Displacement Rank of the Drazin Inverse *

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Abstract

In this paper, we study the displacement rank of the Drazin inverse. Both Sylvester displacement and the generalized displacement are discussed. We present upper bounds for the ranks of the displacements of the Drazin inverse. The general results are applied to the group inverse of a structured matrix such as close-to-Toeplitz, generalized Cauchy, Toeplitz-plus-Hankel, and Bezoutians.

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1 Introduction

Displacement gives a quantitative way of identifying the structure of a matrix. Consider an n-by-n Toeplitz matrix

$$T = \begin{bmatrix} t_0 & t_{-1} & \cdots & t_{-n+1} \\ t_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & \cdots & t_1 & t_0 \end{bmatrix},$$

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in which all the elements on the same diagonal are equal. A displacement is defined as the difference $T - ZTZ^{\mathrm{H}}$, where

$$Z = \begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix} \tag{1.1}$$

is the shift-down matrix and Z^{H} is the complex conjugate and transpose of Z. Define a displacement operator ∇_Z:

$$\nabla_Z T = T - ZTZ^{\mathrm{H}}.$$

It is easy to check that the rank of the displacement $\nabla_Z T$ is at most two, independent of the order n and low compared with n. There are several versions of displacement structures. Given matrices U and V, Sylvester UV-displacement is $AU - VA$ and Stein UV-displacement is $A - VAU$. If the rank of a displacement is low comparing with the order of the matrix, then the matrix is called structured with respect to the displacement [4, 6, 7, 8, 12]. Thus, a Toeplitz matrix T is structured with respect to the displacement $\nabla_Z T$. Low displacement rank can be exploited to construct fast algorithms for triangular factorization, inversion, among others [6].

This paper discusses the displacement rank of the Drazin inverse. For an n-by-n matrix A, the index of A is the smallest nonnegative integer k such that $\mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k)$. The Drazin inverse [1, 3], denoted by A^{D}, of A is the unique matrix satisfying

$$A^{\mathrm{D}} A A^{\mathrm{D}} = A^{\mathrm{D}}, \quad A A^{\mathrm{D}} = A^{\mathrm{D}} A, \quad A^{k+1} A^{\mathrm{D}} = A^k,$$

where k is the index of A. When the index of A is one, A^{D} is called the group inverse of A and is denoted by A^{G}. The Drazin inverse plays an important role in numerical analysis [2, 9, 10].

The index of a matrix is characterized as the order of the largest Jordan block with zero eigenvalues. From Jordan canonical form theory [2], for any complex n-by-n matrix A of index k and $\mathrm{rank}(A^k) = r$, there exits a n-by-n nonsingular matrix R such that

$$A = R \begin{bmatrix} S & 0 \\ 0 & N \end{bmatrix} R^{-1}, \tag{1.2}$$

where S is an r-by-r nonsingular matrix and N is nilpotent, $N^k = 0$. Note that if $\mathrm{index}(A) = 1$, then N is a zero matrix. Now we can write the Drazin inverse of A in the form:

$$A^{\mathrm{D}} = R \begin{bmatrix} S^{-1} & 0 \\ 0 & 0 \end{bmatrix} R^{-1}. \tag{1.3}$$

Denote

$$Q = AA^{\mathrm{D}} \quad \text{and} \quad P = I - AA^{\mathrm{D}}, \tag{1.4}$$

then Q and P are oblique projections onto $\mathrm{Range}(A^k)$ and $\mathrm{Null}(A^k)$ respectively. It is easy to check that

$$\mathrm{Range}(Q) = \mathrm{Range}(A^{\mathrm{D}}) = \mathrm{Range}(A^k) = \mathrm{Range}(Q^{\mathrm{H}}), \tag{1.5}$$

and

$$\mathrm{Null}(A^{\mathrm{D}}) = \mathrm{Null}(A^k) = \mathrm{Range}(P) = \mathrm{Null}(Q). \tag{1.6}$$

This paper is organized as follows. We show an upper bound for the Sylvester displacement rank of the Drazin inverse in Section 2. Then, in Section 3, we give an estimate for the generalized displacement rank of the Drazin inverse. In Section 4, we present a case study of several versions of displacement rank of the group inverse of a structured matrix such as close-to-Toeplitz, generalized Cauchy, Toeplitz-plus-Hankel, and Bezoutian.

2 Sylvester displacement rank

In this section, we study Sylvester displacement rank of the Drazin inverse. We first establish a relation between the Sylvester displacements of a matrix A and its Drazin inverse A^{D}. Then we show that the Sylvester displacement rank of A^{D} is bounded above by the sum of the Sylvester displacement ranks of A and A^k.

Define the Sylvester displacement operator $\Delta_{U,V}$:

$$\Delta_{U,V} A = AU - VA.$$

When A is nonsingular, the Sylvester displacement ranks of A and its inverse A^{-1} are related by

$$\mathrm{rank}(\Delta_{V,U} A^{-1}) = \mathrm{rank}(\Delta_{U,V} A),$$

since $AU - VA = A(UA^{-1} - A^{-1}V)A$. This says that if A is a structured matrix, then A^{-1} is also structured with respect to Sylvester displacement.

Now, we consider the displacement of the Drazin inverse. First, we establish a relation between the displacements of A and A^D.

Proposition 1 *Let $A \in C^{n\times n}$ be of index k, then*

$$\Delta_{V,U} A^{\mathrm{D}} = A^{\mathrm{D}} VP - PUA^{\mathrm{D}} - A^D(\Delta_{U,V} A)A^{\mathrm{D}} \tag{2.1}$$

where P is defined in (1.4).

Proof. It follows from the following identity:

$$A^{\mathrm{D}}(AU - VA)A^{\mathrm{D}} = (I - P)UA^{\mathrm{D}} - A^{\mathrm{D}}V(I - P)$$

since $P = I - A^{\mathrm{D}}A$. □

Now, we consider the ranks of the matrices on the right side of the equation (2.1). From (1.5), we have

$$\mathrm{rank}(PUA^{\mathrm{D}}) = \dim(PU\mathrm{Range}(A^{\mathrm{D}})) = \dim(PU\mathrm{Range}(Q)) = \mathrm{rank}(PUQ).$$

Similarly, from (1.6), we have $\mathrm{Range}(Q^{\mathrm{H}}) = \mathrm{Range}((A^{\mathrm{D}})^{\mathrm{H}})$ and

$$\mathrm{rank}(A^{\mathrm{D}}VP) = \dim((VP)^{\mathrm{H}}\mathrm{Range}((A^{\mathrm{D}})^{\mathrm{H}})) = \mathrm{rank}((VP)^{\mathrm{H}}Q^{\mathrm{H}}) = \mathrm{rank}(QVP).$$

It then follows from (2.1) that

$$\mathrm{rank}(A^{D}V - UA^{D}) \leq \mathrm{rank}(AU - VA) + \mathrm{rank}(QVP) + \mathrm{rank}(PUQ). \tag{2.2}$$

However, QVP and PUQ are dependent on A^{D}. We claim that

$$\mathrm{rank}(QVP) + \mathrm{rank}(PUQ) \leq \mathrm{rank}(UA^{k} - A^{k}V). \tag{2.3}$$

Thus we have the following theorem of Sylvester displacement rank of the Drazin inverse.

Theorem 1 *For any $A \in C^{n\times n}$ of index k,*

$$\mathrm{rank}(\Delta_{V,U}A^{\mathrm{D}}) \leq \mathrm{rank}(\Delta_{U,V}A) + \mathrm{rank}(\Delta_{V,U}A^{k}).$$

This theorem shows that the Sylvester VU-displacement rank of A^{D} is bounded above by the sum of the Sylvester UV-displacement rank of A and the Sylvester VU-displacement of A^k, where k is the index of A. So, if both A and A^k are structured with respect to Sylvester displacement, then A^{D} is also structured.

Now we prove (2.3). Following the dimensions in the Jordan canonical form (1.2), we partition

$$R^{-1}UR = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \quad \text{and} \quad R^{-1}VR = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}.$$

Using the canonical form (1.2), we get

$$\begin{aligned} R^{-1}(UA^{k} - A^{k}V)R &= R^{-1}URR^{-1}A^{k}R - R^{-1}A^{k}RR^{-1}VR \\ &= \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}\begin{bmatrix} S^{k} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} S^{k} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \\ &= \begin{bmatrix} U_{11}S^{k} - S^{k}V_{11} & -S^{k}V_{12} \\ U_{21}S^{k} & 0 \end{bmatrix}. \end{aligned}$$

Since $-S^{k}V_{12}$ and $U_{21}S^{k}$ are submatrices of the last matrix in the above equation and S is nonsingular, we have

$$\mathrm{rank}(UA^{k} - A^{k}V) \geq \mathrm{rank}(S^{k}V_{12}) + \mathrm{rank}(U_{21}S^{k}) = \mathrm{rank}(V_{12}) + \mathrm{rank}(U_{21}).$$

On the other hand, $\mathrm{rank}(V_{12}) = \mathrm{rank}(QVP)$ and $\mathrm{rank}(U_{21}) = \mathrm{rank}(PUQ)$, since, from the canonical forms (1.2) and (1.3),

$$Q = AA^{\mathrm{D}} = R\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}R^{-1} \quad \text{and} \quad P = I - Q = R\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}R^{-1}.$$

This proves (2.3).

3 Generalized displacement

Heinig and Hellinger [4] unified and generalized Sylvester and Stein displacements. The generalized displacement operator $\Delta_{a(V,U)}$ is defined by

$$\Delta_{a(V,U)}A = a_{00}A + a_{01}AU + a_{10}VA + a_{11}VAU,$$

where a_{ij} $(i,j = 0,1)$ are the elements of the 2-by-2 nonsingular matrix a. In particular, when

$$a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

the generalized displacement operator $\Delta_{a(V,U)}$ is Stein UV-displacement operator; when

$$a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

the generalized displacement operator $\Delta_{a(V,U)}$ reduces to Sylvester UV-displacement operator.

In this section, we generalize the result on Sylvester displacement rank to the generalized displacement rank of the Drazin inverse. Specifically, analogous to Theorem 1, we show that the generalized displacement rank of the Drazin inverse A^D is bounded above by the sum of the generalized displacement ranks of A and A^k, where k is the index of A.

Theorem 2 *For any $A \in C^{n\times n}$ of index k and nonsingular $a, x \in C^{2\times 2}$,*

$$\mathrm{rank}(\Delta_{a(U,V)}A^D) \le \mathrm{rank}(\Delta_{a^{\mathrm{T}}(V,U)}A) + \mathrm{rank}(\Delta_{x(U,V)}A^k), \tag{3.1}$$

where a^{T} is the transpose of a.

The following lemma [4] establishes a relation between the generalized displacement and Sylvester displacement.

Lemma 1 *Given n-by-n matrices U, V, and 2-by-2 nonsingular matrix a, there exist 2-by-2 matrices $b = [b_{ij}]$ and $c = [c_{ij}]$ $(i,j = 0,1)$ such that $b_{00}I + b_{01}V$ and $c_{00}I + c_{01}U$ are nonsingular and*

$$a = b^{\mathrm{T}}dc, \quad \text{where} \quad d = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

For any n-by-n matrix A, the generalized displacement

$$\Delta_{a(V,U)}A = (b_{00}I + b_{01}V)(\Delta_{f_c(U),f_b(V)}A)(c_{00}I + c_{01}U),$$

where $f_s(W)$ is a matrix function defined by

$$f_s(W) = (s_{00} + s_{01}W)^{-1}(s_{10}I + s_{11}W) = (s_{10}I + s_{11}W)(s_{00}I + s_{01}W)^{-1} \tag{3.2}$$

for a matrix W and a 2-by-2 nonsingular matrix $s = [s_{ij}]$ $(i,j = 0,1)$ such that $s_{00}I + s_{01}W$ is nonsingular.

In particular, since we are interested in displacement ranks, Lemma 1 gives a useful relation between the generalized displacement rank and Sylvester displacement rank:

$$\mathrm{rank}(\Delta_{a(V,U)}A) = \mathrm{rank}(\Delta_{f_c(U),f_b(V)}A). \tag{3.3}$$

Now, we prove Theorem 2. From Lemma 1, there exist 2-by-2 matrices b, c, y, and z such that

$$a = b^{\mathrm{T}}dc \quad \text{and} \quad x = y^{\mathrm{T}}dz, \qquad \text{where } d = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and from (3.3),

$$\mathrm{rank}(\Delta_{a(U,V)}A^D) = \mathrm{rank}(\Delta_{f_c(V),f_b(U)}A^{\mathrm{D}}), \tag{3.4}$$

$$\mathrm{rank}(\Delta_{a^{\mathrm{T}}(V,U)}A) = \mathrm{rank}(\Delta_{f_b(U),f_c(V)}A), \tag{3.5}$$

and

$$\mathrm{rank}(\Delta_{x(U,V)}A^k) = \mathrm{rank}(\Delta_{f_z(V),f_y(U)}A^k).$$

Applying (2.2) to (3.4), we further transform Sylvester displacement rank of A^{D} to that of A:

$$\mathrm{rank}(\Delta_{f_c(V),f_b(U)}A^{\mathrm{D}}) \leq \mathrm{rank}(\Delta_{f_b(U),f_c(V)}A) + \mathrm{rank}(Qf_c(V)P) + \mathrm{rank}(Pf_b(U)Q). \tag{3.6}$$

Thus, substituting $\mathrm{rank}(\Delta_{a(U,V)}A^{\mathrm{D}})$ and $\mathrm{rank}(\Delta_{a^{\mathrm{T}}(V,U)}A)$ in (3.1) with (3.6) and (3.5) respectively, it then remains to show that

$$\mathrm{rank}(Qf_c(V)P) + \mathrm{rank}(Pf_b(U)Q) \leq \mathrm{rank}(\Delta_{f_z(V),f_y(U)}A^k).$$

From (2.3), we have

$$\mathrm{rank}(\Delta_{f_z(V),f_y(U)}A^k) \geq \mathrm{rank}(Qf_z(V)P) + \mathrm{rank}(Pf_y(U)Q).$$

The following proposition shows that $\mathrm{rank}(Qf_s(V)P) = \mathrm{rank}(QVP)$ and $\mathrm{rank}(Pf_s(U)Q) = \mathrm{rank}(PUQ)$. It then follows that $\mathrm{rank}(Qf_c(V)P) = \mathrm{rank}(QVP) = \mathrm{rank}(Qf_z(V)P)$ and $\mathrm{rank}(Pf_b(U)Q) = \mathrm{rank}(PUQ) = \mathrm{rank}(Pf_y(U)Q)$, which completes the proof of Theorem 2.

In particular, if we choose $a = x$ in (3.1), then

$$\mathrm{rank}(\Delta_{a(U,V)}A^{\mathrm{D}}) \leq \mathrm{rank}(\Delta_{a^{\mathrm{T}}(V,U)}A) + \mathrm{rank}(\Delta_{a(U,V)}A^k).$$

This means that if both A and A^k are structured with respect to some displacement, then A^{D} is also structured.

Proposition 2 *Given W, let k be the index of A and $s = [s_{ij}]$ $(i, j = 0, 1)$ nonsingular such that $s_{00}I + s_{01}W$ is nonsingular, then*

$$\mathrm{rank}(QWP) = \mathrm{rank}(Qf_s(W)P) \quad \textit{and} \quad \mathrm{rank}(PWQ) = \mathrm{rank}(Pf_s(W)Q),$$

where $f_s(W)$ is defined by (3.2).

Proof. Since, from (1.6), $\mathrm{Range}(P) = \mathrm{Null}(A^k)$, we consider $QW|_{\mathrm{Null}(A^k)}$ as an operator restricted on $\mathrm{Null}(A^k)$. Then the null space of this operator is

$$\mathcal{N} = \mathrm{Null}(A^k) \cap \mathrm{Null}(A^k W),$$

since $\mathrm{Null}(Q) = \mathrm{Null}(A^k)$, which implies that $\mathrm{Null}(QW) = \mathrm{Null}(A^k W)$. It then follows that the rank of the operator $QW|_{\mathrm{Null}(A^k)}$ equals the dimension of the quotient space

$$\mathcal{Q} = \mathrm{Null}(A^k) \ominus \mathcal{N}.$$

In other words,

$$\mathrm{rank}(QWP) = \dim(\mathcal{Q}).$$

Following the above argument, if we define the subspaces

$$\tilde{\mathcal{N}} = \mathrm{Null}(A^k) \cap \mathrm{Null}(A^k f_s(W)) \quad \text{and} \quad \tilde{\mathcal{Q}} = \mathrm{Null}(A^k) \ominus \tilde{\mathcal{N}},$$

then $\mathrm{rank}(Qf_s(W)P) = \dim(\tilde{\mathcal{Q}})$. From the definitions of the quotient spaces $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$, if we can show that $\dim(\mathcal{N}) = \dim(\tilde{\mathcal{N}})$, then we have $\dim(\mathcal{Q}) = \dim(\tilde{\mathcal{Q}})$, which implies $\mathrm{rank}(QWP) = \mathrm{rank}(Qf_s(W)P)$.

Indeed, for any $\mathbf{x} \in \mathcal{N}$, which means $A^k\mathbf{x} = A^k W\mathbf{x} = 0$, there is $\mathbf{z} = (s_{00}I + s_{01}W)\mathbf{x} \in \tilde{\mathcal{N}}$, since $A^k\mathbf{z} = A^k(s_{00}I + s_{01}W)\mathbf{x} = 0$ and $A^k f_s(W)\mathbf{z} = A^k(s_{10}I + s_{11}W)\mathbf{x} = 0$. Conversely, for any $\mathbf{z} \in \tilde{\mathcal{N}}$, which means $A^k\mathbf{z} = A^k f_s(W)\mathbf{z} = 0$, we define $\mathbf{x} = (s_{00}I + s_{01}W)^{-1}\mathbf{z}$. Then $A^k f_s(W)\mathbf{z} = 0$ implies that $A^k(s_{10}I + s_{11}W)\mathbf{x} = 0$ and $A^k\mathbf{z} = 0$ implies that $A^k(s_{00}I + s_{01}W)\mathbf{x} = 0$. Thus $A^k\mathbf{x} = 0$ and $A^k W\mathbf{x} = 0$, i.e., $\mathbf{x} \in \mathcal{N}$, since the matrix s is nonsingular.

Similarly, we can prove that $\mathrm{rank}(PWQ) = \mathrm{rank}(Pf_s(W)Q)$ by noting that $\mathrm{rank}(PWQ) = \mathrm{rank}(Q^{\mathrm{H}}W^{\mathrm{H}}P^{\mathrm{H}})$ and $\mathrm{rank}(Pf_s(W)Q) = \mathrm{rank}(Q^{\mathrm{H}}(f_s(W))^{\mathrm{H}}P^{\mathrm{H}})$ and considering the subspaces $\mathrm{Null}((A^k)^{\mathrm{H}})$, $\mathrm{Null}((A^k)^{\mathrm{H}}W^{\mathrm{H}})$, and $\mathrm{Null}((A^k)^{\mathrm{H}}(f_s(W))^{\mathrm{H}})$. □

Example 3.1 Let U be the shift-down matrix Z in (1.1), V the shift-up matrix Z^{H},

$$a = x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and A a singular Toeplitz matrix of index two:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

We get

$$A^{\mathrm{D}} = \frac{1}{2}\begin{bmatrix} -1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 1 & 0 & -3 & -1 & 1 & 3 & -1 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 & -1 & 1 \\ -1 & 0 & 5 & 1 & -1 & -3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

and

$$\operatorname{rank}(\Delta_{a^{\mathrm{T}}(V,U)}A) = 2, \quad \operatorname{rank}(\Delta_{a(U,V)}A^2) = 4, \quad \operatorname{rank}(\Delta_{a(U,V)}A^{\mathrm{D}}) = 6.$$

This example shows that the upper bound given by Theorem 2 is sharp.

4 Case study

In this section we present a case study of Theorem 2. We consider the case when the index of A is one and study the displacement rank of the group inverse of close-to-Toeplitz matrix, generalized Cauchy matrix, Toeplitz-plus-Hankel matrix, and Bezoutian. In this case, (3.1) becomes

$$\operatorname{rank}(\Delta_{a(U,V)}A^{\mathrm{G}}) \le \operatorname{rank}(\Delta_{a^{\mathrm{T}}(V,U)}A) + \operatorname{rank}(\Delta_{x(U,V)}A].$$

4.1 Close-to-Toeplitz matrix

Let U be the shift-down matrix Z in (1.1), V the shift-up matrix Z^{H}, and

$$a = x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

then we have

$$\Delta_{a(U,V)}A^{\mathrm{G}} = A^{\mathrm{G}} - ZA^{\mathrm{G}}Z^{\mathrm{H}}, \quad \Delta_{a^{\mathrm{T}}(V,U)}A = A - Z^{\mathrm{H}}AZ, \quad \text{and} \quad \Delta_{a(U,V)}A = A - ZAZ^{\mathrm{H}}.$$

A matrix is said to be close-to-Toeplitz if the displacement ranks

$$r_+ = \operatorname{rank}(A - Z^{\mathrm{H}}AZ) \quad \text{and} \quad r_- = \operatorname{rank}(A - ZAZ^{\mathrm{H}})$$

are low. Thus, we have

$$\operatorname{rank}(A^{\mathrm{G}} - ZA^{\mathrm{G}}Z^{\mathrm{H}}) \le r_+ + r_-, \tag{4.1}$$

which means that the group inverse of a close-to-Toeplitz matrix is structured. In particular, for a Toeplitz matrix, r_+ and r_- are at most two, we have the following theorem of the displacement rank of the group inverse of a Toeplitz matrix.

Theorem 3 *For a Toeplitz matrix A of index one,*

$$\mathrm{rank}(A^{\mathrm{G}} - ZA^{\mathrm{G}}Z^{\mathrm{H}}) \le 4.$$

For a general matrix, we have $|r_+ - r_-| \le 2$ [5, 6]. The following theorem follows from (4.1).

Theorem 4 *Let $A \in C^{n\times n}$ be of index one, then*

$$\mathrm{rank}(A^{\mathrm{G}} - ZA^{\mathrm{G}}Z^{\mathrm{H}}) \le 2\ \mathrm{rank}(A - Z^{\mathrm{H}}AZ) + 2.$$

The above theorem shows that the group inverse of a close-to-Toeplitz matrix is also close-to-Toeplitz. So the group inverse can be computed by fast algorithms such as Newton method presented in [7, 8].

Also, from (4.1), we get the following decomposition of the group inverse.

Theorem 5 *The group inverse A^{G} of $A \in C^{n\times n}$ can be decomposed as*

$$A^{\mathrm{G}} = \sum_{i=1}^{r} L(\mathbf{c}_i)R(\mathbf{r}_i), \tag{4.2}$$

where $r = r_+ + r_-$, $L(\mathbf{c}_i)$ and $R(\mathbf{r}_i)$ are respectively lower and upper triangular Toeplitz matrices of order n and $\mathbf{c}_i$ and $\mathbf{r}_i$ are the first column and the first row of $L(\mathbf{c}_i)$ and $R(\mathbf{r}_i)$ respectively.

This theorem shows that for a close-to-Toeplitz matrix, the group inverse solution $A^{\mathrm{G}}b$ [9] can be computed in $O(n\log(n))$ operations if the FFT is used in Toeplitz matrix-vector multiplications.

Example 4.1.1 Let $n = 6$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

a singular Toeplitz matrix ($\mathrm{rank}(A) = 5$) of index one. The group inverse of A is

$$A^{\mathrm{G}} = \frac{1}{8}\begin{bmatrix} 1 & -3 & 5 & 2 & 1 & -5 \\ 3 & -1 & -1 & 2 & -1 & 1 \\ -2 & 6 & -2 & 0 & 2 & 2 \\ -1 & 3 & 3 & -2 & -1 & 5 \\ -1 & 3 & 3 & 6 & -1 & -3 \\ 3 & -1 & -1 & -2 & 3 & 1 \end{bmatrix}$$

and $\mathrm{rank}(A^{\mathrm{G}} - ZA^{\mathrm{G}}Z^{\mathrm{H}}) = 4$. So, the the upper bound given by Theorem 3 is tight.

We computed the vectors $\mathbf{c}_i$ and $\mathbf{r}_j$ in the decomposition (4.2) in Matlab:

$$\mathbf{c}_1 = \begin{bmatrix} -0.9967 \\ -0.1251 \\ 0.4467 \\ 0.7616 \\ -0.0151 \\ 0.6450 \end{bmatrix}, \ \mathbf{r}_1 = \begin{bmatrix} -0.0611 \\ 0.4721 \\ -0.5978 \\ -0.2518 \\ -0.1890 \\ 0.5629 \end{bmatrix}^{\mathrm{T}}; \ \mathbf{c}_2 = \begin{bmatrix} -0.0004 \\ -0.6652 \\ 0.3630 \\ 0.3034 \\ 0.6058 \\ -0.7250 \end{bmatrix}, \ \mathbf{r}_2 = \begin{bmatrix} -0.4655 \\ 0.5100 \\ 0.0236 \\ 0.6255 \\ 0.3586 \\ -0.0530 \end{bmatrix}^{\mathrm{T}};$$

$$\mathbf{c}_3 = \begin{bmatrix} 0.1465 \\ 0.1683 \\ -0.0883 \\ 0.1535 \\ 0.3368 \\ 0.1469 \end{bmatrix}, \mathbf{r}_3 = \begin{bmatrix} 0.4502 \\ 0.6206 \\ 0.0634 \\ -0.0141 \\ -0.3561 \\ -0.5302 \end{bmatrix}^{\mathrm{T}}; \mathbf{c}_4 = \begin{bmatrix} 0.0271 \\ 0.2458 \\ 0.1896 \\ 0.0865 \\ -0.0609 \\ -0.1453 \end{bmatrix}, \mathbf{r}_4 = \begin{bmatrix} -0.0737 \\ 0.1784 \\ 0.7335 \\ 0.0485 \\ -0.4075 \\ 0.5063 \end{bmatrix}^{\mathrm{T}}.$$

and measured the error

$$\|A^{\mathrm{G}} - \sum_{i=1}^{4} L(\mathbf{c}_i)R(\mathbf{r}_i)\|_2 = 1.293 \times 10^{-15}.$$

4.2 Generalized Cauchy matrix

In this section, we study the displacement rank of the group inverse of a generalized Cauchy matrix. A matrix A is called a generalized Cauchy matrix if for some vectors $\mathbf{c} = [c_i]$ and $\mathbf{d} = [d_i]$,

$$r = \mathrm{rank}(A\,\mathrm{diag}(\mathbf{d}) - \mathrm{diag}(\mathbf{c})A) \tag{4.3}$$

is small compared with the order of A. In case $c_i \neq d_j$, for all i and j, a generalized Cauchy matrix has the following form:

$$A = \left[\frac{\mathbf{f}_i^{\mathrm{H}}\mathbf{g}_j}{c_i - d_j}\right], \quad \text{for} \quad i, j = 1, ..., n, \tag{4.4}$$

where $\mathbf{f}_i, \mathbf{g}_j \in C^r$.

When $r = 1$ and $\mathbf{f}_i = \mathbf{g}_j = 1$, A is the classical Cauchy matrix. Another important case is the class of Loewner matrices:

$$A = \left[\frac{a_i - b_j}{c_i - d_j}\right], \quad \text{for} \quad i, j = 1, ..., n,$$

whose displacement rank $r = 2$.

Substituting U and V in Theorem 1 with $\mathrm{diag}(\mathbf{d})$ and $\mathrm{diag}(\mathbf{c})$ respectively, we have

$$\begin{aligned} & \mathrm{rank}(A^{\mathrm{G}}\,\mathrm{diag}(\mathbf{c}) - \mathrm{diag}(\mathbf{d})\,A^{\mathrm{G}}) \\ \leq\ & \mathrm{rank}(A\,\mathrm{diag}(\mathbf{d}) - \mathrm{diag}(\mathbf{c})\,A) + \mathrm{rank}(A\,\mathrm{diag}(\mathbf{c}) - \mathrm{diag}(\mathbf{d})\,A). \end{aligned}$$

Under the assumption that

$$c_i - d_j = c_j - d_i, \quad \text{for} \quad i, j = 1, .., n, \tag{4.5}$$

we have

$$\mathrm{rank}(A\,\mathrm{diag}(\mathbf{c}) - \mathrm{diag}(\mathbf{d})\,A) = \mathrm{rank}(A\,\mathrm{diag}(\mathbf{d}) - \mathrm{diag}(\mathbf{c})\,A).$$

Thus, we have the following estimate for the displacement rank of the group inverse of a generalized Cauchy matrix.

Theorem 6 *For the generalized Cauchy matrix A in (4.4), if the index of A is one and the assumption (4.5) holds then*

$$\mathrm{rank}(A^{\mathrm{G}}\,\mathrm{diag}(\mathbf{c}) - \mathrm{diag}(\mathbf{d})\,A^{\mathrm{G}}) \leq 2\,\mathrm{rank}(A\,\mathrm{diag}(\mathbf{d}) - \mathrm{diag}(\mathbf{c})\,A).$$

It follows from the above theorem that the group inverse of the generalized Cauchy matrix of displacement rank r is also a generalized Cauchy matrix and its displacement rank is $2r$.

Theorem 7 *Suppose that the assumption (4.5) holds. The group inverse A^{G} of the generalized Cauchy matrix in (4.4) has the form:*

$$A^{\mathrm{G}} = \left[\frac{\mathbf{x}_i^{\mathrm{H}}\mathbf{y}_j}{d_i - c_j}\right], \quad \textit{for} \quad i, j = 1, ..., n, \tag{4.6}$$

where $\mathbf{x}_i, \mathbf{y}_j \in C^{2r}$ and r is the displacement rank of A defined in (4.3).

Since the multiplication of a generalized Cauchy matrix of order n by a vector can be carried out in $O(n\ \log(n))$ operations, the above theorem shows that the group inverse solution $A^{\mathrm{G}}b$ [9] for a generalized Cauchy system can be computed in $O(n\ \log(n))$ by using (4.6).

Example 4.2.1 Let

$$\mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{bmatrix} = \mathbf{c}, \quad \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{bmatrix} = -\mathbf{c},$$

then $\mathbf{c}$ and $\mathbf{d}$ satisfy the assumption (4.5) and

$$A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 1/3 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

is singular and of index one. The group inverse

$$A^{\mathrm{G}} = \frac{1}{16}\begin{bmatrix} -1 & -6 & -1 \\ -6 & 12 & -6 \\ -1 & -6 & -1 \end{bmatrix}.$$

It can be found that

$$\operatorname{rank}(A^{\mathrm{G}}D(c) - D(d)A^{\mathrm{G}}) = \operatorname{rank}(AD(c) - D(d)A) = \operatorname{rank}(AD(d) - D(c)A) = 1.$$

It can be verified that vectors $\mathbf{x}_i$ and $\mathbf{y}_j$

$$[\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad [\mathbf{y}_1\ \mathbf{y}_2\ \mathbf{y}_3] = \frac{1}{16}\begin{bmatrix} 1 & -6 & 1 \\ -6 & 36 & -6 \end{bmatrix}$$

satisfy (4.6).

4.3 Toeplitz-plus-Hankel Matrix

A matrix is Toeplitz-plus-Hankel if it is the sum of a Toeplitz and a Hankel. This kind of matrix has Sylvester WW-displacement rank at most 4, where $W = Z + Z^{\mathrm{H}}$. Applying Theorem 1, we have the following theorem.

Theorem 8 *If a Toeplitz-plus-Hankel matrix is of index one, then its group inverse has Sylvester WW-displacement rank at most 8, where $W = Z + Z^{\mathrm{H}}$.*

Example 4.3 Let the Toeplitz-plus-Hankel matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
= \begin{bmatrix} 1&1&1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 0&1&1&1&1&1&1&1&1&1 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&0&-1&-1\\ 0&0&0&0&0&0&0&-1&-1&-1\\ 0&0&0&0&0&0&-1&-1&-1&-1\\ 0&0&0&0&0&-1&-1&-1&-1&-1\\ 0&0&0&0&-1&-1&-1&-1&-1&-1\\ 0&0&0&-1&-1&-1&-1&-1&-1&-1\\ 0&0&-1&-1&-1&-1&-1&-1&-1&-1\\ 0&-1&-1&-1&-1&-1&-1&-1&-1&-1 \end{bmatrix},
$$

then we have $\operatorname{rank}(AW - WA) = 4$. Its group inverse

$$
A^{\mathrm{G}} = \begin{bmatrix} 0&0&0&0&0&0&0&0&1/2&0\\ 0&0&0&0&0&0&0&0&1/2&0\\ 0&0&0&0&0&0&0&1&-1&0\\ 0&0&0&0&0&0&1&-1&0&0\\ 0&0&0&0&0&1&-1&0&0&0\\ 0&0&0&0&1&-1&0&0&0&0\\ 0&0&0&1&-1&0&0&0&0&0\\ 0&0&1&-1&0&0&0&0&0&0\\ 1/2&1/2&-1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}.
$$

We find that

$$
\operatorname{rank}(A^{\mathrm{G}}W - WA^{\mathrm{G}}) = 6.
$$

The upper bound for the displacement rank given by Theorem 8 holds.

4.4 Bezoutian

An n-by-n matrix $A = [a_{ij}]$ $(i, j = 0, ..., n-1)$ is called a (Hankel) r-Bezoutian if its generating function

$$
A(\lambda, \mu) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} a_{ij} \lambda^i \mu^j
$$

has the form:

$$
A(\lambda, \mu) = \frac{1}{\lambda - \mu} \sum_{k=1}^{r} a_k(\lambda) b_k(\mu),
$$

where $a_k(\lambda)$ and $b_k(\mu)$ are polynomials. In case of $r = 2$, $b_1 = a_2$, and $b_2 = -a_1$, A is a (Hankel of real line) *Bezoutian* in the classical sense.

Let ∇A denote the matrix with the generating function $(\lambda-\mu)A(\lambda,\mu)$, then A is called an r-Bezoutian if $\text{rank}(\nabla A) \le r$. We introduce the $(n+1)$-by-$(n+1)$ matrix

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

then

$$\nabla A = Z_{n+1}\tilde{A} - \tilde{A}Z_{n+1}^{\mathrm{H}}.$$

Hence, for an r-Bezoutian we have

$$\text{rank}(Z_{n+1}\tilde{A} - \tilde{A}Z_{n+1}^{\mathrm{H}}) \le r.$$

Also, we have the estimate

$$\text{rank}(Z_{n+1}^{\mathrm{H}}\tilde{A} - \tilde{A}Z_{n+1}) \le \text{rank}(\nabla A) \le r.$$

Applying Theorem 1, we obtain

$$\text{rank}(Z_{n+1}\tilde{A}^{\mathrm{G}} - \tilde{A}^{\mathrm{G}}Z_{n+1}^{\mathrm{H}}) \le 2r.$$

We know that

$$\tilde{A}^{\mathrm{G}} = \begin{bmatrix} A^{\mathrm{G}} & 0 \\ 0 & 0 \end{bmatrix}.$$

We conclude this section by the following theorem.

Theorem 9 (Bezoutian) *The group inverse of an r-Bezoutian is a $2r$-Bezoutian.*

5 Concluding remarks

In this paper, we study the displacement rank of the Drazin inverse. We show that Sylvester displacement rank of the Drazin inverse of a matrix A is the sum of the Sylvester displacement ranks of A and A^k, where k is the index of A. We generalize the result to the generalized displacement. Finally, we present a case study of the displacement rank of the group inverse. It is natural to ask if we can extend our results to linear operators in Hilbert spaces [11]. This will be the future research.

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