

Generalized Inverses: Theory and Computations

Guorong Wang

Yimin Wei

Sanzheng Qiao

MATHEMATICS & SCIENCE COLLEGE, SHANGHAI NORMAL UNIVERSITY, SHANGHAI 200234, P.R. CHINA

E-mail address: grwang@online.sh.cn

DEPARTMENT OF MATHEMATICS, FUDAN UNIVERSITY, SHANGHAI 200433, P.R. CHINA

E-mail address: ymwei@fudan.edu.cn

DEPARTMENT OF COMPUTING AND SOFTWARE, MCMASTER UNIVERSITY, HAMILTON, ONTARIO L8S4L7, CANADA

E-mail address: qiao@mcmaster.ca

Graduate Textbook of Shanghai

Science Press

Brief Introduction

Generalized inverses arise in various applications in statistics, science and engineering, such as least squares approximation, singular differential and difference equations, Markov chains and ill-posed problems. This book contains the latest developments in the theory and computations of the generalized inverses.

This book was written for researchers in matrix theory, numerical linear algebra, parallel computations and, particularly, the generalized inverses with applications. And it can also be used as a text or reference for a graduate course. This book is based on basic linear algebra, matrix theory and functional analysis.

Preface

The concept of generalized inverses was first introduced by I. Fredholm [57] in 1903, where a generalized inverse of an integral operator was given and was called “pseudoinverse”. Generalized inverses of differential operators were implied in D. Hilbert’s [78] discussion of generalized Green’s functions in 1904. See W. Reid’s [131] paper in 1931 for a history of generalized inverses of differential operators.

The generalized inverse of matrices was first introduced by E.H. Moore [113] in 1920, who defined a unique generalized inverse by means of projectors of matrices. Little was done in the next 30 years until mid-1950s when discoveries of the least-squares properties of certain generalized inverses and the relationship of generalized inverses to solutions of linear systems brought new interests in the subject. In particular, R. Penrose [119] showed in 1955 that the Moore’s inverse is the unique matrix satisfying four matrix equations. This important discovery revived the study of generalized inverses. In honor of Moore and Penrose’s contribution, this unique generalized inverse is called the Moore-Penrose inverse.

The theory, applications and computational methods of generalized inverses have been developing rapidly during the last 50 years. One milestone is the publication of several monographs ([7], [13], [65] and [129]) on the subject in 1970s, particularly, the excellent volume by Ben-Israel and Greville [7] which has made a long lasting impact on the subject; the other milestone is the publications of two volumes of proceedings. The first is the volume of proceedings [114] of the Advanced Seminar on Generalized Inverses and Applications held at the University of Wisconsin-Madison in 1973 edited by M.Z. Nashed. It is an excellent and extensive survey book. It contains 14 survey papers on the theory, computations and applications of generalized inverses and an exhaustive bibliography that includes all related references up to 1975. The other is the volume of proceedings [11] of the AMS Regional Conference held in Columbia, South Carolina in 1976 edited by S.L. Campbell. It is a new survey book containing 12 papers on the latest applications of generalized inverses. The volume describes changes in research directions and types of generalized inverses since mid-1970s. Prior to this period, due to the applications in statistics, research often centered on generalized inverses for solving linear systems and generalized inverses with least-squares properties. Recent studies focus on such topics as: infinite dimensional theory, numerical computation, matrices of special type (Boolean, integral), matrices over algebraic structures other than real or complex fields, systems theory and non-equation solving generalized inverses.

I have been conducting teaching and research in generalized inverses of matrices since 1976. I gave a course “Generalized Inverses of Matrices” and held

many seminars for graduate students majoring in Computational Mathematics in our department. Since 1979, my colleagues and I with graduate students have obtained a number of results on generalized inverses in the areas of perturbation theory, condition numbers, recursive algorithms, finite algorithms, imbedding algorithms, parallel algorithms, generalized inverses of rank- r modified matrices and Hessenberg matrices, extensions of the Cramer rules and the representation and approximation of generalized inverses of linear operators. Dozens of papers are published in refereed journals in China and other countries. They draw attentions from researchers around world. I have received letters from more than ten universities in eight countries, U.S.A., Germany, Sweden, etc. requesting papers or seeking academic contacts. Colleagues in China show strong interests and support in our work, and request systematic presentation of our work. With the support of the Academia Sinica Publishing Foundation and the National Natural Science Foundation of China, Science Press published my book "Generalized Inverses of Matrices and Operators" [169] in Chinese in 1994. That book is noticed and welcomed by researchers and colleagues in China. It has been adopted by several universities as textbook or reference book for graduate students. The book was reprinted in 1998.

In order to improve graduate teaching and international academic exchange, I was encouraged to write this English version based on the Chinese version. This English version is not a direct translation of the Chinese version. In addition to the contents in the Chinese version, this book includes the contents from more than 100 papers since 1994. The final product is an entirely new book, while the spirit of the Chinese version still lives. For example, Sections 2, 3 and 5 of Chapter 3, Section 1 of Chapter 6, Sections 4 and 5 of Chapter 7, Sections 1, 4 and 5 of Chapter 8, Chapters 4, 10 and 11 are all new.

Dr. Yimin Wei of Fudan University in China and Dr. Sanzheng Qiao of McMaster University in Canada were two of my former excellent students. They have made many achievements in the area of generalized inverses and are recognized internationally. I would not possibly finish this book without their collaborations.

We would like to thank Professor A. Ben-Israel, Dr. Jianming Miao of Rutgers University, and Professors R. E. Hartwig, S. L. Campbell and C. D. Meyer, Jr. of North Carolina State University, and Professor C. W. Groetsch of University of Cincinnati. The texts [7], [13] and [65] undoubtedly have had an influence on this book. We also thank Professor Erxiong Jiang of Shanghai University, Professor Zhihao Cao of Fudan University, Professor Musheng Wei of East-China Normal University and Professor Yonglin Chen of Nanjing Normal University for their help and advice in the subject for many years, and my doctoral student Yaoming Yu for typing this book.

I appreciate any comments and corrections from the readers.

Finally, I am indebted to the support by the Graduate Textbook Publishing Foundation of Shanghai Education Committee and Shanghai Normal University.

Guorong Wang
Shanghai Normal University
June 2003

Contents

Preface	iv
List of Notations	vi
Chapter 1. Equation Solving Generalized Inverses	1
1.1. The Moore-Penrose inverse	1
1.2. $\{i, j, k\}$ inverses	8
1.3. The generalized inverses with prescribed range and null space	15
1.4. Weighted Moore-Penrose inverse	25
1.5. Bott-Duffin inverse and generalized Bott-Duffin inverse	31
Chapter 2. Drazin Inverse	47
2.1. Drazin inverse	47
2.2. Group inverse	55
2.3. W-weighted Drazin inverse	61
Chapter 3. The Generalization of Cramer Rule and the Minors of the Generalized Inverses	66
3.1. The nonsingularity of bordered matrices	66
3.2. Cramer rule for the solution of a linear equation	72
3.3. Cramer rule for the solution of a matrix equation	84
3.4. The determinantal expressions of the generalized inverses and projectors	96
3.5. The determinantal expressions of the minors of the generalized inverses	99
Chapter 4. The Reverse Order Law and Forward Order Law for the Generalized Inverses $A_{T,S}^{(2)}$	113
4.1. Introduction	113
4.2. Reverse order law	118
4.3. Forward order law	121
Chapter 5. Computational Aspects of the Generalized Inverses	129
5.1. Methods based on full rank factorizations	130
5.2. Singular value decompositions and (M, N) singular value decompositions	137
5.3. Generalized inverses of sums and partitioned matrices	143
5.4. Imbedding methods	157
5.5. Finite algorithms	162
Chapter 6. The Parallel Algorithms for Computing the Generalized Inverses	166

6.1. The model of parallel processors	167
6.2. Measures of the performance of parallel algorithms	170
6.3. Parallel algorithms	172
6.4. Equivalence theorem	186
Chapter 7. Perturbation Analysis of the Moore-Penrose Inverse and the Weighted Moore-Penrose Inverse	191
7.1. Perturbation bound	191
7.2. Continuity	200
7.3. Rank-preserving modification	202
7.4. Condition number	204
7.5. Expression for the perturbation of the weighted Moore-Penrose inverse	208
Chapter 8. Perturbation Analysis of the Drazin Inverse and the Group Inverse	212
8.1. Perturbation bound for the Drazin inverse	212
8.2. Continuity of the Drazin inverse	215
8.3. Core-rank preserving modification of the Drazin inverse	217
8.4. Condition number of the Drazin inverse	220
8.5. Perturbation bound for the group inverse	222
Chapter 9. The Moore-Penrose Inverse of Linear Operators	225
9.1. Definition and basic properties	225
9.2. Representation theorem	231
9.3. Computational methods	233
Chapter 10. Drazin Inverse of Operators	241
10.1. Definition and basic properties	241
10.2. Representation theorem	245
10.3. Computational procedures	247
10.4. Perturbation bound	252
Chapter 11. W -weighted Drazin Inverse of Operators	255
11.1. Definition and basic properties	255
11.2. Representation theorem	258
11.3. Computational procedures	262
11.4. Perturbation bound	268
Bibliography	272
Index	281