A QR Decomposition Based Algorithm for Computing Transfer Functions from State Equations

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Abstract: This paper presents the derivation of a new algorithm for computing transfer functions from state equations with emphasis on the transfer zeros. This algorithm employs a QR decomposition to transform the original state equation to a reduced inverse system of minimum order, therefore it is computationally efficient and numerically stable. Three numerical examples are given to show the procedure and its computational advantages.

1. INTRODUCTION

In system (or circuit) analysis, it is often required to convert from a frequency domain analysis to its time domain analysis or vice versa. The frequency domain analysis usually involves poles and zeros of transfer functions and the time domain analysis usually involves the state equations. Computing the state equation of a linear time-invariant system from a transfer function is quite straightforward, however the reverse task is not simple. Special attention has to be paid to this task for the computation of the zeros because the numerical error in the computation may be severe for high-order systems with ill-conditioned matrices.

Consider the following state equations:

$$\begin{aligned} x^{(j+1)} &= Ax^{(j)} + \mu b \\ y^{(j)} &= c^T x^{(j)} \end{aligned} \tag{1}$$

corresponding to a circuit, where A is an n-by-n constant matrix, b and c are n-by-1 constant vectors, and μ is a scalar. Thus this state-space model is a linear time-invariant system of order n. Based on an inverse system concept, Brockett [1] derived the following form for the transfer function:

$$T(s) = q^{-1}\frac{\det(sI_n - A + qbc^TA^\alpha)}{s^\alpha \det(sI_n - A)} \tag{2}$$

where I_n is the identity matrix of order n, α is the smallest positive integer j such that $c^TA^{j-1}b \neq 0$, and

$$q = \frac{1}{c^TA^{\alpha-1}b}.$$

The algorithm presented in this paper is based on the transformation method [6]. This method applies a transformation to the state vector x to reduce the order of the system. Then the transfer function, especially the transfer zeros, can be found from the reduced system. Effectively, the computation is reduced. There exit several methods for determining the transformation. For example, Pottle [6] proposed a Gaussian elimination based technique and Chen [2] presented a pseudo-inverse approach by using the exchange method. In this paper, we propose a QR decomposition based algorithm. As we know that the QR decomposition is numerically stable, we expect our method is more reliable than the methods in [2] and [6].

2. THE ALGORITHM

Since $\alpha < n$ is the smallest positive integer such that $c^TA^{\alpha-1}b \neq 0$, we have,

$$c^Tb = \cdots = c^TA^{\alpha-2}b = 0.$$

Then from (1), we can show that

$$y^{(j)} = c^TA^jx^{(0)} \qquad \text{for} \quad j = 0, ..., \alpha - 1. \tag{3}$$

Moreover,

$$y^{(\alpha)} = c^TAx^{(\alpha-1)} = \cdots = c^TA^{\alpha-1}x^{(1)} = c^TA^\alpha x^{(0)} + \mu q^{-1} \tag{4}$$

If we define

$$v_\alpha = \begin{pmatrix} y^{(0)} \\ y^{(1)} \\ \vdots \\ y^{(\alpha-1)} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} c^T \\ c^TA \\ \vdots \\ c^TA^{\alpha-1} \end{pmatrix},$$

then from (3) we get

$$v_\alpha = Bx^{(0)}. \tag{5}$$

Note that B is an α-by-n matrix. In the following, we show how to find a transformation to reduce the dimension of the state vector x from n to $n - \alpha$.

Suppose that an orthogonal matrix Q triangularizes B, specifically,

$$BQ = (L \quad 0) \tag{6}$$

where L is α-by-α lower triangular and nonsingular. Equivalently,

$$B^T = Q \begin{pmatrix} L^T \\ 0 \end{pmatrix}$$

is the QR decomposition of B^T. The matrix Q can be formed by a sequence of Householder transformations or a sequence of Givens rotations [4]. Now, we use the orthogonal matrix Q to transform the state vector x into a new vector z:

$$z^{(j)} = Q^T x^{(j)}. \tag{7}$$

Orthogonal transformations have many desirable properties. For example, $Q^T = Q^{-1}$. Partitioning

$$Q = (Q_1 \quad Q_2)$$

where Q_1 consists of the first α columns of Q, we denote

$$z_1^{(j)} = Q_1^T x^{(j)} \quad \text{and} \quad z_2^{(j)} = Q_2^T x^{(j)}.$$

Thus

$$z^{(j)} = \begin{pmatrix} z_1^{(j)} \\ z_2^{(j)} \end{pmatrix}.$$

Note that $z_1^{(j)}$ has dimension α and $z_2^{(j)}$ has dimension $n - \alpha$. Since, $Q_1 Q_1^T + Q_2 Q_2^T = QQ^T = I_n$, we have

$$Q_1 z_1^{(j)} + Q_2 z_2^{(j)} = x^{(j)}. \tag{8}$$

Next, we derive a state-space equation for the vector $z_2^{(j)}$ of dimension $n - \alpha$. Since the system represented by (1) is time-invariant, it suffices to consider the transition between $z_2^{(0)}$ and $z_2^{(1)}$. For simplicity, we denote

$$x = x^{(0)}, \quad \hat{x} = x^{(1)},$$

and

$$z_1 = Q_1^T x, \quad \hat{z}_1 = Q_1^T \hat{x}, \quad z_2 = Q_2^T x, \quad \hat{z}_2 = Q_2^T \hat{x}.$$

Then, from (8), the first equation in (1) is transformed into

$$Q_1 \hat{z}_1 + Q_2 \hat{z}_2 = AQ_1 z_1 + AQ_2 z_2 + \mu b.$$

To derive a state equation for z_2, we eliminate $\hat{z}_1$ in the above equation by premultiplying the both sides with Q_2^T. Since Q is orthogonal, $Q_2^T Q_1 = 0$ and $Q_2^T Q_2 = I_{n-\alpha}$. It then follows that

$$\hat{z}_2 = Q_2^T A Q_2 z_2 + Q_2^T A Q_1 z_1 + \mu Q_2^T b. \tag{9}$$

To complete the state-space equation for the reduced state vector z_2, we further manipulate the right side of the above equation as follows. First, (4) gives

$$\mu = -qc^T A^\alpha x + qy^{(\alpha)}.$$

By using (8), it follows that

$$\mu = -qc^T A^\alpha Q_1 z_1 - qc^T A^\alpha Q_2 z_2 + qy^{(\alpha)}.$$

Substituting μ in (9) with the above equation, we get

$$\hat{z}_2 = Q_2^T H Q_2 z_2 + Q_2^T H Q_1 z_1 + qy^{(\alpha)} Q_2^T b \tag{10}$$

where $H = A - qbc^T A^\alpha$. Second, from (5), (6), and (7), we get

$$v_\alpha = (L \quad 0) Q^T x = L Q_1^T x = L z_1.$$

Substituting z_1 in (10) with $L^{-1} v_\alpha$, we have

$$\hat{z}_2 = Q_2^T H Q_2 z_2 + Q_2^T H Q_1 L^{-1} v_\alpha + qQ_2^T b y^{(\alpha)}.$$

Finally, let

$$v_{\alpha+1} = \begin{pmatrix} v_\alpha \\ y^{(\alpha)} \end{pmatrix},$$

we obtain the new state equation of reduced order

$$\hat{z}_2 = Q_2^T H Q_2 z_2 + Q_2^T (HQ_1 L^{-1} \quad qb) v_{\alpha+1}. \tag{11}$$

Note that $\hat{z}_2$ and z_2 are $(n - \alpha)$ vectors and $Q_2^T H Q_2$ is an $(n - \alpha)$-by-$(n - \alpha)$ matrix. Thus by using the transformation Q, we have reduced the original system (1) of order n to the new system (11) of order $n - \alpha$. The eigenvalues of $Q_2^T H Q_2$ are the poles of the transfer function of the system (11), which are identical to the transfer zeros of the original system (1) [2].

We summarize the above in the following algorithm.

Algorithm. Given the state equations (1), this algorithm computes its transfer zeros as follows:

- 1. Find the smallest positive integer such that $c^T A^{\alpha-1} b \neq 0$;
- 2. Form the matrix

$$B = \begin{pmatrix} c^T \\ c^T A \\ \vdots \\ c^T A^{\alpha-1} \end{pmatrix};$$

- 3. Triangularize B by an orthogonal transformation Q:

$$BQ = (L \quad 0);$$

- 4. Partition $Q = (Q_1\ Q_2)$ where Q_1 consists of the first α columns of Q;
- 5. Set $H = A - qbc^T A^\alpha$ where $q = 1/(c^T A^{\alpha-1} b)$;
- 6. Compute the eigenvalues of $Q_2^T H Q_2$.

3. EXAMPLES

Example 1. Given A, b, c, and μ of the state equations (1) as follow:

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 0.5 & -0.5 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\text{and} \quad \mu = 1.0,$$

then we have

$$c^T b = 0 \quad \text{and} \quad c^T A b = 0.5 \neq 0.$$

Thus we get

$$\alpha = 2, \quad q = (c^T A b)^{-1} = 2,$$

and

$$B = \begin{pmatrix} c^T \\ c^T A \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0.5 & -0.5 & 0 \end{pmatrix},$$

$$H = A - qbc^T A^2 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0.5 & -0.5 & 0 \end{pmatrix}.$$

We find that the orthogonal matrix

$$Q = \begin{pmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \\ 1 & 0 & 0 \end{pmatrix}$$

triangularizes B:

$$BQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & 0 \end{pmatrix}.$$

It then follows that

$$Q_2 = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}$$

and $Q_2^T H Q_2 = -1$ is the only transfer zero of the original system. The poles can be computed by using (2). They are -1 and $-0.5 \pm j0.866$ in addition to the two poles at the origin.

We programmed our algorithm in MATLAB and ran the following two circuit examples on a SUN Ultra1.

Example 2. The circuit shown in Fig. 1 is adopted from [3, Fig. 2].

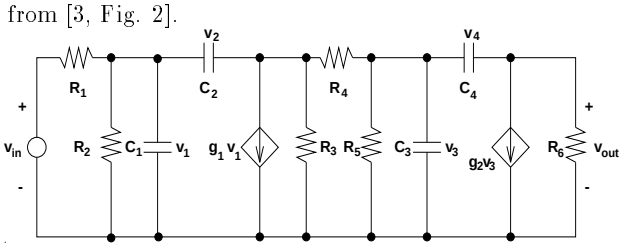

Fig. 1. Cascade of two common emitter amplifiers.

Denoting

$$x = (v_1, v_2, v_3, v_4)^T,$$

we derived from the circuit A, b, c and μ of the state equations (1) as follows:

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$$b = \begin{pmatrix} \frac{1}{C_1 R_1} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mu = 1.0.$$

Substituting the values

$$R_1 = R_2 = R_6 = 1k\Omega, \quad R_3 = 20k\Omega, \quad R_4 = 200\Omega,$$

$$R_5 = 800\Omega, \quad g_1 = g_2 = 0.021\Omega^{-1},$$

$$C_1 = C_3 = 200pF, \quad \textit{and} \quad C_2 = C_4 = 5pF,$$

our algorithm computed $\alpha = 2$ and the two zeros: 4.200×10^9 and 4.200×10^9. Using (2), we got the following nonzero poles:

$$-3.259 \times 10^6, \quad -1.113 \times 10^7, \quad -3.102 \times 10^8,$$

$$\text{and} \quad -1.167 \times 10^9.$$

$$A = \begin{pmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} + g_1\right) & -\frac{1}{C_1}\left(\frac{1}{R_3} + \frac{1}{R_4}\right) & \frac{1}{C_1 R_4} & 0 \\ -\frac{1}{C_2}\left(\frac{1}{R_3} + \frac{1}{R_4} + g_1\right) & -\frac{1}{C_2}\left(\frac{1}{R_3} + \frac{1}{R_4}\right) & \frac{1}{C_2 R_4} & 0 \\ \frac{1}{C_3 R_4} & \frac{1}{C_3 R_4} & -\frac{1}{C_3}\left(\frac{1}{R_4} + \frac{1}{R_5} + \frac{1}{R_6} + g_2\right) & -\frac{1}{C_3 R_6} \\ 0 & 0 & -\frac{1}{C_4}\left(\frac{1}{R_6} + g_2\right) & -\frac{1}{C_4 R_6} \end{pmatrix}$$

Example 3. In this example, we use the phase shift oscillator in [5, Fig. 3]. The circuit is reproduced, with corrections, in Fig. 2.

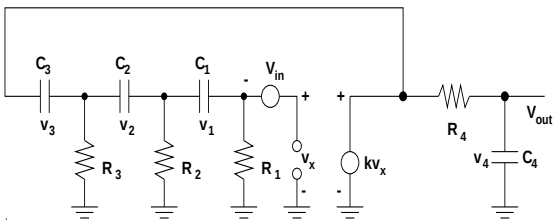

Fig. 2. Phase shift oscillator.

Again, denoting

$$x = (v_1, v_2, v_3, v_4)^T,$$

and

$$\alpha_i = \frac{1}{C_i(1-k)}, \quad \text{for} \quad i = 1, 2, 3, 4,$$

we derived from the circuit A, b, c and μ of the state equations (1) as follows:

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$$b = \begin{pmatrix} -\frac{\alpha_1 k}{R_1} \\ -\alpha_2 k \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \\ -\alpha_3 k \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) \\ \frac{\alpha_4 k}{R_4} \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\text{and} \quad \mu = 1.0.$$

Using the following values:

$$R_1 = R_2 = R_3 = 1k\Omega, \ R_4 = 20k\Omega,$$

$$C_1 = C_2 = C_3 = C_4 = 1\mu F, \text{ and } k = -29,$$

our algorithm computed $\alpha = 1$ and the three zeros: -3.080×10^2, -6.431×10^2, -5.049×10^3. From (2), we computed the four nonzero poles:

$$-5.000 \times 10, \quad -2.000 \times 10^2, \quad \text{and} \quad \pm j4.082 \times 10^2.$$

4. CONCLUSIONS

Applying the minimal inverse system concept and the QR decomposition, a new algorithm has been derived for the computation of transfer zeros from a state equation. Since the transformed new state equation is of minimal order, reduced through the procedure of QR decomposition, the new algorithm has the computational advantages of efficiency and numerical stability which have been demonstrated in the numerical examples.

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$$A = \begin{pmatrix} -\frac{\alpha_1}{R_1} & -\frac{\alpha_1}{R_1} & -\frac{\alpha_1}{R_1} & 0 \\ -\alpha_2 \left(\frac{1}{R_1} + \frac{k}{R_2}\right) & -\alpha_2 \left(\frac{1}{R_1} + \frac{1}{R_2}\right) & -\alpha_2 \left(\frac{1}{R_1} + \frac{1}{R_2}\right) & 0 \\ -\alpha_3 \left(\frac{1}{R_1} + \frac{k}{R_2} + \frac{k}{R_3}\right) & -\alpha_3 \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{k}{R_3}\right) & -\alpha_3 \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) & 0 \\ \frac{\alpha_4 k}{R_4} & \frac{\alpha_4 k}{R_4} & \frac{\alpha_4 k}{R_4} & -\frac{1}{C_4 R_4} \end{pmatrix}$$