{1}{z(t)} \left( A z - z^* A z + z^* \bar{A} z \right).$$

(3.10)

where $z(t) \in \mathbb{C}^l$ is the continuous time counterpart of $z_k$. For Type III, the adaptive gradient descent algorithm is

$$z_{k+1} = z_k - \eta_k (z_k^* z_k) \nabla_{z_k} J(z_k; A) = z_k - \eta_k \left( A z_k - \frac{z_k^* A z_k + z_k^* \bar{A} z_k}{2 z_k^* z_k} \right).$$

(3.11)

and the complex-valued ODE associated with (3.11) is

$$\frac{dz(t)}{dt} = \frac{A z - z^* A z + z^* \bar{A} z}{2z^* z}.$$  (3.12)

For Type IV, the adaptive gradient descent algorithm is

$$z_{k+1} = z_k - \eta_k (z_k^* z_k)^2 \nabla_{z_k} J(z_k; A)$$

$$= z_k - \eta_k (z_k^* z_k) \left( A z_k - \frac{z_k^* A z_k + z_k^* \bar{A} z_k}{2 z_k^* z_k} \right).$$

(3.13)

and the complex-valued ODE associated with (3.13) is

$$\frac{dz(t)}{dt} = \frac{z^* (A z - z^* A z + z^* \bar{A} z)}{2z^* z}.$$  (3.14)

The solutions of the above three complex-valued ODEs and their properties are summarized in the following theorem.

**Theorem 3.2.** Under the conditions of Assumption 3.1, given a symmetric matrix $A \in \mathbb{C}^{l \times l}$ with $-\sigma_1 > -\sigma_2 > \cdots > -\sigma_l > 0$, let $z(t) = \sum_{i=1}^{l} a_i(t) q_i$ be a solution of the complex-valued ODEs (3.10) or (3.12) or (3.14) expressed in the terms of the set $\{q_1, q_2, \ldots, q_l\}$ of all the unitary Takagi vectors of $A$, then for any initial condition $z(0) = z_0 \in \mathbb{C}^l$ with $\Re(z_0 q_i) \neq 0$ and the real and imaginary parts of $a_i(0)$ being nonzero, we have the following statements:

(a) for all $t \geq 0$, the coefficients $a_i(t)$ ($i = 2, 3, \ldots, l$) of the solution for (3.10) are given by

$$\frac{\Re(a_i(t))}{\Re(a_i(0))} = \frac{\Re(a_i(0))}{\Re(a_i(0))} e^{-\sigma_1 - \sigma_i t|z(0)|^2},$$

$$\frac{\Im(a_i(t))}{\Im(a_i(0))} = \frac{\Im(a_i(0))}{\Im(a_i(0))} e^{-\sigma_1 - \sigma_i t|z(0)|^2};$$

(3.15)

(b) for all $t \geq 0$, the coefficients $a_i(t)$ ($i = 2, 3, \ldots, l$) of the solution for (3.12) are given by

$$\frac{\Re(a_i(t))}{\Re(a_i(0))} = \frac{\Re(a_i(0))}{\Re(a_i(0))} e^{-\sigma_1 - \sigma_i t|z(0)|^2},$$

$$\frac{\Im(a_i(t))}{\Im(a_i(0))} = \frac{\Im(a_i(0))}{\Im(a_i(0))} e^{-\sigma_1 - \sigma_i t|z(0)|^2};$$

(3.16)

(c) for all $t \geq 0$, the coefficients $a_i(t)$ ($i = 2, 3, \ldots, l$) of the solution for (3.14) are given by

$$\frac{\Re(a_i(t))}{\Re(a_i(0))} = \frac{\Re(a_i(0))}{\Re(a_i(0))} e^{-\sigma_1 - \sigma_i t|z(0)|^2},$$

$$\frac{\Im(a_i(t))}{\Im(a_i(0))} = \frac{\Im(a_i(0))}{\Im(a_i(0))} e^{-\sigma_1 - \sigma_i t|z(0)|^2};$$

(3.17)
(d) Furthermore, for (3.10), (3.12) and (3.14), we also have
\(\nabla (a_i(t)) \to \pm \|z_0\|_2\) and \(\nabla (a_i(t)) \to 0\) as \(t \to \infty\).

The points \(\pm q_i\) are (uniformly) asymptotically stable. The domain of
contraction of \(q_i\) is \(D(q_i) = \{z \in C^1 : \|a_i q_i\| > 0\}\) and that of
\(-q_i\) is \(D(-q_i) = \{z \in C^1 : \|a_i (z^* q_i)\| < 0\}\).

Proof. Analogous to the proof of Theorem 3.1. \(\Box\)

For Type II (3.10), the equations in (3.15) imply \(\nabla (a_i(t)) \to 0\) and
\(\nabla (a_i(t)) \to 0\), as \(t \to \infty\), for \(i = 2, 3, \ldots, l\). For Type III (3.12),
the equations in (3.16) imply \(\nabla (a_i(t)) \to 0\) and \(\nabla (a_i(t)) \to 0\), as \(t \to \infty\),
for \(i = 2, 3, \ldots, l\). For Type IV (3.14), the equations in (3.17) imply
\(\nabla (a_i(t)) \to 0\) and \(\nabla (a_i(t)) \to 0\), as \(t \to \infty\), for \(i = 2, 3, \ldots, l\).
Thus, if \(\|z_0\|_2 = 1\) for Types II, III and IV, then \(z(t) \to \pm q_i\) as \(t \to \infty\).

Note that if \(z\) satisfies (3.10) or (3.12) or (3.14), then
\[
\frac{d}{dt} \|z(t)\|_2^2 = (z(t)^* dA^*_z \frac{dz}{dt} + z(t)^* dA_z \frac{dz}{dt} - z(t)^* = 0,
\]

implying that \(\|z(t)\|_2 = \|z(0)\|_2\), \(t > 0\), for all the three types. Applying the approximation \(\|z(t)\|_2 \approx \|z(0)\|_2\), to Type II (3.9) and Type IV (3.13), we have the following simplified Types II and IV algorithms

\[
\begin{align*}
(3.9)' : \quad & z_{k+1} = z_k + \eta_k \|z_0\|_2^2 \left( A^*_z - \frac{z^*_A A z_k + z^*_A A z_k}{2z^*_A z_k} \right), \\
(3.13)' : \quad & z_{k+1} = z_k + \eta_k \|z_0\|_2^2 \left( A^*_z - \frac{z^*_A A z_k + z^*_A A z_k}{2z^*_A z_k} \right).
\end{align*}
\]

Although, the three adaptive algorithms (3.9)', (3.11) and (3.13)' differ only in their gain constants, our experiments show that they perform differently.

3.3. Type \(V\)

When \(f(z_k; A)\) is defined by Type \(V\), we have the following adaptive method:

\[
z_{k+1} = z_k + \eta_k \left( -\frac{2A^*_z}{z^*_A A z_k + z^*_A A z_k} - z_k \right).
\]

Our model is a generalization of the previous work by Plumbly [24] and Miao and Hua [19], where real \(A\) and \(z_k\) in (3.18) are considered. The complex-valued ODE associated with the adaptive method (3.18) is

\[
\frac{dz(t)}{dt} = -\frac{2A^*_z}{z^*_A A z_k + z^*_A A z_k} - z(t).
\]

where \(z(t) \in C^i\) is the continuous time counterpart of \(z_k\) with \(t\) representing time. If \(z\) satisfies (3.18), then
\[
\frac{d}{dt} \|z(t)\|_2^2 = (z(t)^* dA^*_z \frac{dz}{dt} + z(t)^* dA_z \frac{dz}{dt} - z(t)^* = 0.
\]

and \(\|z(t)\|_2 = \|z(0)\|_2\), \(t \to \infty\), implying that \(\|z(t)\|_2 = 1\) as \(t \to \infty\).

We now analyze the stable stationary points of the complex-valued ODE (3.19). An energy function for (3.19) is given by
\[
E(z) = z^* z - \frac{1}{2} \ln \left( \frac{z^*_A A z + z^*_A A z}{2} \right)^2.
\]

Since \(\lim_{\|z\|_2 \to 0} E(z) = +\infty\), \(\lim_{\|z\|_2 \to \infty} E(z) = +\infty\), the function \(E(z)\) has global minimum points. The following theorem shows a relation between the principal Takagi vector of \(A\) and the global minima of \(E(z)\).

Theorem 3.3. Under the conditions of Assumption 3.1, let \(A \in C^{l \times l}\) be a symmetric matrix with \(\sigma_1 > \sigma_2 > \cdots > \sigma_l > 0\), then the two converging points \(\pm q_i\) of the complex-valued ODE (3.19) are the global minimum points of the energy function \(E(z)\), which has no other local minimum point. In addition, \(\pm q_i\) are the saddle points of \(E(z)\).

Proof. The gradients of \(E(z)\) with respect to \(z\) and \(z^\ast\) are respectively
\[
\frac{\partial E(z)}{\partial z^\ast} = z - \frac{2A^*_z}{z^*_A A z + z^*_A A z} \quad \text{and} \quad \frac{\partial E(z)}{\partial z} = \frac{2A^*_z}{z^*_A A z + z^*_A A z}.
\]

It then follows that the stationary points of \(E(z)\) are indeed the Takagi vectors \(\pm q_i\), \(i = 2, 3, \ldots, l\). The Hessian matrix of \(E(z)\) is
\[
H(z) := \begin{bmatrix} E_{zz}(z) & E_{z^\ast z}(z) \\ E_{z^\ast z}(z) & E_{zz^\ast}(z) \end{bmatrix}
\]

where
\[
E_{zz}(z) = -\frac{2A}{z^*_A A z + z^*_A A z} + \frac{4A^*_z A}{(z^*_A A z + z^*_A A z)^2}.
\]

Since both the eigenvalues of \(H(\pm q_i)\) are positive, \(H(\pm q_i)\) is positive definite. Thus the stationary points \(\pm q_i\) are the local minimum points of \(E(z)\).

Similarly, it can be verified that
\[
H(\pm q_i) = \begin{bmatrix} \sigma_i - \sigma_1 & \pm q_i \\ \pm q_i & \sigma_1 \end{bmatrix}, \quad i = 2, 3, \ldots, l.
\]

Since \(H(\pm q_i)\), \(2 \leq i \leq l\) has two positive eigenvalues and \((2l - 2)\) negative eigenvalues, it is indefinite. Hence, \(\pm q_i\) are saddle points of \(E(z)\). Since \(E(z)\) has only two local minimal points \(\pm q_i\) and \(E(q_i) = E(-q_i)\), \(\pm q_i\) are the global minima. \(\Box\)

The rate of convergence for (3.18) can be obtained from the equation \(\|z(t)\|_2^2 = 1 + (\|z(t)\|_2^2 - 1)e^{-2t}\). A unique feature of this algorithm is that the time constant for \(\|z(t)\|_2\) is 1 and independent of the Takagi structure of \(A\).

3.4. Type \(VI\)

When \(f(z_k; A)\) is given by Type \(VI\), we have the following adaptive procedure:

\[
z_{k+1} = z_k + \eta_k \left( A^*_z - \mu z_k (z_k^* A z_k - 1) \right),
\]

where \(\mu > 0\). Our method is a generalization the previous work by Chauvin [8], where both \(A\) and \(z_k\) in \(f(z_k; A)\) are assumed to be real. The complex-valued ODE associated with the adaptive method (3.20) is

\[
\frac{dz(t)}{dt} = A^*_z - \mu z_k (z^*_A z_k - 1).
\]
where $\mu > 0$ and $\mathbf{z}(t) \in \mathbb{C}^d$ is the continuous time counterpart of $\mathbf{z}_i$ with $t$ representing time. The solution of (3.21) and its properties are summarized in the theorem below.

**Theorem 3.4.** Under the conditions of Assumption 3.1, for a given symmetric matrix $\mathbf{A} \in \mathbb{C}^{d \times d}$ with $\sigma_1 > \sigma_2 \geq \cdots \geq \sigma_d > 0$, let $\mathbf{z}(t) = \sum_{i=1}^{d} \mathbf{q}_i(t) \mathbf{q}_i^*$ be a solution of the complex-valued ODE (3.21) expressed in terms of the set $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_d\}$ of all the unitary Takagi vectors of $\mathbf{A}$, then for any initial condition $\mathbf{z}(0) = \mathbf{z}_0 \in \mathbb{C}^d$ with $\mathbf{z}_0 \neq \mathbf{0}$ and the real and imaginary parts of $\mathbf{a}(0)$ being nonzero, the coefficients $\mathbf{a}_i(t)$ ($i = 1, 2, \ldots, d$) of the solution $\mathbf{z}(t)$ for $t > 0$ are given by

$$\begin{align*}
\mathbf{a}_i(t) &= \mathbf{a}_i(0) e^{-(\sigma_1 - \sigma_i) t}, \\
\mathbf{a}_i^*(t) &= \mathbf{a}_i(0) e^{-(\sigma_1 + \sigma_i) t}.
\end{align*}$$

(3.25)

Furthermore, we also have $\mathbf{a}_i(t) \to \pm \sqrt{1 + \sigma_1 / \mu}$ and $\mathbf{a}_i^*(t) \to 0$ as $t \to \infty$.

The points $\pm \sqrt{1 + \sigma_1 / \mu}$ are (uniformly) asymptotically stable. The domain of contraction of $\sqrt{1 + \sigma_1 / \mu}$ is $\mathbb{D}(1) = \{ \mathbf{z} \in \mathbb{C}^d : \mathbf{z}(\mathbf{q}_1) > 0 \}$ and that of $-\sqrt{1 + \sigma_1 / \mu}$ is $\mathbb{D}(1) = \{ \mathbf{z} \in \mathbb{C}^d : \mathbf{z}(\mathbf{q}_1) < 0 \}$.

**Proof.** Analogous to the proof of Theorem 3.1. □

It follows from (3.22) that $\mathbf{a}_i(t) \to 0$ and $\mathbf{a}_i^*(t) \to 0$, as $t \to \infty$, for $i = 2, 3, \ldots, d$. Thus $\mathbf{z}(t)$ converges to $\pm \sqrt{1 + \sigma_1 / \mu}$, $\mathbf{q}_1$. In terms of the energy function

$$E(z) = -\frac{z^* \mathbf{A} z + z^* \mathbf{A} z}{2} + \frac{\mu}{2} (z^* z - 1)^2$$

for the adaptive procedure (3.20), the stable stationary points of (3.21) are given in the following theorem.

**Theorem 3.5.** Under Assumption 3.1, let $\mathbf{A} \in \mathbb{C}^{d \times d}$ be a symmetric matrix with $\sigma_1 > \sigma_2 \geq \cdots \geq \sigma_d > 0$, then the two converging points $\pm \sqrt{1 + \sigma_1 / \mu}$ are of the complex-valued ODE (3.21) are the global stable points of $E(z)$, which has no other local minimum point. In addition, $\pm \sqrt{1 + \sigma_1 / \mu}$ are saddle points of $E(z)$.

**Proof.** Analogous to the proof of Theorem 3.3. □

3.5. Types VII and VIII

When $\mathbf{f}(\mathbf{z}_i; \mathbf{A})$ is given by Type VIII, we have the following adaptive procedure:

$$\mathbf{z}_{k+1} = \mathbf{z}_k + \eta_k \left( \mathbf{A} \mathbf{z}_k - \mathbf{z}_k \mathbf{A} \mathbf{z}_k + \frac{\mathbf{z}_k^* \mathbf{A} \mathbf{z}_k}{2} - \mu \mathbf{z}_k (\mathbf{z}_k^* \mathbf{z}_k - 1) \right),$$

(3.23)

where $\mu > 0$. Note that when $\mu = 1$, the above adaptive procedure reduces to Type VII. The complex-valued ODE associated with the adaptive method (3.23) is

$$\frac{d \mathbf{z}(t)}{dt} = \mathbf{A} \mathbf{z}(t) - \mathbf{z}(t) \mathbf{A} \mathbf{z}(t) + \frac{\mathbf{z}(t)^* \mathbf{A} \mathbf{z}(t)}{2} - \mu \mathbf{z}(t) (\mathbf{z}(t)^* \mathbf{z}(t) - 1),$$

(3.24)

where $\mu > 0$ and $\mathbf{z}(t) \in \mathbb{C}^d$ is the continuous time counterpart of $\mathbf{z}_i$ with $t$ representing time. The solution of (3.24) and its properties are summarized in the theorem below.

**Theorem 3.6.** Under the conditions of Assumption 3.1, given a symmetric matrix $\mathbf{A} \in \mathbb{C}^{d \times d}$ with $\sigma_1 > \sigma_2 \geq \cdots \geq \sigma_d > 0$, let $\mathbf{z}(t) = \sum_{i=1}^{d} \mathbf{q}_i(t) \mathbf{q}_i^*$ be a solution of the complex-valued ODE (3.24) expressed in terms of the set $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_d\}$ of all the unitary Takagi vectors $\mathbf{A}$, then for any initial condition $\mathbf{z}(0) = \mathbf{z}_0 \in \mathbb{C}^d$ with $\mathbf{z}_0 \neq \mathbf{0}$ and

In this section, some computer simulation results are shown to illustrate our models. All computations were carried out in MATLAB Version 2016a, which has a unit of roundoff $2^{-52} = 1.1 \times 10^{-16}$, on a laptop with Intel Core i5-3470M CPU (3.20 GHz) and 8.00 GB RAM. All floating-point numbers were rounded to two digits after the decimal point.

Chen et al. [9] presented an iterative algorithm for computing the largest Takagi value of complex symmetric matrices, which is similar to the power method [12] for matrix eigenvalue problems. Here we denote this algorithm by “PM”. In this section, we also compare the proposed methods with PM for different testing complex symmetric matrices.

First, we ran our adaptive algorithms on a single data set with various starting vectors $\mathbf{z}_0$. Then, we generated several data samples and used the same starting vector $\mathbf{z}_0$. We refer to the Takagi values and Takagi vectors of a symmetric matrix $\mathbf{A} \in \mathbb{C}^{d \times d}$ computed by the divide-and-conquer method [36] as the actual values. In order to measure the convergence and accuracy of the algorithms, we computed the percentage direction cosine at kth iteration of each adaptive algorithm defined by

$$\cosine(k) = \frac{100 |\mathbf{z}_0^* \mathbf{f}_1|}{\parallel \mathbf{z}_0 \parallel_2},$$

where $\mathbf{z}_0$ is the approximate principal Takagi vector of $\mathbf{A}$ at kth iteration and $\mathbf{f}_1$ is the actual principal Takagi vector computed by the divide-and-conquer method. Thus, cosine(k) ≤ 100 and the larger the more accurate $\mathbf{z}_0$ is.

**Example 4.1.** A Hankel matrix is a square matrix with constant skew-diagonals (positive sloping diagonals). By definition, an $n \times l$ complex Hankel matrix can be generated by a given sequence $\{x_j \in \mathbb{C} : i = 1, 2, \ldots, 2l - 1\}$ which determines its first column and last row.

For a given positive integer $l$, the first column and the last row of a Hankel matrix $\mathbf{A}$ are given by

$$\begin{align*}
\mathbf{a}_{k+1} &= \mathbf{a}_k, \\
\mathbf{a}_{k+i+1} &= \mathbf{a}_k, \\
\end{align*}$$

where all $\mathbf{a}_k$ are given by [32]:

$x_k = \exp((-0.01 + 0.04 \pi) k) + \exp((-0.02 + 0.44 \pi) k)$.

When $l = 1000$, Fig. 1 shows the error

$$\text{ERR}(k) := \| \mathbf{A} \mathbf{z}_k - \sigma \mathbf{z} \|_2.$$
where $z_k$ is the principal Takage vector and $\sigma_k$ is the corresponding Takage value of $A$ computed at the $k$th iteration of each of the eight algorithms using any given random initial $z_0$ in Example 4.1. These algorithms take 3.18s, 2.52s, 2.94s, 2.73s, 2.47s, 1.70s, 2.53s, 2.63s and 2.70s, respectively. As seen in Fig. 1, Type V (3.18) is the worst. Hence, we do not consider the numerical implementation of the adaptive algorithm with Type V.

When $l = 10$, applying the MATLAB function svd, the first two Takage values of $A$ are 9.06 and 8.24, respectively. The rest of the Takage values are much smaller than the first two. The initial values of all the adaptive algorithms are $z_0 = 0.5r$, where $r \in C^{10}$ is a complex random vector such that the real parts and the imaginary parts of its entries obey the standard normal distribution. For all the algorithms, we set gain factor $\eta_k = 1/(100 + k)$. For Type VI (3.20) and Type VIII (3.23), we set $\mu = 0.1$.

The results are summarized in Table 1, where we report the percentage direction cosine after $k = 250$ and 300 iterations for each algorithm. Table 1 shows that Type IV (3.13) converged faster than the others.

**Example 4.2.** Suppose that $l$ is a positive integer. The testing complex symmetric matrices $A \in C^{l \times l}$ were generated by the following steps [37]:

1. generate a random unitary matrix $Q \in C^{l \times l}$: $[Q \ R] = qr(randn(l) + i \ randn(l));$
2. generate a diagonal matrix $D \in \mathbb{R}^{l \times l}$ with diagonal elements $d_{11} = 25, \ d_{22} = 25/c, \ d_{kk} = rand(1), \ k = 3, 4, \ldots, l$.

### Table 1

The percentage direction cosines of the principal Takage vector of $A$ computed by the adaptive algorithms at iterations $k = 250, 300$ for random initial $z_0$ in Example 4.1.

<table>
<thead>
<tr>
<th>$|z_0|_2$</th>
<th>$k$</th>
<th>Type I</th>
<th>Type II</th>
<th>Type III</th>
<th>Type IV</th>
<th>Type VI</th>
<th>Type VII</th>
<th>Type VIII</th>
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<td>65.31</td>
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<td>84.93</td>
<td>84.95</td>
<td>84.99</td>
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</tbody>
</table>
where \( c \in \{1.10, 1.14, 1.20, 1.23, 1.30, 1.32, 1.40, 1.41, 1.47, 1.50\} \).

The initial value of all the adaptive algorithms was set to \( z_0 = 0.5r \), where \( r \in \mathbb{C}^1 \) is a complex random vector such that the real parts and the imaginary parts of its entries obey the standard normal distribution. For all the algorithms, we set \( \eta_k = 1/(100 + k) \) for Type VI (3.20) and Type VII (3.23), we set \( \mu = 0.1 \).

For the case of \( I = 10 \), Table 2 lists the percentage direction cosines after \( k = 250 \) and 300 iterations for each algorithm. Table 2 shows that Type IV (3.13) converges faster than the others, the convergence behavior of Type I (3.3), Type III (3.11), Type VI (3.20), Type VII (3.23) with \( \mu = 1 \), and Type VIII (3.23) is similar and the convergence behavior of Type II (3.9) is similar. As expected, the algorithm converges faster when the ratio \( \sigma_1/\sigma_2 \) is larger.

**Example 4.3.** The testing complex symmetric matrices were generated the same as **Example 4.2** except for the second step, we set \( \sigma_1 \in \{11.58, 11.63, 11.73, 11.84, 12.14, 12.54, 12.87, 13.57, 14.09, 17.97\} \);
\( \sigma_2 \in \{6.32, 6.49, 6.92, 7.18, 7.64, 8.08, 8.67, 9.33, 11.66\} \).
So, the diagonal entries of \( D \) are
\[ \begin{align*}
&d_{11} = \sigma_1, \quad d_{22} = \sigma_2, \quad d_{kk} = \text{rand}(1), \quad k = 3, 4, \ldots, 10. 
\end{align*} \]

The initial values of all adaptive algorithms were \( z_0 = 0.5r \), where \( r \in \mathbb{C}^{10} \) is a complex random vector such that the real parts and the imaginary parts of its entries obey the standard normal distribution. For all the algorithms, we set \( \eta_k = 1/(100 + k) \) for Type VI (3.20) and Type VIII (3.23), we set \( \mu = 10 \).

The results are listed in Table 3, showing that the convergence behavior of all adaptive algorithms is similar to the results in **Example 4.2**.

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References


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