Componentwise Condition Numbers for Generalized Matrix Inversion and Linear Least Squares *

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Abstract

We present componentwise condition numbers for the problems of Moore-Penrose generalized matrix inversion and linear least squares. Also, the condition numbers for these condition numbers are given.

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1 Introduction

Condition number is a measurement of the sensitivity of a problem to the perturbation in its inputs. In general, consider a function $f(x)$. Suppose that the input x is perturbed by Δx. The condition number κ for the problem $f(x)$ quantifies the magnification of the relative errors caused by the perturbation. Specifically, κ satisfies

$$\frac{|f(x+\Delta x)-f(x)|}{|f(x)|} \leq \kappa \frac{|\Delta x|}{|x|}.$$

Assuming $|\Delta x| \leq \epsilon |x|$, we can define the condition number

$$\kappa = \lim_{\epsilon \to 0^+} \sup_{|\Delta x| \leq \epsilon |x|} \frac{|f(x+\Delta x)-f(x)|}{\epsilon |f(x)|}.$$

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In the problem of inverting a nonsingular matrix A, the condition number

$$\kappa(A) = \|A\|\,\|A^{-1}\|$$

represents the ratio between the relative errors in A and its inverse:

$$\frac{\|(A+\Delta A)^{-1} - A^{-1}\|}{\|A^{-1}\|} \le \frac{\kappa(A)}{1-\kappa(A)\|\Delta A\|/\|A\|}\frac{\|\Delta A\|}{\|A\|},$$

assuming the perturbation ΔA is small relative to A [4]. In this paper, $\|\cdot\|$ denotes the 2-norm. The condition number for solving a nonsingular system of linear equations $Ax = b$ is also $\kappa(A) = \|A\|\,\|A^{-1}\|$ in that

$$\frac{\|(A+\Delta A)^{-1}(b+\Delta b) - A^{-1}b\|}{\|A^{-1}b\|} \le \kappa(A)\left(\frac{\|\Delta A\|}{\|A\|} + \frac{\|\Delta b\|}{\|b\|}\right) + O(\epsilon^2),$$

for ΔA and Δb such that $\|\Delta A\| \le \epsilon\,\|A\|$, $\|\Delta b\| \le \epsilon\,\|b\|$, and $A + \Delta A$ is nonsingular [4].

In the general case when A can be rectangular or rank-deficient, the Moore-Penrose generalized inver $A^\dagger$ of A is introduced. It can be defined as the unique matrix satisfying the follow four matrix equations for X [2]:

$$AXA = A, \quad XAX = X, \quad (AX)^{\mathrm{T}} = AX, \quad (XA)^{\mathrm{T}} = XA.$$

The condition number for the generalized matrix inversion is given by $\|A\|\,\|A^\dagger\|$ [6]. For the problem of linear least squares

$$\min_x \|b - Ax\|, \tag{1.1}$$

the minimal norm solution is $A^\dagger b$ and the condition number is approximately $\|A\|\,\|A^\dagger\|$ when the residual $r = b - Ax$ is small and $\|A\|^2\|A^\dagger\|^2$ otherwise [6]. The condition numbers for weighted Moore-Penrose inverse and weighted least squares are discussed in [8, 9]. The condition numbers for structured least squares are given in [10].

The above condition numbers are called normwise condition numbers, because they are in the forms of matrix norms. The normwise analysis has two major drawbacks: It is norm dependent; it gives no information about the sensitivity of individual components [7]. Rohn [7] presented componentwise condition numbers for matrix inversion and nonsingular system of linear equations. Let $A = [A_{ij}]$. Denoting $|A| = [|A_{ij}|]$, we say $|A| \le |B|$ when $|A_{ij}| \le |B_{ij}|$ for all i and j. The componentwise condition number for matrix inversion is defined by

$$c_{ij}(A) = \lim_{\epsilon\to 0+} \sup\left\{\frac{|(A+\Delta A)^{-1} - A^{-1}|_{ij}}{\epsilon|A^{-1}|_{ij}},\ |\Delta A| \le \epsilon|A|\right\},$$

for nonsingular $A + \Delta A$. Rohn proposed

$$c_{ij}(A) = \frac{(|A^{-1}|\,|A|\,|A^{-1}|)_{ij}}{|A^{-1}|_{ij}}. \tag{1.2}$$

For the nonsingular system $Ax = b$ of linear equations, Rohn defined

$$c_i(A,b) = \lim_{\epsilon\to 0+} \sup \left\{ \frac{|(A+\Delta A)^{-1}(b+\Delta b) - A^{-1}b|_i}{\epsilon |A^{-1}b|_i},\ |\Delta A| \le \epsilon |A|,\ |\Delta b| \le \epsilon |b| \right\},$$

for nonsingular $A + \Delta A$, and proposed

$$c_i(A,b) = \frac{(|A^{-1}|\,|A|\,|A^{-1}b| + |A^{-1}|\,|b|)_i}{|A^{-1}b|_i}. \tag{1.3}$$

In this paper, we present a componentwise condition number for the Moore-Penrose generalized inversion and a componentwise condition number for the minimal norm linear least squares in Sections 2 and 3 respectively. These condition numbers are generalizations of those in [7] in that our condition numbers become (1.2) and (1.3) in the nonsingular cases. Then in Section 4, we show condition numbers, called level-2 condition numbers, for our componentwise condition numbers.

2 Generalized Inversion

The following theorem shows a componentwise condition number for the generalized matrix inversion.

Theorem 2.1 *Let the componentwise condition number for the Moore-Penrose generalized matrix inversion be defined by*

$$c_{ij}(A) = \lim_{\epsilon\to 0+} \sup \left\{ \frac{|(A+\Delta A)^{\dagger} - A^{\dagger}|_{ij}}{\epsilon |A^{\dagger}|_{ij}},\ |\Delta A| \le \epsilon |A| \right\}, \tag{2.1}$$

for $R(\Delta A) \subseteq R(A)$ and $R(\Delta A^{\mathrm{T}}) \subseteq R(A^{\mathrm{T}})$, then

$$c_{ij}(A) \le \frac{(|A^{\dagger}|\,|A|\,|A^{\dagger}|)_{ij}}{|A^{\dagger}|_{ij}}$$

and this bound is achievable.

Proof. It is shown in [1] that if $R(\Delta A) \subseteq R(A)$, $R(\Delta A^{\mathrm{T}}) \subseteq R(A^{\mathrm{T}})$, and $\|A^{\dagger}\|\,\|\Delta A\| < 1$, then

$$(A+\Delta A)^{\dagger} = (I + A^{\dagger}\Delta A)^{-1}A^{\dagger}.$$

It then follows from $|\Delta A| \le \epsilon |A|$ and the expansion of $(I + A^{\dagger}\Delta A)^{-1}$ that

$$(A+\Delta A)^{\dagger} - A^{\dagger} = -A^{\dagger}\Delta A A^{\dagger} + O(\epsilon^2)E, \tag{2.2}$$

where E is the matrix of which all entries equal one. Thus, componentwisely, we have

$$|(A+\Delta A)^{\dagger} - A^{\dagger}|_{ij} = |A^{\dagger}\Delta A A^{\dagger}|_{ij} + O(\epsilon^2).$$

Since

$$|A^\dagger \Delta A A^\dagger|_{ij} \le (|A^\dagger|\,|\Delta A|\,|A^\dagger|)_{ij} \le \epsilon(|A^\dagger|\,|A|\,|A^\dagger|)_{ij},$$

from the definition (2.1), we have the inequality

$$c_{ij}(A) \le \frac{(|A^\dagger|\,|A|\,|A^\dagger|)_{ij}}{|A^\dagger|_{ij}}.$$

This bound can be achieved for any matrix A such that $A = |A|$ and $A^\dagger = |A^\dagger|$. Indeed, let $\Delta A_0 = -\epsilon A$, then ΔA_0 satisfies

$$|\Delta A_0| = \epsilon |A|, \quad R(\Delta A_0) = R(A), \quad \text{and} \quad R(\Delta A_0^{\mathrm{T}}) = R(A^{\mathrm{T}}).$$

Now, using (2.1), $(A + \Delta A_0)^\dagger = (1-\epsilon)^{-1} A^\dagger$ implies that

$$\begin{aligned}
c_{ij}(A) &= \lim_{\epsilon \to 0+} \sup \left\{ \frac{|(A+\Delta A)^\dagger - A^\dagger|_{ij}}{\epsilon |A^\dagger|_{ij}},\ |\Delta A| \le \epsilon |A| \right\} \\
&\ge \lim_{\epsilon \to 0+} \frac{|(A+\Delta A_0)^\dagger - A^\dagger|_{ij}}{\epsilon |A^\dagger|_{ij}} \\
&= \lim_{\epsilon \to 0+} \frac{\epsilon |A^\dagger|_{ij}}{(1-\epsilon)\epsilon |A^\dagger|_{ij}} \\
&= 1.
\end{aligned}$$

On the other hand, since $A = |A|$ and $A^\dagger = |A^\dagger|$,

$$\frac{(|A^\dagger|\,|A|\,|A^\dagger|)_{ij}}{|A^\dagger|_{ij}} = \frac{|A^\dagger A A^\dagger|_{ij}}{|A^\dagger|_{ij}} = 1.$$

This completes the proof. $\square$

For example, the matrix E, of which all entries equal one, satisfies $|E| = E$ and $|E^\dagger| = n^{-2}E = E^\dagger$, where n is the order of E.

Since

$$|A^\dagger| = |A^\dagger A A^\dagger| \le |A^\dagger|\,|A|\,|A^\dagger|,$$

we have $(|A^\dagger|\,|A|\,|A^\dagger|)_{ij}/|A^\dagger|_{ij} \ge 1$. Thus, from Theorem 2.1, we propose

$$c_{ij}(A) = \frac{(|A^\dagger|\,|A|\,|A^\dagger|)_{ij}}{|A^\dagger|_{ij}} \tag{2.3}$$

as the componentwise condition number for the generalized matrix inversion and define

$$c(A) = \max_{i,j}(c_{ij}(A)). \tag{2.4}$$

The condition number (2.3) is a generalization of (1.2), since $A^\dagger = A^{-1}$ when A is nonsingular.

3 Linear Least Squares

Analogous to the componentwise condition number for the generalized matrix inversion presented in the previous section, we have the following result for the componentwise condition number for the minimal norm linear least squares problem.

Theorem 3.1 *Let the componentwise condition number for the least squares problem (1.1) be defined by*

$$c_i(A,b) = \lim_{\epsilon\to 0+} \sup \left\{ \frac{|(A+\Delta A)^\dagger(b+\Delta b) - A^\dagger b|_i}{\epsilon |A^\dagger b|_i}, \quad |\Delta A| \le \epsilon |A|,\ |\Delta b| \le \epsilon |b| \right\}, \tag{3.1}$$

for $R(\Delta A) \subseteq R(A)$ and $R(\Delta A^{\mathrm{T}}) \subseteq R(A^{\mathrm{T}})$, then

$$c_i(A,b) \le \frac{(|A^\dagger|\,|A|\,|A^\dagger b| + |A^\dagger|\,|b|)_i}{|A^\dagger b|_i}$$

and this bound is achievable.

Proof. Applying (2.2), we get

$$(A+\Delta A)^\dagger(b+\Delta b) - A^\dagger b = A^\dagger \Delta b - A^\dagger \Delta A A^\dagger b + O(\epsilon^2)e,$$

where e is the vector of which all the components equal one. Then, in componentwise form, we have

$$|(A+\Delta A)^\dagger(b+\Delta b) - A^\dagger b|_i = |A^\dagger \Delta b - A^\dagger \Delta A A^\dagger b|_i + O(\epsilon^2).$$

Since

$$|A^\dagger \Delta b - A^\dagger \Delta A A^\dagger b|_i \le (|A^\dagger \Delta b| + |A^\dagger \Delta A A^\dagger b|)_i \le \epsilon(|A^\dagger|\,|A|\,|A^\dagger b| + |A^\dagger|\,|b|)_i,$$

from the definition (3.1), we have

$$c_i(A,b) \le \frac{(|A^\dagger|\,|A|\,|A^\dagger b| + |A^\dagger|\,|b|)_i}{|A^\dagger b|_i}.$$

Again, the above bound is achievable for any A and b such that $A = |A|$, $A^\dagger = |A^\dagger|$, and $b = |b|$. In fact, setting

$$\Delta A_0 = -\epsilon A \quad \text{and} \quad \Delta b_0 = \epsilon b,$$

we get

$$|\Delta A_0| = \epsilon |A|, \quad R(\Delta A_0) = R(A), \quad R(\Delta A_0^{\mathrm{T}}) = R(A^{\mathrm{T}}), \quad \text{and} \quad |\Delta b_0| = \epsilon |b|.$$

Then, from (3.1), $(A+\Delta A_0)^\dagger = (1-\epsilon)^{-1}A^\dagger$ and $b+\Delta b_0 = (1+\epsilon)b$ imply that

$$\begin{aligned} c_i(A,b) &= \lim_{\epsilon\to 0+} \sup\left\{ \frac{|(A+\Delta A)^\dagger(b+\Delta b) - A^\dagger b|_i}{\epsilon|A^\dagger b|_i}, \quad |\Delta A|\le\epsilon|A|,\ |\Delta b|\le\epsilon|b| \right\} \\ &\ge \lim_{\epsilon\to 0+} \frac{|(A+\Delta A_0)^\dagger(b+\Delta b_0) - A^\dagger b|_i}{\epsilon|A^\dagger b|_i} \\ &= \lim_{\epsilon\to 0+} \frac{2}{1-\epsilon} \\ &= 2. \end{aligned}$$

On the other hand, since $A=|A|$, $A^\dagger = |A^\dagger|$, and $b=|b|$,

$$\frac{(|A^\dagger|\,|A|\,|A^\dagger b| + |A^\dagger|\,|b|)_i}{|A^\dagger b|_i} = \frac{|A^\dagger A A^\dagger b + A^\dagger b|_i}{|A^\dagger b|_i} = 2.$$

This completes the proof. □

From Theorem 3.1, we propose

$$c_i(A,b) = \frac{(|A^\dagger|\,|A|\,|A^\dagger b| + |A^\dagger|\,|b|)_i}{|A^\dagger b|_i} \tag{3.2}$$

as the componentwise condition number for the least squares and define

$$c(A,b) = \max_i(c_i(A,b)). \tag{3.3}$$

The condition number (3.2) is a generalization of (1.3), since $A^\dagger = A^{-1}$ when A is nonsingular.

4 Level-2 Condition Numbers

In the previous two sections, we proposed the componentwise condition numbers. How sensitive are these condition numbers to the perturbations? Demmel [3] introduced the concept of condition number of the condition number and showed that for certain problems condition number of the condition number is the condition number up to a constant factor. Higham [5] investigated the condition numbers, called level-2 condition numbers, for the condition numbers for matrix inversion and nonsingular linear systems. In this section, we present level-2 condition numbers for the generalized inversion and least squares. Our results are generalizations of those in [5] in that they are the same as those in [5] for the nonsingular cases.

Theorem 4.1 *Let the level-2 condition number for the componentwise condition number $c_{ij}(A)$ for the Moore-Penrose generalized matrix inversion be defined by*

$$c_{ij}^{[2]}(A) = \lim_{\epsilon\to 0+} \sup\left\{ \frac{|c_{ij}(A+\Delta A) - c_{ij}(A)|}{\epsilon c_{ij}(A)},\ |\Delta A|\le\epsilon|A| \right\}, \tag{4.1}$$

for $R(\Delta A) \subseteq R(A)$ *and* $R(\Delta A^{\mathrm{T}}) \subseteq R(A^{\mathrm{T}})$, *then*

$$c_{ij}^{[2]}(A) \le 1 + 3\,c(A).$$

Proof. We first derive lower and upper bounds for $|(A+\Delta A)^{\dagger}|$. From (2.2) and $|\Delta A| \le \epsilon\,|A|$, we get

$$\Big||(A+\Delta A)^{\dagger}| - |A^{\dagger}|\Big| \le \epsilon\,|A^{\dagger}|\,|A|\,|A^{\dagger}| + O(\epsilon^2)E. \tag{4.2}$$

It then follows from the definition (2.3) of $c_{ij}(A)$ that

$$(1-\epsilon\,c_{ij}(A))|A^{\dagger}|_{ij} \le |(A+\Delta A)^{\dagger}|_{ij} \le (1+\epsilon\,c_{ij}(A))|A^{\dagger}|_{ij}.$$

From (2.4), $c(A) \ge c_{ij}(A) \ge 1$ for all i and j, hence

$$(1-\epsilon\,c(A))|A^{\dagger}|_{ij} \le |(A+\Delta A)^{\dagger}|_{ij} \le (1+\epsilon\,c(A))|A^{\dagger}|_{ij}, \tag{4.3}$$

for all i and j, which means

$$(1-\epsilon\,c(A))|A^{\dagger}| \le |(A+\Delta A)^{\dagger}| \le (1+\epsilon\,c(A))|A^{\dagger}|. \tag{4.4}$$

Then, using (4.4) and $|A+\Delta A| \le (1+\epsilon)|A|$, we have the upper bound:

$$\begin{aligned}
& (|(A+\Delta A)^{\dagger}|\,|A+\Delta A|\,|(A+\Delta A)^{\dagger}|)_{ij} \\
\le\ & (1+\epsilon\,c(A))^2(1+\epsilon)(|A^{\dagger}|\,|A|\,|A^{\dagger}|)_{ij} + O(\epsilon^2) \\
=\ & (1+\epsilon+2\epsilon\,c(A))((|A^{\dagger}|\,|A|\,|A^{\dagger}|)_{ij} + O(\epsilon^2)
\end{aligned} \tag{4.5}$$

Similarly, we can obtain the lower bound

$$(|(A+\Delta A)^{\dagger}|\,|A+\Delta A|\,|(A+\Delta A)^{\dagger}|)_{ij} \ge (1-\epsilon-2\epsilon\,c(A))((|A^{\dagger}|\,|A|\,|A^{\dagger}|)_{ij} + O(\epsilon^2). \tag{4.6}$$

Now, using (4.3) and (4.5), we get

$$\begin{aligned}
c_{ij}(A+\Delta A) &= \frac{(|(A+\Delta A)^{\dagger}|\,|A+\Delta A|\,|(A+\Delta A)^{\dagger}|)_{ij}}{|(A+\Delta A)^{\dagger}|_{ij}} \\
&\le \frac{(1+\epsilon+2\epsilon\,c(A))(|A^{\dagger}|\,|A|\,|A^{\dagger}|)_{ij}}{(1-\epsilon\,c(A))|A^{\dagger}|_{ij}} + O(\epsilon^2) \\
&= (1+\epsilon+2\epsilon\,c(A))(1+\epsilon\,c(A))c_{ij}(A) + O(\epsilon^2) \\
&= (1+\epsilon+3\epsilon\,c(A))c_{ij}(A) + O(\epsilon^2),
\end{aligned}$$

which implies that

$$\frac{c_{ij}(A+\Delta A) - c_{ij}(A)}{\epsilon\,c_{ij}(A)} \le 1 + 3c(A) + O(\epsilon).$$

Similarly, using (4.3) and (4.6), we get

$$\frac{c_{ij}(A+\Delta A) - c_{ij}(A)}{\epsilon\,c_{ij}(A)} \ge -1 - 3c(A) + O(\epsilon).$$

This completes the proof. $\square$

Analogous to the level-2 condition number for the componentwise condition number for the Moore-Penrose generalized inverse, we can also get level-2 condition number for the componentwise condition number for the least square problem as follows.

Theorem 4.2 *Let the level-2 condition number for the componentwise condition number $c_i(A,b)$ for the minimal norm linear least square problems defined by*

$$c_i^{[2]}(A,b) = \lim_{\epsilon\to 0+} \sup\left\{ \frac{|c_i(A+\Delta A, b+\Delta b) - c_i(A,b)|}{\epsilon c_i(A,b)}\ ,\ |\Delta A| \le \epsilon |A|,\ |\Delta b| \le \epsilon |b| \right\}$$

for $R(\Delta A) \subseteq R(A)$ and $R(\Delta A^T) \subseteq R(A^T)$ then

$$c_i^{[2]}(A,b) \le 2c(A,b) + c(A) + 1.$$

Proof. For the least square problem (1.1), the minimal norm solution is $x = A^\dagger b$, where $A^\dagger$ is the Moore-Penrose generalized inverse. Let $x + \Delta x$ be the minimal norm solution of the perturbed least squares problem $\min_y \|(b+\Delta b) - (A+\Delta A)y\|_2$, then, from (2.2),

$$\begin{aligned} x + \Delta x &= (A+\Delta A)^\dagger (b+\Delta b) \\ &= (A^\dagger - A^\dagger \Delta A A^\dagger + O(\epsilon^2)E)(b+\Delta b) \\ &= A^\dagger b + A^\dagger \Delta b - A^\dagger \Delta A A^\dagger b + O(\epsilon^2)e. \end{aligned}$$

When $|\Delta A| \le \epsilon|A|$ and $|\Delta b| \le \epsilon|b|$, we have the following upper and lower bounds for $|x+\Delta x|$:

$$|A^\dagger b| - \epsilon|A^\dagger||b| - \epsilon|A^\dagger||A||A^\dagger b| + O(\epsilon^2) \le |x+\Delta x| \le |A^\dagger b| + \epsilon|A^\dagger||b| + \epsilon|A^\dagger||A||A^\dagger b| + O(\epsilon^2).$$

It then follows the definition (3.2) of $c_i(A,b)$ that

$$|x|_i(1-\epsilon c_i(A,b)) + O(\epsilon^2) \le |x+\Delta x|_i \le |x|_i(1+\epsilon c_i(A,b)) + O(\epsilon^2).$$

Since $c(A,b) \ge c_i(A,b)$, from (3.3), we obtain

$$|x|_i(1-\epsilon c(A,b)) + O(\epsilon^2) \le |x+\Delta x|_i \le |x|_i(1+\epsilon c(A,b)) + O(\epsilon^2),$$

for all i, which implies

$$|x|(1-\epsilon c(A,b)) + O(\epsilon^2) \le |x+\Delta x| \le |x|(1+\epsilon c(A,b)) + O(\epsilon^2). \tag{4.7}$$

Then, using (3.2), (4.4), (4.7), $|\Delta A| \le \epsilon|A|$, and $|\Delta b| \le \epsilon|b|$, we get the following upper bound for $c_i(A+\Delta A, b+\Delta b)$,

$$c_i(A+\Delta A,\ b+\Delta b) \quad = \quad \frac{(|(A+\Delta A)^\dagger|\ |A+\Delta A|\ |x+\Delta x|)_i}{|x+\Delta x|_i} + \frac{(|(A+\Delta A)^\dagger|\ |b+\Delta b|)_i}{|x+\Delta x|_i}$$

$$
\begin{aligned}
&\leq \frac{((1+\epsilon c(A))|A^\dagger|(1+\epsilon)|A|(1+\epsilon c(A,b))|x|)_i}{|x|_i(1-\epsilon c(A,b))} \\
&\quad + \frac{((1+\epsilon c(A))|A^\dagger|(1+\epsilon)|b|)_i}{|x|_i(1-\epsilon c(A,b))} \\
&= \frac{(1+\epsilon+\epsilon c(A)+\epsilon c(A,b))(1+\epsilon c(A,b))(|A^\dagger|\ |A|\ |A^\dagger b|)_i}{|x|_i} \\
&\quad + \frac{(1+\epsilon+\epsilon c(A))(1+\epsilon c(A,b))(|A^\dagger|\,|b|)_i}{|x|_i} + O(\epsilon^2) \\
&= \frac{(1+\epsilon+\epsilon c(A)+2\epsilon c(A,b))(|A^\dagger|\ |A|\ |A^\dagger b|)_i}{|x|_i} + O(\epsilon^2) \\
&\quad + \frac{(1+\epsilon+\epsilon c(A)+\epsilon c(A,b))(|A^\dagger|\,|b|)_i}{|x|_i} + O(\epsilon^2) \\
&\leq (1+\epsilon+\epsilon c(A)+2\epsilon c(A,b))c_i(A,b) + O(\epsilon^2). \qquad (4.8)
\end{aligned}
$$

Similarly, we can get the lower bound for $c_i(A+\Delta A, b+\Delta b)$

$$
\begin{aligned}
c_i(A+\Delta, b+\Delta b) &\geq \frac{(1-\epsilon c(A)-\epsilon-2\epsilon c(A,b))(|A^\dagger|\ |A|\ |x|)_i}{|x|_i} \\
&\quad + \frac{(1-\epsilon-\epsilon c(A)-\epsilon c(A,b))(|A^+|\ |b|)_i}{|x|_i} + O(\epsilon^2) \\
&\geq (1-\epsilon-\epsilon c(A)-2\epsilon c(A,b))c_i(A,b) + O(\epsilon^2). \qquad (4.9)
\end{aligned}
$$

Hence, using (4.8) and (3.2), we obtain

$$
\frac{c_i(A+\Delta A, b+\Delta b)-c_i(A,b)}{\epsilon c_i(A,b)} \leq 1 + c(A) + 2c(A,b) + O(\epsilon).
$$

Similarly, using (4.9) and (3.2), we also have

$$
\frac{c_i(A+\Delta A, b+\Delta b)-c_i(A,b)}{\epsilon c_i(A,b)} \geq -1 - c(A) - 2c(A,b) + O(\epsilon).
$$

This completes the proof. □

Conclusion We have presented the componentwise condition numbers for the generalized inversion and least squares. They include the componentwise condition numbers for matrix inversion and nonsingular linear system proposed by Rohn [7] as special cases. Also, we have generalized the level-2 condition numbers by Higham [5] to the generalized inversion and least squares and showed that condition numbers of our componentwise condition numbers are the componentwise condition numbers.

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