Decaying Exponentials and Signal Decomposition

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1 Introduction

We consider a signal decomposition problem: Given a sequence $\{s_k\}$ of signals, we decompose them into a small set of decaying complex exponentials z_i ($i = 1, ..., r$):

$$s_k = \sum_{i=1}^{r} c_i z_i^{k-1}, \tag{1.1}$$

assuming $c_i \neq 0$ for all i and z_i distinct. For example, a signal s_k composed of sinusoids can be expressed by

$$s_k = \sum_{i=1}^{r} a_i e^{j((k-1)\omega_i + \phi_i)}$$

where a_i are amplitudes, ω_i frequencies, and ϕ_i phases. Setting

$$c_i = a_i e^{j\phi_i} \quad \text{and} \quad z_i = e^{j\omega_i},$$

we get the expression (1.1) for s_k. This problem arises from spectral analysis [2]. The decomposition (1.1) is determined by the integer r, the complex coefficients c_i, and the complex exponentials z_i ($i = 1, ..., r$). We will concentrate on the determination of r and z_i, since c_i can be readily found by the linear least squares data fitting technique. Specifically, let $s = (s_1, ..., s_m)^T$ be the signal vector and

$$V = \begin{pmatrix} e^T \\ e^T D_z \\ \vdots \\ e^T D_z^{m-1} \end{pmatrix}$$

where $e = (1, 1, ..., 1)^T$ is an r-by-1 vector and D_z denotes an r-by-r diagonal matrix $\mathrm{diag}(z_1, ..., z_r)$, then we find the coefficient vector $c = (c_1, ..., c_r)^T$ by minimizing

$$\|s - Vc\|_2.$$

In this paper, we first show an association between the problem of finding r exponentials z_i and the problem of solving the generalized eigenproblem in the following section. Then we present an algorithm for computing the exponentials in §3 and an updating scheme in §4. In §5, we present another algorithm by exploiting the symmetry of the data matrix. This method can lead to an algorithm for the generalized eigenproblem of complex and symmetric matrices.

2 Background

Suppose that we are given m signals: $s_1, ..., s_m$. It is reasonable to assume that $m > 2r$, since we have $2r + 1$ unknowns, namely r, $c_1, ..., c_r$, $z_1, ..., z_r$, to be determined. We arrange these m signals into a Hankel matrix

$$H = \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ s_2 & s_3 & \cdots & s_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ s_k & s_{k+1} & \cdots & s_m \end{pmatrix} \tag{2.1}$$

where $k = m - n + 1 > n \geq r$. The first inequality $k > n$ means that H has more columns than rows. The second inequality $n \geq r$ implies that r can be the rank of H. In the following, we will show that r is indeed the rank. Using (1.1), we can verify the following decomposition

$$H = \begin{pmatrix} e^T \\ e^T D_z \\ \vdots \\ e^T D_z^{k-1} \end{pmatrix} D_c (e \; D_z e \; \cdots \; D_z^{n-1} e)$$

where e and D_z are defined previously and $D_c = \mathrm{diag}(c_1, ..., c_r)$. This decomposition shows that r is the rank of H. Partitioning

$$H = \begin{pmatrix} g^T \\ H_1 \end{pmatrix} = \begin{pmatrix} H_2 \\ h^T \end{pmatrix} \tag{2.2}$$

where g and h are n-by-1 vectors and denoting

$$V_1 = \begin{pmatrix} e^T \\ e^T D_z \\ \vdots \\ e^T D_z^{k-2} \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} e^T \\ e^T D_z \\ \vdots \\ e^T D_z^{n-1} \end{pmatrix},$$

we get a pair of decompositions:

$$H_1 = V_1 D_c D_z V_2^T \quad \text{and} \quad H_2 = V_1 D_c V_2^T.$$

Note that V_1 and V_2 are Vandermonde matrices. Thus, both V_1 and V_2 are of full column rank, since z_i are distinct. This implies that H, H_1 and H_2 all have rank r. Therefore, r can be determined by applying any rank-revealing techniques to H, H_1, or H_2.

Now we consider the problem of computing the exponentials z_i. We will show an association between the exponentials and the generalized eigenvalues of the matrix pair (H_1, H_2). Let

$$V_1 = Q_1 R_1 \quad \text{and} \quad V_2 = Q_2 R_2$$

be the QR decompositions, where $Q_1 \in C^{(k-1)\times r}$ and $Q_2 \in C^{n\times r}$ have orthonormal columns, i.e., $Q_1^T Q_1 = Q_2^T Q_2 = I_r$, and R_1 and R_2 (r-by-r) are upper triangular and nonsingular. Then we have the following simultaneous decompositions:

$$H_1 = Q_1 R_1 D_c D_z R_2^T Q_2^T \quad \text{and} \quad H_2 = Q_1 R_1 D_c R_2^T Q_2^T.$$

Consequently,

$$H_1 Q_2 R_2^{-T} = H_2 Q_2 R_2^{-T} D_z. \tag{2.3}$$

Recall that if a scalar λ satisfies

$$Ax = \lambda Bx,$$

for some $x \neq 0$, then it is called the generalized eigenvalue of the pair of matrices A and B and x is the associated eigenvector [1]. It then follows from (2.3) that z_i are the r nonzero generalized eigenvalues of the pair (H_1, H_2) and the columns of $Q_2 R_2^{-T}$ are the corresponding eigenvectors. Thus, from the above derivation, we have transformed the problem of finding the exponentials z_i into the problem of computing the nonzero generalized eigenvalues of the matrix pair (H_1, H_2).

So, given H_1 and H_2, how can we compute the generalized eigenvalues of (H_1, H_2)? A basic tenet in matrix computation is that an original problem is first transformed into a simpler problem by decomposing the original matrices into simpler matrices. In our case, we first decompose H_1 and H_2, say,

$$H_1 = FAG \quad \text{and} \quad H_2 = FBG$$

such that F is of full column rank, G is nonsingular, and A and B have simpler structure than H_1 and H_2. Then the generalized eigenproblem of H_1 and H_2 is equivalent to that of A and B. In general, to compute the generalized eigenvalues of (A, B), we solve for a nonsingular Y and an upper triangular T in the matrix equation:

$$AY = BYT. \tag{2.4}$$

Then the diagonal elements of T are the desired generalized eigenvalues. Note that the equation (2.4) may not have a solution. For example, if

$$A = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

then there does not exist Y and T (scalars) satisfying (2.4).

In conclusion, in the absence of noise, this signal decomposition problem can be solved by techniques for revealing the rank of H, H_1, or H_2 and finding the nonzero generalized eigenvalues of (H_1, H_2). The singular value decomposition (SVD) provides a high-resolution method for solving this problem [2, 4]. However, the SVD method is unsuitable for adaptive processing and makes no use of the symmetric structure of the Hankel matrix H (2.1). In the following section, we present a particular pair of decompositions $H_1 = FAG$ and $H_2 = FBG$ which leads to an adaptive algorithm for computing r and the exponentials z_i. In §5, we show another particular pair of decompositions $H_1 = FAG$, and $H_2 = FBG$ which exploits the symmetry of the Hankel matrix H.

3 Incorporating the URV Decomposition

The first method for finding r and the exponentials z_i incorporates the rank-revealing URV decomposition [5]. As shown in the previous section, in the absence of noise, the Hankel matrix H in (2.1) has rank r, so any rank-revealing method can be used to find r. Suppose that, the following URV decomposition reveals the rank r

$$H = U \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} V^H \tag{3.1}$$

where U (k-by-n) has orthonormal columns, i.e., $U^H U = I_n$, R (r-by-r) is upper triangular and nonsingular, and V (n-by-n) is unitary. As described in the previous section, to find the complex exponentials z_i, we first decompose H_1 and H_2. Partitioning

$$U = \begin{pmatrix} v^T \\ U_1 \end{pmatrix} = \begin{pmatrix} U_2 \\ u^T \end{pmatrix}$$

where u and v are n-by-1 vectors, from the partition (2.2) and the decomposition (3.1), we get

$$H_1 = U_1 \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} V^H \quad \text{and} \quad H_2 = U_2 \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} V^H.$$

Furthermore, we perform the QR decomposition

$$U_2 = Q \begin{pmatrix} S \\ 0 \end{pmatrix}$$

where Q ($(k-1)$-by-$(k-1)$) is unitary and S (n-by-n) is upper triangular. If we partition

$$S = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}$$

where S_{11} is r-by-r, then we have the simultaneous decompositions:

$$H_1 = QQ^H U_1 \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & I_{n-r} \end{pmatrix} V^H$$

and

$$H_2 = Q \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} V^H = Q \begin{pmatrix} S_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & I_{n-r} \end{pmatrix} V^H.$$

Thus the problem of finding the complex exponentials z_i, i.e., the generalized eigenvalues of H_1 and H_2, is equivalent to the problem of finding the generalized eigenvalues of

$$A = Q^H U_1 \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} S_{11} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.2}$$

We can show that any nonzero generalized eigenvalue of (A, B) is an eigenvalue of the r-by-r matrix

$$X = S_{11}^{-1} (I_r \ 0) Q^H U_1 \begin{pmatrix} I_r \\ 0 \end{pmatrix}. \tag{3.3}$$

Suppose that λ is a nonzero generalized eigenvalue of (A, B), i.e.,

$$Az = \lambda Bz$$

for some vector z. Partitioning

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

where z_1 is r-by-1, from (3.2), we get

$$Q^H U_1 \begin{pmatrix} z_1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} S_{11} z_1 \\ 0 \end{pmatrix}.$$

It follows that

$$(I_r \ 0) Q^H U_1 \begin{pmatrix} I_r \\ 0 \end{pmatrix} z_1 = \lambda S_{11} z_1.$$

This shows that λ is an eigenvalue of X in (3.3). Denoting

$$\hat{U}_1 = U_1 \begin{pmatrix} I_r \\ 0 \end{pmatrix} \quad \text{and} \quad \hat{Q} = Q \begin{pmatrix} I_r \\ 0 \end{pmatrix},$$

which are composed of the first r columns of U_1 and Q respectively, we have

$$X = S_{11}^{-1}\hat{Q}^H\hat{U}_1. \tag{3.4}$$

Thus, we have reduced the generalized eigenproblem of (H_1, H_2) to the eigenproblem of the r-by-r matrix X.

Although the above discussion assumes the absence of noise, it is readily amendable to the case when noise is present. In summary, we present the following algorithm for computing r and the exponentials.

Algorithm 1. *Given the Hankel matrix H composed of the signals, this algorithm computes the r exponentials.*

1. *Perform the URVD:*
$$H = U\begin{pmatrix} R_{11} & E_1 \\ 0 & E_2 \end{pmatrix} V^H$$
and determine the rank r;

2. *Partition*
$$U = \begin{pmatrix} v^T \\ U_1 \end{pmatrix} = \begin{pmatrix} U_2 \\ u^T \end{pmatrix};$$

3. *Compute the QRD*
$$U_2 = Q\begin{pmatrix} S \\ 0 \end{pmatrix};$$

4. *Multiply $\hat{Q}^H\hat{U}_1$, where $\hat{Q}$ and $\hat{U}_1$ consist of the first r columns of Q and U_1 respectively;*

5. *The exponentials are the eigenvalues of the r-by-r matrix $X = S_{11}^{-1}(\hat{Q}^H\hat{U}_1)$, where S_{11} is the rth order leading principal submatrix of S.*

4 Adding a Row

When a new row is added to the Hankel matrix H, its URV decomposition in the step 1 of Algorithm 1 can be efficiently updated, see [5] for details. However, whenever a row is added to H, the U matrix in the URV decomposition (3.1) also grows by one row. As rows are added to H, the matrix U grows indefinitely. As shown by the steps 3 and 4 of Algorithm 1, U_1 and U_2, the submatrices of U, are needed for computing the exponentials. Thus, it is undesirable to store the matrices U_1 and U_2. Fortunately, in this application, we are interested in the eigenvalues of X in (3.4), which is given by S_{11} and $\hat{Q}^H\hat{U}_1$. The matrix S_{11} is the rth order leading principal submatrix of S, which is the upper triangular factor in the QR decomposition of U_2 (see Algorithm 1, step 3). Moreover, the product $\hat{Q}^H\hat{U}_1$ is the rth order leading principal submatrix of $Q^H U_1$ (Algorithm 1, step 4), where Q is the unitary factor in the QR decomposition of U_2 (Algorithm 1, step 3). Thus, to update X, it suffices to update S and $Q^H U_1$, both are $n \times n$ matrices. We note that it is necessary to update the $n \times n$ matrices S and $Q^H U_1$ instead of the $r \times r$ matrices S_{11} and $\hat{Q}^H \hat{U}_1$, since the rank r can potentially increase up to n as rows are added to H. So, we need all information in S and $Q^H U_1$. In the following we propose schemes for updating S and $Q^H U_1$ without storing U_1 and U_2.

The building blocks in updating the URV decomposition are the application of right- and left-rotations to the triangular factor R in the URV decomposition. When a rotation is applied to the right of R, it is simultaneously applied to the right of the V factor. This is irrelevant to our underlying matrices S and $Q^H U_1$. So, we concentrate our discussion on the application of left-rotations to R. We show how to update R, S, and $Q^H U_1$ without explicitly computing U or U_1 or U_2.

Suppose that a row vector h^T is added to H, we write

$$\hat{H} \stackrel{\text{def}}{=} \begin{pmatrix} H \\ h^T \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R \\ h^T V \end{pmatrix} V^H \stackrel{\text{def}}{=} \hat{U} \hat{R} \hat{V}^H.$$

Partitioning

$$\hat{U} = \begin{pmatrix} \hat{v}^T \\ \hat{U}_1 \end{pmatrix} = \begin{pmatrix} \hat{U}_2 \\ \hat{u}^T \end{pmatrix} \tag{4.1}$$

and recalling that

$$U = \begin{pmatrix} v^T \\ U_1 \end{pmatrix} = \begin{pmatrix} U_2 \\ u^T \end{pmatrix},$$

we get

$$\hat{U}_1 = \begin{pmatrix} U_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{U}_2 = \begin{pmatrix} U_2 & 0 \\ u^T & 0 \end{pmatrix}, \quad \text{and} \quad \hat{u} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Consequently, from the step 3 of Algorithm 1, we initialize the QR decomposition

$$\hat{U}_2 = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & 0 \\ u^T & 0 \end{pmatrix} \stackrel{\text{def}}{=} \hat{Q} \hat{S}. \tag{4.2}$$

Thus, initializing

$$\hat{V} = V, \quad \hat{R} = \begin{pmatrix} R \\ h^T V \end{pmatrix}, \quad \hat{S} = \begin{pmatrix} S & 0 \\ 0 & 0 \\ u^T & 0 \end{pmatrix}, \quad \hat{u} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{4.3}$$

from (4.1), we have the decompositions

$$\hat{H} = \begin{pmatrix} \hat{v}^T \\ \hat{U}_1 \end{pmatrix} \hat{R} \hat{V}^H = \begin{pmatrix} \hat{Q} \hat{S} \\ \hat{u}^T \end{pmatrix} \hat{R} \hat{V}^H. \tag{4.4}$$

In our updating scheme, instead of storing U_1 and U_2, we keep $Q^H U_1$, which has a fixed order n. For updating $Q^H U_1$, we initialize

$$\hat{Q}^H \hat{U}_1 = \begin{pmatrix} Q^H U_1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.5}$$

As stated before, in updating the URV decomposition, when we apply a right-rotation to $\hat{R}$, we simply apply this rotation to the right to $\hat{V}$. When we apply a left-rotation P to $\hat{R}$, from (4.4) and (4.5), we apply P^H to the right of $\hat{S}$, $\hat{u}^T$, and $\hat{U}_1$. Since we keep $\hat{Q}^H \hat{U}_1$ instead of $\hat{U}_1$, we update $\hat{Q}^H \hat{U}_1$ by postmultiplying it with P^H. Similarly, in restoring the triangular structure of $\hat{S}$, when we apply a rotation P to the left of $\hat{S}$, from (4.2), we should apply P^H to the right of $\hat{Q}$. Since we store $\hat{Q}^H \hat{U}_1$ instead of $\hat{Q}$, we apply P^H to the left of $\hat{Q}^H \hat{U}_1$.

Algorithm 2. *Suppose that we have the following variables available*
R: the triangular factor in the URV decomposition,
S: the triangular factor in the QR decomposition of U_2,
u^T: the bottom row of the unitary factor U in the URVD,
$Q^H U_1$: see steps 2 and 3 of Algorithm 1 for the definitions of U_1 and Q,
and a new row h^T is added to H, the following steps show how to update these variables and compute the exponentials.

1. *Initialization:*

$$\hat{V} = V; \quad \hat{R} = \begin{pmatrix} R \\ h^T V \end{pmatrix}; \quad \hat{S} = \begin{pmatrix} S & 0 \\ 0 & 0 \\ u^T & 0 \end{pmatrix}; \quad \hat{u} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix};$$

and

$$\hat{Q}^H \hat{U}_1 = \begin{pmatrix} Q^H U_1 & 0 \\ 0 & 1 \end{pmatrix};$$

2. *In updating the URV decomposition,*
   (a) *When $\hat{R}$ is post-multiplied by a rotation P, i.e., $\hat{R} \leftarrow \hat{R}P$, update $\hat{V} \leftarrow \hat{V}P$;*
   (b) *When $\hat{R}$ is pre-multiplied by a rotation P, i.e., $\hat{R} \leftarrow P\hat{R}$, update*
   $$\hat{S} \leftarrow \hat{S}P^H, \quad \hat{u}^T \leftarrow \hat{u}^T P^H, \quad \hat{Q}^H \hat{U}_1 \leftarrow \hat{Q}^H \hat{U}_1 P^H;$$
   (c) *In triangularizing $\hat{S}$, when a rotation P is applied to the left of $\hat{S}$, i.e., $\hat{S} \leftarrow P\hat{S}$, update $\hat{Q}^H \hat{U}_1 \leftarrow P(\hat{Q}^H \hat{U}_1)$;*

3. *After the URV decomposition is updated, the new rank $\hat{r}$ is determined, and $\hat{S}$ is triangularized, the exponentials are the eigenvalues of $\hat{X} = \hat{S}_{11}^{-1}(\tilde{Q}^H \tilde{U}_1)$ where $\hat{S}_{11}$ and $\tilde{Q}^H \tilde{U}_1$ are the $\hat{r}$th order leading principal submatrices of $\hat{S}$ and $\hat{Q}^H \hat{U}_1$ respectively.*

5 Exploiting Symmetry

When $m = 2n$, the Hankel matrices

$$H_1 = \begin{pmatrix} s_2 & s_3 & \cdots & s_{n+1} \\ s_3 & s_4 & \cdots & s_{n+2} \\ \vdots & \vdots & \cdots & \vdots \\ s_{n+1} & s_{k+2} & \cdots & s_m \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ s_2 & s_3 & \cdots & s_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ s_n & s_{n+1} & \cdots & s_{m-1} \end{pmatrix}$$

are square and symmetric. While the algorithm presented in the previous section is adaptive, it makes no use of the symmetry. In this section, we propose another algorithm which exploits the symmetry.

In [3], we propose a rank-revealing factorization of symmetric matrices. Here, we apply this technique to H_1 and H_2. To exploit the symmetry, we first decompose a symmetric matrix into a product of a lower triangular matrix and its transpose. Assuming the signals $s_i \neq 0$ for all i and denoting

$$\hat{R}_1 = D_1^{-1/2}\text{triu}(H_1), \quad \text{and} \quad \tilde{R}_1 = \hat{R}_1 - D_1^{1/2}$$

where D_1 is diagonal matrix whose diagonal equals the diagonal of H_1 and $\text{triu}(H_1)$ is the upper triangular part of H_1. Thus $\tilde{R}_1$ is the strict upper triangular part of $\hat{R}_1$. It can be verified that

$$H_1 = \hat{R}_1^T \hat{R}_1 - \tilde{R}_1^T \tilde{R}_1 = \begin{pmatrix} \hat{R}_1 \\ j\tilde{R}_1 \end{pmatrix}^T \begin{pmatrix} \hat{R}_1 \\ j\tilde{R}_1 \end{pmatrix}.$$

Then we find a (complex) orthogonal transformation P such that $P^T P = I$ and

$$P \begin{pmatrix} \hat{R}_1 \\ j\tilde{R}_1 \end{pmatrix} = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}$$

is upper triangularized. Thus we have the symmetric decomposition

$$H_1 = R_1^T R_1$$

where R_1 is upper triangular. Similarly, we compute the decomposition

$$H_2 = R_2^T R_2.$$

The construction of the (complex) orthogonal transformations which introduce zeros into vectors is presented in Appendix.

Next we compute the symmetric rank-revealing factorization [3]

$$R_2 = U_2 S_2 V^T \tag{5.1}$$

where U_2, $V \in C^{n\times n}$ are orthogonal, i.e., $U_2^T U_2 = V^T V = I_n$. Thus $S_2^T S_2$ and H_2 have same eigenvalues. In the absence of noise, the upper triangular matrix S_2 satisfies the rank-revealing condition

$$S_2^T S_2 = \begin{pmatrix} \hat{S}_2^T \hat{S}_2 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\hat{S}_2$ is the rth order leading principal submatrix of S_2. Consequently, we have the symmetric rank-revealing factorization

$$H_2 = V \begin{pmatrix} \hat{S}_2^T \hat{S}_2 & 0 \\ 0 & 0 \end{pmatrix} V^T.$$

This shows that H_2 and $\hat{S}_2^T \hat{S}_2$ have the same set of non-zero eigenvalues, since $V^T = V^{-1}$. Simultaneously, for R_1, we compute the decomposition

$$R_1 = U_1 S_1 V^T \tag{5.2}$$

where $U_1 \in C^{n \times n}$ is orthogonal and S_1 upper triangular. Note that the decompositions (5.1) and (5.2) share same matrix V. Now the problem of finding the generalized eigenvalues of H_1 and H_2 is equivalent to that of finding the generalized eigenvalues of

$$A = S_1^T S_1 \quad \text{and} \quad B = S_2^T S_2.$$

Similar to (3.3), these generalized eigenvalues are the eigenvalues of the r-by-r matrix

$$(\hat{S}_2^T \hat{S}_2)^{-1} (\hat{S}_1^T \hat{S}_1) \qquad \text{or} \qquad (\hat{S}_1 \hat{S}_2^{-1})^T (\hat{S}_1 \hat{S}_2^{-1}),$$

where $\hat{S}_1$ and $\hat{S}_2$ are the rth order leading principal submatrices of S_1 and S_2 respectively. In summary, we present the following algorithm for computing the r exponentials.

Algorithm 3. *Given Hankel matrices H_1 and H_2 where $m = 2n$, this algorithm computes the r exponentials.*

1. *Decompose $H_1 = \hat{R}_1^T \hat{R}_1 - \tilde{R}_1^T \tilde{R}_1$, where $\hat{R}_1 = D_1^{-1/2} \mathrm{triu}(H_1)$, $\tilde{R}_1 = \hat{R}_1 - D_1^{1/2}$, and D_1 is the diagonal part of H_1;*
2. *Obtain the symmetric factorization $H_1 = R_1^T R_1$ through an orthogonal transformation P such that*
$$P^T \begin{pmatrix} \hat{R}_1 \\ i\tilde{R}_1 \end{pmatrix} = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}$$
is upper triangularized;
3. *Similarly, factorize $H_2 = R_2^T R_2$, where R_2 is $n \times n$ and upper triangular;*
4. *Compute the symmetric rank-revealing factorization*
$$R_2 = U_2 S_2 V^T$$
where U_2, and V are orthogonal and S_2 is upper triangular and satisfies the rank-revealing condition; Simultaneously, decompose
$$R_1 = U_1 S_1 V^T$$
where U_1 is orthogonal and S_1 upper triangular;

5. *Let r be the rank revealed in the previous step and $\hat{S}_1$ and $\hat{S}_2$ are the rth order leading principal submatrices of S_1 and S_2 respectively, then the exponents are the eigenvalues of the r-by-r symmetric matrix $(\hat{S}_1\hat{S}_2^{-1})^T(\hat{S}_1\hat{S}_2^{-1})$.*

Finally, we remark that in step 5, we can further exploit the symmetric structure in $(\hat{S}_1\hat{S}_2^{-1})^T(\hat{S}_1\hat{S}_2^{-1})$ by computing the following decomposition

$$\hat{S}_1\hat{S}_2^{-1} = \hat{U}\hat{\Sigma}\hat{V}^T$$

where $\hat{U}$, $\hat{V} \in C^{n\times n}$ are orthogonal, i.e., $\hat{U}^T\hat{U} = \hat{V}^T\hat{V} = I_r$ and $\hat{\Sigma}$ is diagonal. Then the diagonal elements of $\hat{\Sigma}^2$ are the r eigenvalues of $(\hat{S}_1\hat{S}_2^{-1})^T(\hat{S}_1\hat{S}_2^{-1})$. The whole procedure is analogous to the computation of the singular value decomposition $S = U\Sigma V^H$ to obtain the eigenvalues of the Hermitian and positive definite matrix $S^H S$. The major difference is that here we use the complex orthogonal transformations whereas in the SVD we use unitary transformations. The detailed discussion of the procedure is out of the scope of this paper.

6 Appendix

Algorithm. *Complex Symmetric Reflector*
Given x such that $x^T x \neq 0$, compute a symmetric reflector $H_S = I - \delta_S^{-1} u u^T$ so that $H_S x = -\alpha e_1$ and $H_S^T H_S = I$.

$\alpha = \sqrt{x^T x}$;
$u = x + \alpha e_1$;
$\delta_S = \alpha u_1$.

Algorithm. *Complex Symmetric Rotation*
Given a, $b \in C$ such that $a^2 + b^2 \neq 0$, compute the c and s in a symmetric rotation

$$G_S = \begin{pmatrix} c & s \\ s & -c \end{pmatrix}$$

so that

$$G_S \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \quad \text{and} \quad G_S^T G_S = I_2.$$

if $(|a| > |b|)$
  $\tau = b/a$; $c = 1/\sqrt{1+\tau^2}$; $s = c\tau$;
else
  $\tau = a/b$; $s = 1/\sqrt{1+\tau^2}$; $c = s\tau$.

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