Application of the LLL Algorithm in Sphere Decoding

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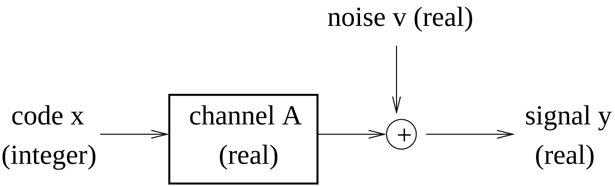
Outline

Application



A communication channel

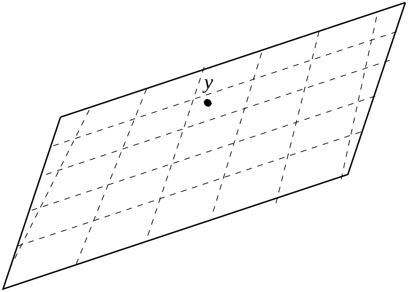
x: code vector, integer
A: channel matrix, real
y: received signal, $y = Ax + v$

$$\min_{x \in Z^m} \|Ax - y\|_2^2$$

Integer Least Squares

$$\min_{x \in Z^m} \|Ax - y\|_2^2$$

A: Generating matrix, n-by-m, $n \geq m$, real
y: n-vector, real
x: m-vector, solution, integer



A Naive Approach

A seemingly simple approach, Babai solution

$$x = \lceil A^{\dagger} y \rfloor$$

Example

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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Example

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{real LS solution} \quad \begin{bmatrix} -0.3333 \\ 0.3333 \end{bmatrix} \text{ rounded to } \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

giving residual $\|Ax - y\|_2 = \sqrt{3}$.

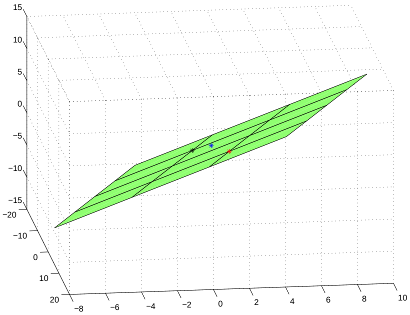
A Naive Approach (cont.)

The integer least squares solution

$$x = \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

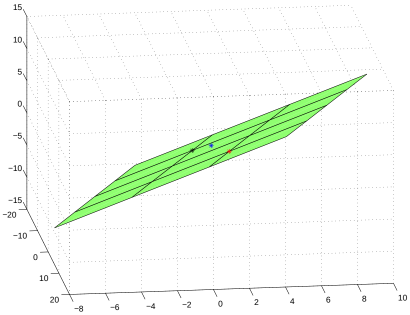
giving residual $\|Ax - y\|_2 = \sqrt{2}$.

Graph



A graph of the naive approach

Graph



A graph of the naive approach

In general, integer least squares problem is non-polynomial (NP) hard.

Outline

Problem Setting

1. Search for all lattice points inside the sphere

$$\|Ax - y\|_2 \leq \rho$$

   of radius ρ.
2. Among the lattice points inside the sphere, find the one that minimizes $\|Ax - y\|_2$.

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Choosing a radius ρ

- Too large, too many lattice points inside sphere, expensive
- Too small, no lattices points inside sphere

Reducing Dimension

QR decomposition

$$A = [Q_1 \;\; Q_2] \begin{bmatrix} R \\ 0 \end{bmatrix}$$

$[Q_1 \;\; Q_2]$: orthogonal
R: upper triangular, m-by-m

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Then

$$\|Ax - y\|_2^2 = \|Rx - Q_1^T y\|_2^2 + \|Q_2^T y\|_2^2$$

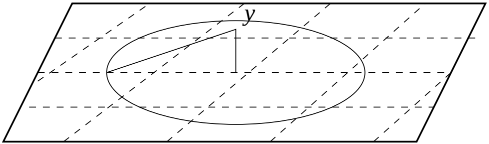
Reducing Dimension (cont.)

$$\|Ax - y\|_2^2 \leq \rho^2$$

becomes the triangular ILS problem:

$$\|Rx - \hat{y}\|_2^2 \leq \hat{\rho}^2$$

$$\hat{y} = Q_1^T y$$
$$\hat{\rho}^2 = \rho^2 - \|Q_2^T y\|_2^2$$



Searching

Partition

$$Rx - \hat{y} = \begin{bmatrix} R_{1:m-1,1:m-1} & r_{1:m-1,m} \\ 0 & r_{m,m} \end{bmatrix} \begin{bmatrix} x_{1:m-1} \\ x_m \end{bmatrix} - \begin{bmatrix} \hat{y}_{1:m-1} \\ \hat{y}_m \end{bmatrix}$$

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$$\begin{aligned} \|Rx - \hat{y}\|_2^2 &= \|R_{1:m-1,1:m-1}x_{1:m-1} - (\hat{y}_{1:m-1} - x_m r_{1:m-1,m})\|_2^2 \\ &\quad + (r_{m,m}x_m - \hat{y}_m)^2 \\ &\leq \hat{\rho}^2 \end{aligned}$$

Searching (cont.)

Two necessary conditions:

1. $|r_{m,m}x_m - \hat{y}_m| \leq \hat{\rho}$
2. $\|R_{1:m-1,1:m-1}x_{1:m-1} - (\hat{y}_{1:m-1} - x_m r_{1:m-1,m})\|_2^2 \leq \tilde{\rho}^2$,
   $\tilde{\rho}^2 = \hat{\rho}^2 - (r_{m,m}x_m - \hat{y}_m)^2$

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Sphere decoding:
Find all integers satisfying cond1;
For each integer solve cond2 recursively. (DFS)

Choosing ρ

Hassibi and Vikalo, 2005
In communications

$$y = Ax + v$$

v: white noise, variance σ^2

Given a probability p,
1. Find α satisfying

$$p = \int_0^{\alpha n/2} \frac{\lambda^{n/2-1}}{\Gamma(n/2)} e^{-\lambda} \mathrm{d}\lambda$$

2. $\rho^2 = \alpha n \sigma^2$

Choosing ρ (cont.)

- The solution lies in the sphere of radius ρ with probability p.
- The expected complexity is polynomial, often roughly cubic.
- Works well when σ^2 is small.

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- The expected complexity is polynomial, often roughly cubic.
- Works well when σ^2 is small.
- Channel matrix A is not taken into consideration (assuming some statistical characteristics).

Choosing ρ (cont.)

We propose:

1. Solve for real LS solution $\hat{x} = R^{-1}\hat{y}$
2. $\hat{\rho}^2 = \|R\lceil \hat{x} \rfloor - \hat{y}\|_2^2$

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At least one lattice point in sphere, deterministic.
Both R (A) and $\hat{y}$ (v) are taken into account.

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Error in the computed $R^{-1}\hat{y}$ must be addresses.

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What is the LLL algorithm?

A.K. Lenstra, H.W. Lenstra, and L. Lovász (1982)

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1. $|r_{i,j}| \leq |r_{i,i}|/2, \quad j > i$
2. $r^2_{i+1,i+1} \geq \omega r^2_{i,i} - r^2_{i,i+1}, \quad 0.25 \leq \omega \leq 1$

What is the LLL algorithm? (cont.)

Application:
Cryptography (integer arithmetic)

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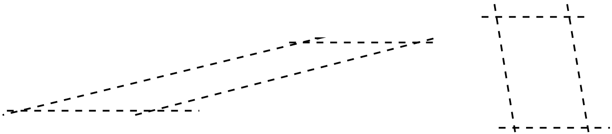
Luk and SQ (2007), numerical properties

What does the LLL algorithm do?

Example ($\omega = 0.75$)

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = QRZ^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}^{-1}$$



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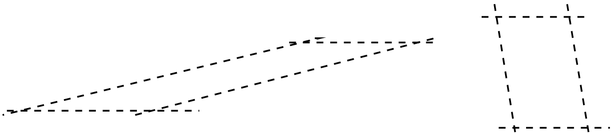

Making a lattice grid closer to orthogonal.

How may the LLL algorithm help?

Two ways:
Reduce search radius
Reduce the number of search paths

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Example

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$$\text{ILS solution } \mathbf{z} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\text{distance } \|A\mathbf{z} - \mathbf{b}\|_2 = \sqrt{2}$$

Reducing search radius

QR decomposition

$$R = \begin{bmatrix} 3.7417 & 8.5524 \\ 0 & 1.9640 \end{bmatrix} \quad \hat{\mathbf{b}} = \begin{bmatrix} 1.6036 \\ 0.6547 \end{bmatrix}$$

LLL algorithm ($\omega = 0.75$)

$$\tilde{R} = \begin{bmatrix} 2.2361 & -0.4472 \\ 0 & 3.2864 \end{bmatrix} \quad \tilde{\mathbf{b}} = \begin{bmatrix} 1.3416 \\ 1.0955 \end{bmatrix}$$

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Suppose we use

$$\begin{aligned} \rho &= \|R\lceil R^{-1}\hat{\mathbf{b}}\rfloor - \hat{\mathbf{b}}\|_2 \\ \tilde{\rho} &= \|\tilde{R}\lceil R^{-1}\tilde{\mathbf{b}}\rfloor - \tilde{\mathbf{b}}\|_2 \end{aligned}$$

as the search radii, then

$$\rho = 1.7321 \quad \text{and} \quad \tilde{\rho} = 1.4142$$

Reducing the number of search paths

$$R = \begin{bmatrix} 3.7417 & 8.5524 \\ 0 & 1.9640 \end{bmatrix} \quad \hat{\mathbf{b}} = \begin{bmatrix} 1.6036 \\ 0.6547 \end{bmatrix}$$

There are two integers $x_2 = 0, 1$ satisfying

$$|r_{2,2}x_2 - \hat{b}_2| \leq \rho \; (1.7321)$$

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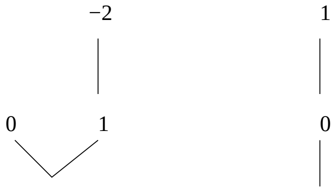
Even if we use 1.7321 as the radius here, there is still one integer 0.

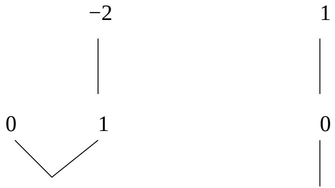
Search trees



$$\tilde{Q}\tilde{R} = RZ, \qquad Z = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = Z \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Search trees



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Reducing the number of search paths in the early stages of a DFS can significantly reduce the total number of search paths.

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Our preliminary experiments show:
The combination of our technique for choosing search radius and the LLL algorithm can reduce running time by almost 50%.

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Future work

- complex
- consider computational error in calculating search radius
- extensive experiments on various A and b to investigate numerical behavior

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Questions?