Unifying Unitary and Hyperbolic Rotations and Reflectors*

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Abstract

In Cholesky updating, Givens rotations or Householder transformations are used. In Cholesky downdating, where the matrix resulted from downdating is positive definite, hyperbolic rotations or hyperbolic Householder transformations are used. There are applications, for example, fast Toeplitz triangularization, where the matrix resulted from downdating may be indefinite. In this paper, we unify the unitary and hyperbolic rotations and Householder transformations (reflectors) and present a unified algorithm for both updating and general downdating where the downdated matrix can be either positive definite or indefinite.

Key Words. Cholesky downdating, hyperbolic rotation, hyperbolic Householder transformation, fast Toeplitz triangularization.

1 Introduction

Consider an n-by-n Hermitian and positive definite matrix A and its Cholesky factorization

$$A = R^{\mathrm{H}} R \tag{1}$$

where R is upper triangular. The Cholesky downdating problem is to compute the Cholesky factorization of a new matrix $\hat{A}$ obtained by subtracting $X^{\mathrm{H}}X$ from A, where X is p-by-n. In other words, the Cholesky downdating calculates an upper triangular $\hat{R}$ such that

$$\hat{A} = \hat{R}^{\mathrm{H}}\hat{R} \qquad \text{where} \quad \hat{A} = A - X^{\mathrm{H}}X$$

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by operating on R and X. Note that here we assume that the downdated matrix $\hat{A}$ is also positive definite. Using the Cholesky factorization (1), we write

$$\hat{A} = R^{\mathrm{H}}R - X^{\mathrm{H}}X = \begin{pmatrix} R \\ X \end{pmatrix}^{\mathrm{H}} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} R \\ X \end{pmatrix}. \tag{2}$$

The building blocks in Cholesky downdating are the transformations, denoted by P, which introduce zeros into the bottom part of the vector:

$$P \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} \hat{r} \\ 0 \end{pmatrix} \quad \text{and} \quad P^{\mathrm{H}} \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix} P = \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}$$

where r represents a diagonal element of the upper triangular R, x is a column in X in (2), and $\hat{r}$ gives a diagonal element of $\hat{R}$. Hyperbolic rotations [2] or hyperbolic Householder transformations (reflectors) [6, 7] are this kind of transformations.

In [2], a real hyperbolic rotation is defined by

$$Q = \begin{pmatrix} c & -s \\ -s & c \end{pmatrix}, \qquad c^2 - s^2 = 1.$$

The hyperbolic rotation Q satisfies

$$Q^{\mathrm{T}} S Q = S, \qquad \text{where} \quad S = \mathrm{diag}(1,\ -1),$$

and the above property is called S-orthogonality. A hyperbolic rotation can be constructed to introduce a zero into a 2-by-1 vector:

$$Q \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}.$$

The following algorithm [2] computes c and s in Q given a vector $x = (x_1, x_2)^{\mathrm{T}}$.

Algorithm 1 (Hyperbolic Rotation) *Given a real vector* $x = (x_1, x_2)^{\mathrm{T}}$ $(|x_1| \neq |x_2|)$*, this algorithm computes the cosh-sinh pair in the hyperbolic rotation.*

> **if** $x_2 = 0$
> $s = 0$; $c = 1$;
> **else**
>   **if** $|x_1| > |x_2|$
> $\tau = x_2/x_1$; $c = 1/\sqrt{1-\tau^2}$; $s = c\tau$;
>   **elseif** $|x_1| < |x_2|$
> $\tau = x_1/x_2$; $s = 1/\sqrt{1-\tau^2}$; $c = s\tau$;
>   **end**
> **end**

Actually, for a well suited Cholesky downdating, the downdated matrix is positive definite, thus we always have $|x_1| > |x_2|$. We note that this algorithm is applicable only to Cholesky downdating, where $|x_1| > |x_2|$. When $|x_1| < |x_2|$, the cosh-sinh pair (c, s) computed by the above algorithm fails to satisfy the equation $c^2 - s^2 = 1$ and, consequently, the hyperbolic rotation Q fails to satisfy the S-orthogonality. We also note that it is not obvious how to generalize the above algorithm to the complex case.

Given a complex vector x, Rader and Steinhardt [6, 7] proposed a hyperbolic Householder transformation

$$P_{\mathrm{RS}} = S - \delta_{\mathrm{RS}}^{-1} u u^{\mathrm{H}} \tag{3}$$

where

$$S = \mathrm{diag}(\sigma_i), \qquad \sigma_i = \pm 1,$$

called signature matrix, and

$$u = Sx + \alpha e_1 \ (\alpha = \sqrt{x^{\mathrm{H}} S x}), \quad \text{and} \quad \delta_{RS} = (u^{\mathrm{H}} S u)/2.$$

The hyperbolic Householder transformation defined above is S-unitary, i.e., $P_{\mathrm{RS}}^{\mathrm{H}} S P_{\mathrm{RS}} = S$, and introduces zeros into x: $P_{\mathrm{RS}} x = -\alpha e_1$. However, as pointed out in [7], the above definition requires that

$$x^{\mathrm{H}} S x = \sigma_1 |\alpha|^2. \tag{4}$$

In other words, $x^{\mathrm{H}} S x$ and σ_1 must have the same sign. In Cholesky downdating, $S = \mathrm{diag}(1, -I)$, i.e., $\sigma_1 = 1$, and the downdated matrix is positive definite implying that $x^{\mathrm{H}} S x > 0$. Consequently, (4) is satisfied.

The downdating problem occurs not only in Cholesky factorization, but also in other applications where the downdated matrices can be indefinite. For example, in the symmetric Toeplitz triangularization, for a given real and symmetric Toeplitz matrix T, we first decompose T using the displacement structure [3]

$$T = R_1^{\mathrm{T}} R_1 - R_2^{\mathrm{T}} R_2 \tag{5}$$

where R_1 and R_2 are upper triangular Toeplitz. Rewriting (5) in matrix form, we get

$$T = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}.$$

Then we apply hyperbolic transformations to triangularize the matrix

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \longrightarrow \begin{pmatrix} R \\ 0 \end{pmatrix}$$

When T is positive definite, the hyperbolic transformations defined above can be applied. Since R_1 and R_2 can be computed efficiently, this procedure yields fast triangularization algorithms for Toeplitz matrices [5]. However, if T is indefinite, the hyperbolic transformations given above fail to work.

In this paper, we propose new definitions of rotations and reflectors (Householder transformations), which unify the unitary and hyperbolic rotations and reflectors in the following sense:

- They include unitary rotations and Householder transformations as special cases;
- In the case of Cholesky downdating, they are the same as the hyperbolic transformations described avove;
- They can be applied to general case where the downdated matrix is indefinite.

2 Rotations

In the previous section, we described a definition of a real hyperbolic rotation and an algorithm [2]. We have pointed out that Algorithm 1 is applicable to Cholesky downdating. In a general case, the downdated matrix may be indefinite. Consider a simple downdating problem:

$$4^2 - 5^2 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

The downdated matrix, writing in matrix form,

$$-9 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

is no longer positive definite. The above form shows that the middle signature matrix changes from $\mathrm{diag}(1, -1)$ to $\mathrm{diag}(-1, 1)$. This suggests that the S-orthogonality requirement, i.e., $Q^{\mathrm{T}} S Q = S$, be inappropriate for the general downdating problems. We propose the following definition of a rotation. Since this definition includes both unitary (Givens) and hyperbolic rotations as special cases, we call it quadratic rotation or simply rotation. Moreover, we consider complex data.

Definition 1 (Rotation) *Given a signature matrix $S = \mathrm{diag}(\sigma_1, \sigma_2)$, a (quadratic) rotation has the form:*

$$Q = \begin{pmatrix} c^* & (\sigma_2/\sigma_1) s^* \\ -s & c \end{pmatrix} \tag{6}$$

where the pair (c, s) satisfies

$$\sigma_1 |c|^2 + \sigma_2 |s|^2 = \hat{\sigma}_1, \quad \hat{\sigma}_1 = \pm 1.$$

Denoting $\hat{\sigma}_2 = (\sigma_2/\sigma_1)\hat{\sigma}_1$, we have

$$\hat{\sigma}_1 |c|^2 + \hat{\sigma}_2 |s|^2 = \sigma_1 \quad \text{and} \quad \hat{\sigma}_2 |c|^2 + \hat{\sigma}_1 |s|^2 = \sigma_2.$$

Let $\hat{S} = \mathrm{diag}(\hat{\sigma}_1, \hat{\sigma}_2)$, then we can verify that

$$Q^{\mathrm{H}} \hat{S} Q = S.$$

When $\sigma_1 = \sigma_2 = 1$ (or -1), we have $\hat{\sigma}_1 = \hat{\sigma}_2 = 1$ (or -1, respectively). Thus $Q^{\mathrm{H}} Q = I$ and Q is a unitary (Givens) rotation. When $S = \mathrm{diag}(1, -1)$, i.e., $\sigma_1 = 1$ and $\sigma_2 = -1$, and $|c|^2 > |s|^2$, we have $\hat{\sigma}_1 = 1$ and $\hat{\sigma}_2 = -1$. That is $\hat{S} = S$ and Q is S-unitary, $Q^{\mathrm{H}} S Q = S$, as described in the previous section. Thus this definition includes both the unitary rotation and the hyperbolic rotation defined in [2] as special cases. This definition is more general in that $|c|^2 - |s|^2 = \hat{\sigma}_1$ can be 1 or -1. This relaxed condition is useful in the general downdating applications. Consider the following general downdating problem:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{\mathrm{H}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We want to find a rotation Q in the form of (6), where $\sigma_1 = 1$ and $\sigma_2 = -1$, such that

$$Q \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix} \quad \text{and} \quad Q^{\mathrm{H}} \hat{S} Q = \mathrm{diag}(1, -1).$$

We choose

$$c = \frac{x_1}{\sqrt{||x_1|^2 - |x_2|^2|}} \quad \text{and} \quad s = \frac{x_2}{\sqrt{||x_1|^2 - |x_2|^2|}}.$$

It follows that

$$\hat{\sigma}_1 = \mathrm{sign}(|x_1|^2 - |x_2|^2) \quad \text{and} \quad \hat{\sigma}_2 = -\hat{\sigma}_1.$$

For the simple downdating problem considered earlier:

$$\begin{pmatrix} 4 \\ 5 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix},$$

we have

$$Q = \begin{pmatrix} 4/3 & -5/3 \\ -5/3 & 4/3 \end{pmatrix}.$$

We can verify that

$$Q \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad \text{and} \quad Q^{\mathrm{T}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In summary, we have the following algorithm for computing the rotation.

Algorithm 2 (Rotation) *Given the updating or general downdating problem:*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{\mathrm{H}} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

this algorithm computes a rotation

$$Q = \begin{pmatrix} c^* & (\sigma_2/\sigma_1)s^* \\ -s & c \end{pmatrix}$$

and a signature matrix $\hat{S} = \mathrm{diag}(\hat{\sigma}_1, \hat{\sigma}_2)$ *so that*

$$Q \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix} \quad \textit{and} \quad Q^{\mathrm{H}} \hat{S} Q = \mathrm{diag}(\sigma_1, \sigma_2).$$

if $x_2 = 0$
 $c = 1$; $s = 0$; $\hat{\sigma}_1 = \sigma_1$; $\hat{\sigma}_2 = \sigma_2$; return
end
if $\sigma_1|x_1| = -\sigma_2|x_2|$
 error("No hyperbolic rotation when $|x_1| = |x_2|$")
end
if $|x_1| > |x_2|$
 $\tau = x_2/x_1$; $\delta = 1 + (\sigma_2/\sigma_1)|\tau|^2$;
 $c = \mathrm{sign}(x_1)/\sqrt{|\delta|}$; $s = \tau c$;
else
 $\tau = x_1/x_2$; $\delta = \sigma_2/\sigma_1 + |\tau|^2$;
 $s = \mathrm{sign}(x_2)/\sqrt{|\delta|}$; $c = \tau s$;
end
if $\delta > 0$
 $\hat{\sigma}_1 = \sigma_1$; $\hat{\sigma}_2 = \sigma_2$;
else
 $\hat{\sigma}_1 = -\sigma_1$; $\hat{\sigma}_2 = -\sigma_2$;
end

It can be verified that when $\sigma_1\sigma_2 = 1$, the above algorithm generates a unitary rotation, when $\sigma_1\sigma_2 = -1$ and $\sigma_1|x_1|^2 + \sigma_2|x_2|^2 > 0$, which includes the Cholesky downdating as a special case, it generates the hyperbolic rotation described in Section 1; when $\sigma_1\sigma_2 = -1$ and $\sigma_1|x_1|^2 + \sigma_2|x_2|^2 < 0$, it generates a rotation Q such that $Q^{\mathrm{H}}(-S)Q = S$.

3 Reflectors

In Section 1, we introduced the hyperbolic Householder transformation proposed in [6, 7]. It has the form (3) and is S-unitary. But their transformation has the limitation (4) and is applicable only to well suited Cholesky downdating problems, where the downdated matrix is positive definite. In this section, we present a definition of reflectors which includes both their transformation and the unitary reflector (Householder transformation) as special cases. It is applicable to both updating and general downdating problems.

Definition 2 (Reflector) *Given a signature matrix S and a complex vector u such that $u^{\mathrm{H}}Su \neq 0$ and it is scaled so that $|u^{\mathrm{H}}Su| = 2$, a reflector is of the form PJ where*

$$P = S - \mathrm{sign}(u^{\mathrm{H}}Su)uu^{\mathrm{H}} \tag{7}$$

and J is the matrix representing the permutation $(1, j)$ for some index j.

It can be verified that P in (7) is S-unitary, i.e., $P^{\mathrm{H}}SP = S$. When $S = I$ (or $-I$) and $J = I$, $PJ = P$ is the unitary Householder transformation. When $S = \mathrm{diag}(1, -I)$ and $u^{\mathrm{H}}Su > 0$, i.e., $\mathrm{sign}(u^{\mathrm{H}}Su) = \sigma_1 = 1$, we choose $J = I$, then $PJ = P$ is the hyperbolic Householder defined in [6, 7].

Now, we show how to apply this reflector to the updating and general downdating problems represented by

$$x^{\mathrm{H}}Sx. \tag{8}$$

In general, we want to find a reflector PJ such that

$$(PJ)x = -\alpha e_1 \quad \text{and} \quad (PJ)^{\mathrm{H}}\hat{S}(PJ) = S$$

for some signature matrix $\hat{S}$. When S in (8) is I or $-I$, this is the updating problem. When $S = \mathrm{diag}(1, -I)$ and $x^{\mathrm{H}}Sx > 0$, this is the Cholesky downdating. We first assume that $\mathrm{sign}(x^{\mathrm{H}}Sx) = \sigma_1$ and choose $J = I$. Since P is S-unitary, we have

$$x^{\mathrm{H}}Sx = (Px)^{\mathrm{H}}S(Px) = |\alpha|^2\sigma_1. \tag{9}$$

This is consistent with our assumption $\mathrm{sign}(x^{\mathrm{H}}Sx) = \sigma_1$. Also, we get

$$|\alpha| = \sqrt{|x^{\mathrm{H}}Sx|}.$$

If $Px = (S - \mathrm{sign}(u^{\mathrm{H}}Su)uu^{\mathrm{H}})x = -\alpha e_1$, then u is a scalar multiple of $Sx + \alpha e_1$. To avoid cancellation, we choose

$$\alpha = \mathrm{sign}(\sigma_1 x_1)\sqrt{|x^{\mathrm{H}}Sx|} = \sigma_1 \mathrm{sign}(x_1)\sqrt{|x^{\mathrm{H}}Sx|}. \tag{10}$$

Next, we scale u so that $|u^{\mathrm{H}}Su| = 2$. A straightforward, yet inefficient, way is to scale u with $\sqrt{2}/\sqrt{|u^{\mathrm{H}}Su|}$. In the following, we derive an efficient way of computing the scalar $|u^{\mathrm{H}}Su|/2$. Let $u = Sx + \alpha e_1$, then

$$u^{\mathrm{H}}Su = (x^{\mathrm{H}}S + \alpha^* e_1^{\mathrm{T}})Su = x^{\mathrm{H}}u + \sigma_1\alpha^* u_1.$$

On the other hand, using (9) and (10), we get

$$\alpha x_1^* = \alpha^* x_1 = \sigma_1 |x_1| \sqrt{|u^{\mathrm{H}} S u|} \tag{11}$$

and then

$$x^{\mathrm{H}} u = x^{\mathrm{H}} (Sx + \alpha e_1) = \sigma_1 |\alpha|^2 + \alpha x_1^* = \sigma_1 \alpha^* (\alpha + \sigma_1 x_1).$$

That is $x^{\mathrm{H}} u = \sigma_1 \alpha^* u_1$. It then follows that

$$u^{\mathrm{H}} S u = 2\sigma_1 \alpha^* u_1. \tag{12}$$

Therefore, the scalar $|u^{\mathrm{H}} S u|/2$ can be efficiently computed by

$$|u^{\mathrm{H}} S u|/2 = |\alpha^* u_1|\,.$$

Finally, we consider $\mathrm{sign}(u^{\mathrm{H}} S u)$ in (7). It follows from (12) that

$$\mathrm{sign}(u^{\mathrm{H}} S u) = \sigma_1 \mathrm{sign}(\alpha^* u_1).$$

Since $u = Sx + \alpha e_1$, from (11), we have

$$\alpha^* u_1 = \alpha^* (\sigma_1 x_1 + \alpha) = \sigma_1 \alpha^* x_1 + |\alpha|^2 > 0.$$

Thus

$$\mathrm{sign}(u^{\mathrm{H}} S u) = \sigma_1 = \mathrm{sign}(x^{\mathrm{H}} S x).$$

Denoting $\delta = \sigma_1 \alpha^* u_1$, we have the reflector

$$P = S - \delta^{-1} u u^{\mathrm{H}} \qquad \text{where} \quad u = Sx + \alpha e_1$$

satisfying $Px = -\alpha e_1$.

Now we consider the case when $\mathrm{sign}(x^{\mathrm{H}} S x) = -\sigma_1$. In this case, we have $S \neq \pm I$, because if $S = \pm I$ then $\mathrm{sign}(x^{\mathrm{H}} S x) = \sigma_1$. So, there exists an index j such that $\sigma_j = -\sigma_1 = \mathrm{sign}(x^{\mathrm{H}} S x)$. Let J be the matrix representing the permutation $(1, j)$ and

$$y = Jx \quad \text{and} \quad \hat{S} \equiv \mathrm{diag}(\hat{\sigma}_i) = JSJ,$$

then $\mathrm{sign}(y^{\mathrm{H}} \hat{S} y) = x^{\mathrm{H}} S x = \sigma_j = \hat{\sigma}_1$. Using the above procedure, we can find a reflector $P = \hat{S} - \delta^{-1} u u^{\mathrm{H}}$ such that $Py = -\alpha e_1$ and $P^{\mathrm{H}} \hat{S} P = \hat{S}$. In other words,

$$(PJ)x = -\alpha e_1 \quad \text{and} \quad (PJ)^{\mathrm{H}} \hat{S} (PJ) = S.$$

In summary, we have the following algorithm for computing the unified reflector.

Algorithm 3 (Reflector) *Given a complex vector x and a signature matrix $S = \mathrm{diag}(\sigma_i)$, this algorithm computes the vector u, scalars α and δ, and permutation $(1, j)$ in the reflector (Householder transformation)*

$$P = \hat{S} - \delta^{-1} u u^{\mathrm{H}}.$$

Let matrix J represent the permutation $(1, j)$, then $\hat{S} = JSJ$,

$$(PJ)x = -\alpha e_1 \quad \textit{and} \quad (PJ)^{\mathrm{H}} \hat{S} (PJ) = S.$$

$m = \max\{|x_i|\}$; $x = x/m$;
if $\mathrm{sign}(x^{\mathrm{H}}Sx) \neq \sigma_1$
    find j so that $\sigma_j = \mathrm{sign}(x^{\mathrm{H}}Sx)$;
end
$y = Jx$; $\hat{S} = JSJ$;
$\alpha = \mathrm{sign}(\hat{\sigma}_1 y_1)\sqrt{\left|y^{\mathrm{H}}\hat{S}y\right|}$;
$u = \hat{S}y + \alpha e_1$;
$\delta = \hat{\sigma}_1 \alpha^* u_1$;
$\alpha = m\alpha$;

4 Example

In this section we show an example of applying the unified rotation to fast Toeplitz triangularization. Suppose the symmetric (indefinite) Toeplitz matrix is

$$T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}.$$

We first decompose T using its displacement structure:

$$T = R_1^{\mathrm{T}} R_1 - R_2^{\mathrm{T}} R_2, \tag{13}$$

where R_1 and R_2 are upper triangular and Toeplitz. Their first rows are

$$r_1 = (1,\ 2,\ 3) \quad \text{and} \quad r_2 = (0,\ 2,\ 3)$$

respectively. We want to find an upper triangular matrix R and a signature matrix D such that

$$R^{\mathrm{T}} D R = R_1^{\mathrm{T}} R_1 - R_2^{\mathrm{T}} R_2. \tag{14}$$

Writing (13) in matrix form, we get

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}.$$

This is a general downdating problem instead of the Cholesky downdating, because T is indefinite.

Using the unified rotation, we can compute R and D in $O(n^2)$ operations, where n is the order of T. In the first iteration, we pair the second row of R_1 and the first row of R_2 and eliminate the leading nonzero element in the first row of R_2 using the unified rotation.

Since both R_1 and R_2 are Toeplitz, we effectively eliminate the whole superdiagonal in R_2 where the leading nonzero element resides. This procedure is illustrated as following

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & \sqrt{3} & 4/\sqrt{3} \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & 1/\sqrt{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rotation is

$$Q = \begin{pmatrix} -1/\sqrt{3} & 2/\sqrt{3} \\ 2/\sqrt{3} & -1/\sqrt{3} \end{pmatrix},$$

which satisfies

$$Q \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix} \quad \text{and} \quad Q^{\mathrm{T}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The subsequent iterations are similar. The final R and D in the decomposition (14) are

$$R = \begin{pmatrix} 1 & 2 & 3 \\ 0 & \sqrt{3} & 4/\sqrt{3} \\ 0 & 0 & \sqrt{8}/\sqrt{3} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since R_1 and R_2 are Toeplitz, we need only to store and operate on the first rows of R_1 and R_2, leading to an $O(n^2)$ algorithm.

This fast Toeplitz triangularization reveals the inertia. In this example, the middle signature matrix D shows that T has one positive eigenvalue and two negative eigenvalues. This triangularization procedure can be used as the preprocessing in the eigenvalue decomposition.

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