Elementary Petri Nets

SE 3BB4

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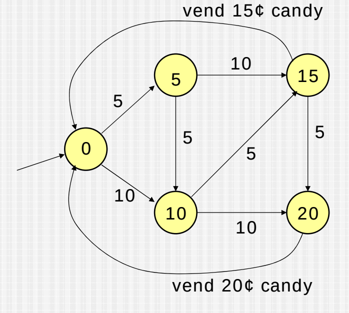
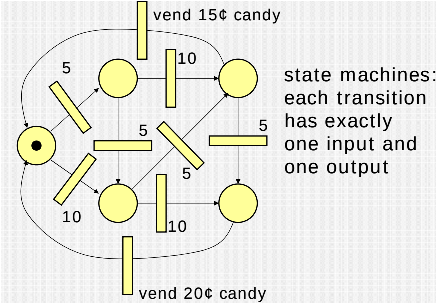
From State Machines to Elementary Petri Nets

Elementary Petri Nets are directed bipartite graphs with

- *places*, represented by circles or ovals (represent some type of resource)
- *transitions*, represented by rectangles or lines (consume and produce resources)
- *arcs* (from places to transitions or transitions to places)
- *tokens*, placed in places
- *initial marking*, tokens is some places



Finite State Machine



Petri Net

Modeling Concurrency

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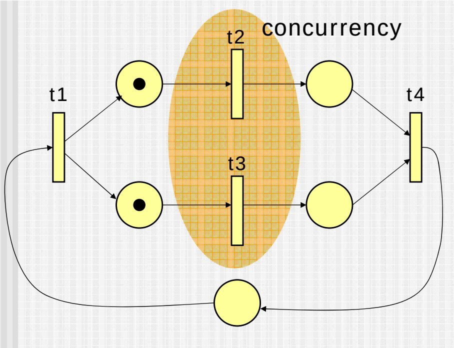
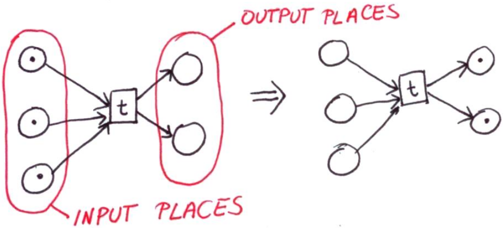
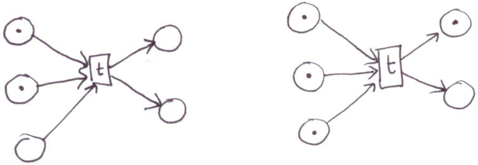


Firing Rules for Elementary Petri Nets

- A transition t can be *fired* if and only if it has tokens in all its input places and all output places are empty.
- After firing t, all its input places become empty and all its output places contain tokens.



Before fining t After firing t

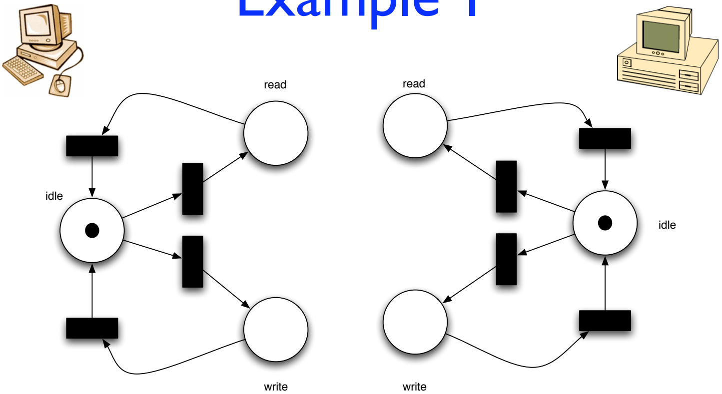


t can not be fired

Modeling Mutual Exclusion

- Two computers, one printer/data base, etc.
- Without any synchronization, individual viewpoints of computers.

Example I



The token tells us the state of the process

Modeling Mutual Exclusion

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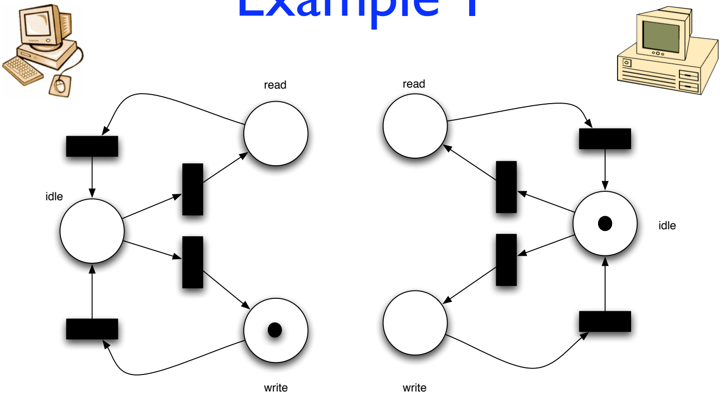

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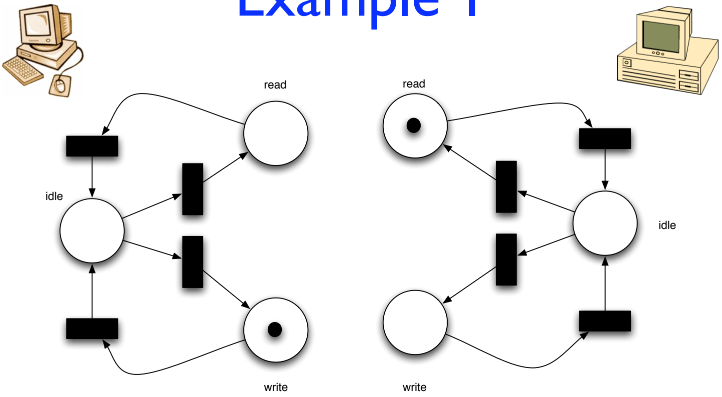

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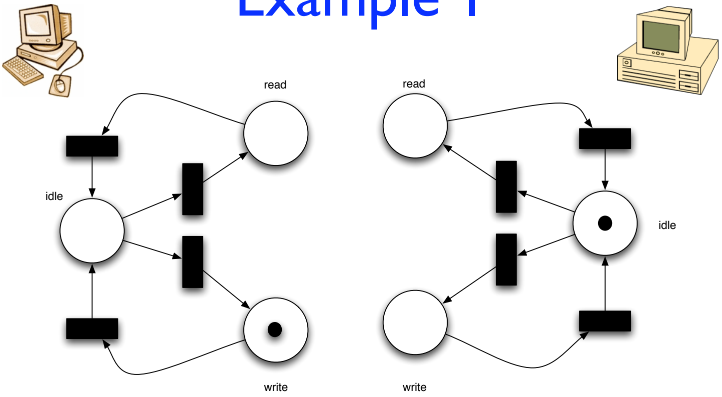
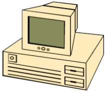
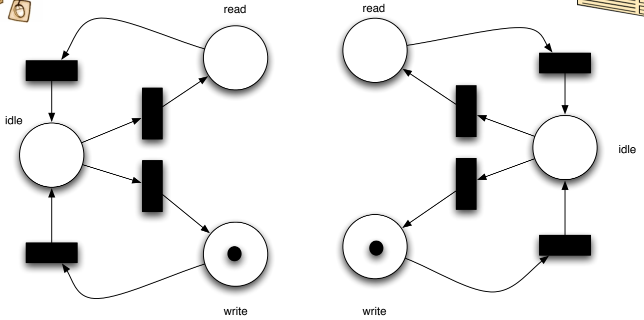


The token tells us the state of the process

- Two computers, one printer/data base, etc.
- Without any synchronization, individual viewpoints of computers.
- PROBLEM, both computers want to write but there is only one printer/data base, etc.

Example I



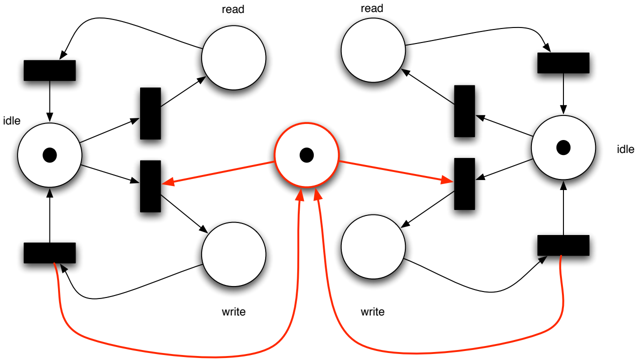


The token tells us the state of the process

Modeling Mutual Exclusion

- Two computers, one printer/data base, etc.
- Synchronization is added.

Example I

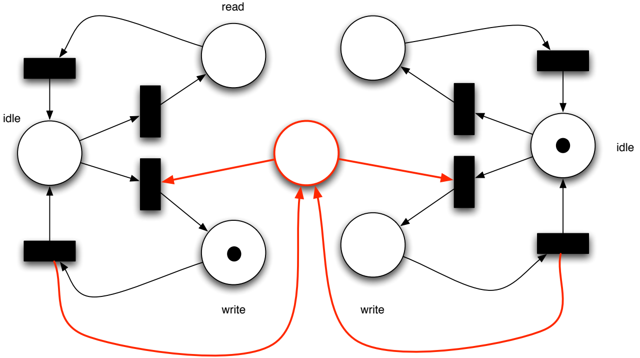


Add a lock to ensure mutual exclusion

Modeling Mutual Exclusion

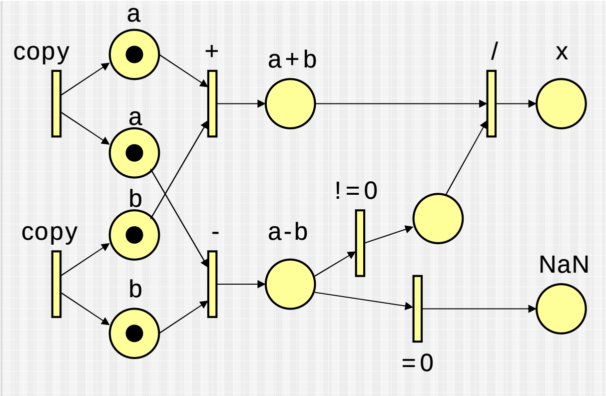
- Two computers, one printer/data base, etc.
- Synchronization is added.

Example I



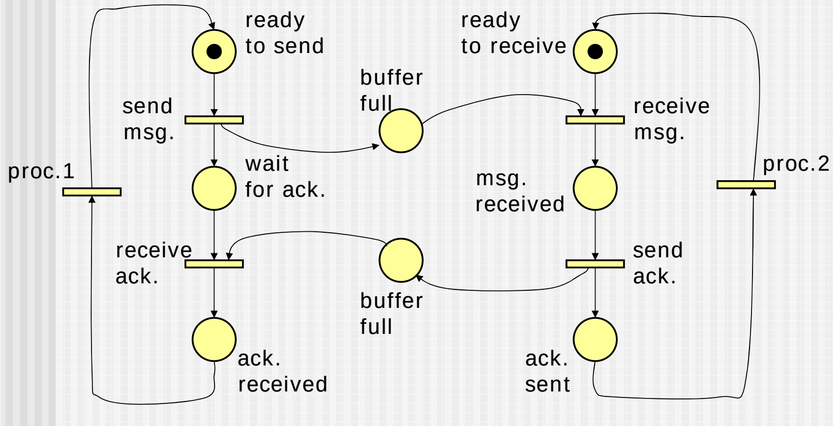
Modeling Dataflow Computation

- Petri nets allow modeling **without** decomposing the whole system into sequential component!
- $x = (a + b)/(a - b)$



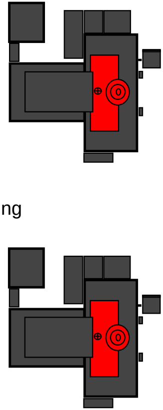
- The two *copy* transitions can be removed, they represent inputs from environment.

Modeling Communication Protocols



Loading and Unloading Two Machines

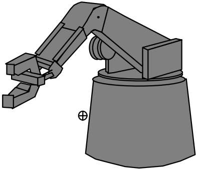
Machine 1



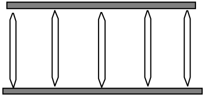
Machine 2

Machine States:
Loading
Processing
Waiting for unloading
Unloading

Robot

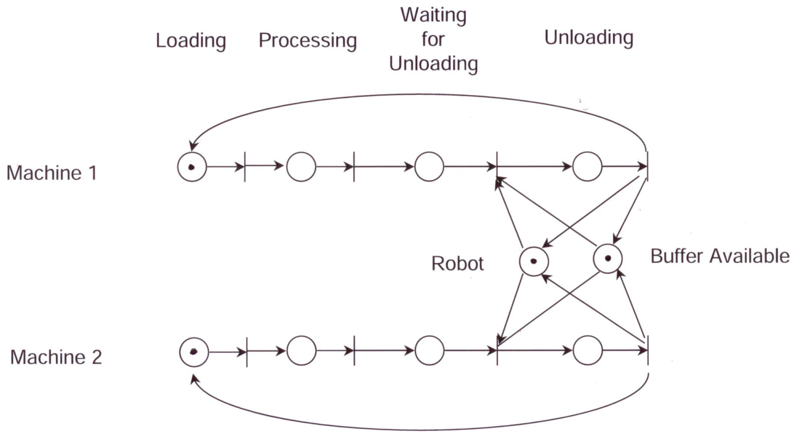


Buffer



Buffer State:
Space availability

Loading and Unloading Two Machines

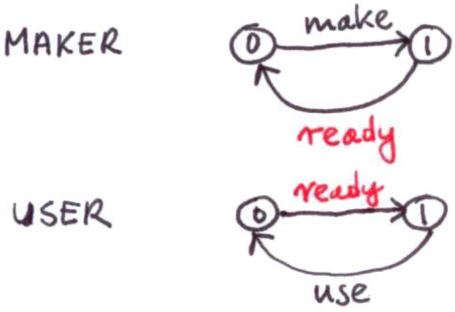


Composition of LTS (Maker-User Example)

$$MAKER = make \rightarrow ready \rightarrow MAKER$$

$$USER = ready \rightarrow use \rightarrow USER$$

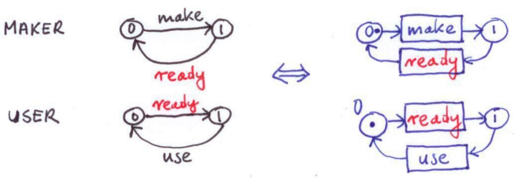
$$\| \; MAKER\_USER = Maker \parallel USER$$



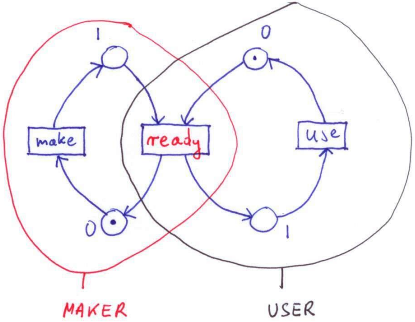
LTS:

LTS via Elementary Petri Nets

1. Represent each LTS as a Petri Net

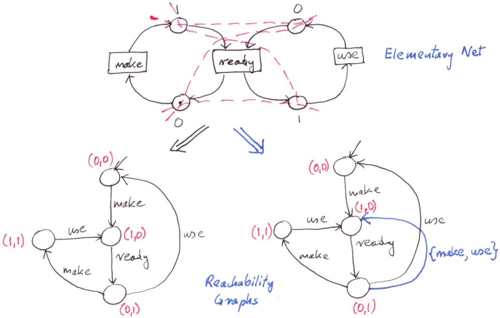


2. 'Glue' together both nets through the common transition *ready*.



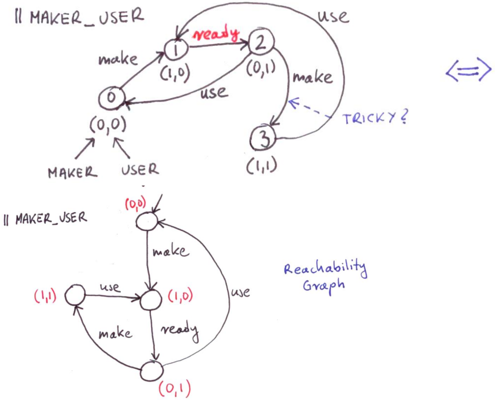
Reachability Graphs

- Reachability graphs are finite state machines that represent the behaviours of Petri nets.
- Each state of the reachability graph represent a 'marking' of Petri net.
- Simultaneous executions (*steps*), like $\{make, use\}$, may be allowed.



Reachability Graphs and $\|$ operator

- Reachability graphs are the same as final LTS obtained via operator '$\|$'!
- Getting final LTS vis Petri nets is more natural than via standard procedure!



Formal Definition (1)

Definition

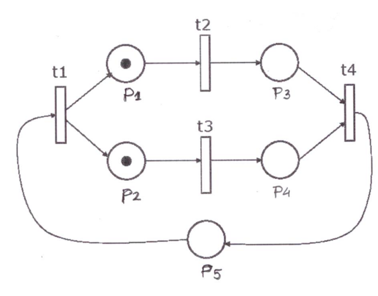
An **Elementary Net** is a tuple

$$\mathtt{N} = (P, T, F, C_{init})$$

such that

1. P and T are finite and disjoint sets of places and transitions represented, respectively, as circles and rectangles;
2. $F \subseteq (P \times T) \cup (T \times P)$ is the flow relation of $\mathtt{N}$ - represented as directed arcs between places and transitions;
3. $C_{init} \subseteq P$ is the initial marking (or initial configuration) of $\mathtt{N}$.

Example

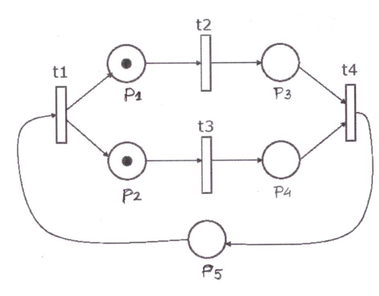


$$
\begin{aligned}
P &= \{p_1, p_2, p_3, p_4, p_5\},\\
T &= \{t_1, t_2, t_3, t_4\},\\
F &= \{(p_1, t_2), (p_2, t_3), (p_3, t_4), (p_4, t_4), (p_5, t_1),\\
&\qquad (t_1, p_1), (t_1, p_2), (t_2, p_3), (t_3, p_4), (t_4, p_5)\},\\
C_{init} &= \{p_1, p_2\}.
\end{aligned}
$$

Formal Definition (2)

- For every $x \in P \cup T$, the set ${}^\bullet x = \{y \mid (y,x) \in F\}$ denotes the *input* nodes of x, and
- the set $x^\bullet = \{y \mid (x,y) \in F\}$ denotes the *output* nodes of x.
- The dot-notation extends to sets in the natural way, e.g. the set $X^\bullet$ comprises all outputs of the nodes in X.
- We often (but not always) assume that for every $t \in T$, both ${}^\bullet t$ and $t^\bullet$ are non-empty and disjoint.

Example



$^\bullet p_1 = \{t_1\}$, $^\bullet p_2 = \{t_1\}$, $^\bullet p_3 = \{t_2\}$, $^\bullet p_4 = \{t_3\}$, $^\bullet p_5 = \{t_4\}$,
$p_1^\bullet = \{t_2\}$, $p_2^\bullet = \{t_3\}$, $p_3^\bullet = \{t_4\}$, $p_4^\bullet = \{t_4\}$, $p_2^\bullet = \{t_1\}$,
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$^\bullet\{t_1, t_3\} = \{p_2, p_5\}$, $^\bullet\{t_2, t_3\} = \{p_1, p_2\}$,
$\{t_1, t_3\}^\bullet = \{p_1, p_2, p_4\}$, $\{t_2, t_3\}^\bullet = \{p_3, p_4\}$.

- A transition t is enabled at a configuration C if ${}^\bullet t \subseteq C$ and $t^\bullet \cap C = \emptyset$.
- An enabled transition t can fire leading to a new configuration $C' = (C \setminus {}^\bullet t) \cup t^\bullet$.
- We denote this by $C[t\rangle C'$, or by $C[t\rangle_{\mathrm{N}} C'$, if C, C' and t may belong to different nets.
- We will also write $C[t_1 \dots t_n\rangle C'$ if $C[t_1\rangle C_1 \dots C_{n-1}[t_n\rangle C'$ for some configurations $C_1, \dots, C_{n-1}$.

Definition

A **firing sequence** of an Elementary Petri Net is any sequence of transitions $t_1, \dots, t_n$ for which there are markings $C_1, \dots, C_n$ satisfying:

$$C_{init}[t_1\rangle C_1[t_2\rangle C_2 \dots [t_n\rangle C_n.$$

Step-sequence Semantics

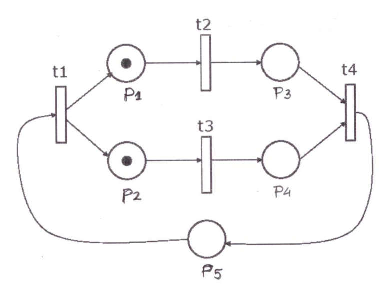
- Let $A \subseteq T$ be a non-empty set such that for all distinct $t_1, t_2 \in A$:
$$(t_1^\bullet \cup {}^\bullet t_1) \cap (t_2^\bullet \cup {}^\bullet t_2) = \emptyset.$$
- Then A is enabled at a marking C if ${}^\bullet A \subseteq C$ and $A^\bullet \cap C = \emptyset$.
- We also denote this by $C[A\rangle C'$, or $C[A\rangle_{\mathrm{N}} C'$ when C, C' and A may belong to different nets, where $C' = (C \setminus {}^\bullet A) \cup A^\bullet$.

Definition

A **firing step sequence** is a sequence of sets (or steps) $A_1, \ldots, A_n$ for which there are markings $C_1, \ldots, C_n$ satisfying:

$$C_{init}[A_1\rangle C_1[A_2\rangle C_2 \ldots [A_n\rangle C_n.$$

Example



- Some firing sequences: $t_2 t_3 t_4 t_1$ since $\{p_1, p_2\}[t_2\rangle\{p_2, p_3\}[t_3\rangle\{p_3, p_4\}[t_4\rangle\{p_5\}[t_1\rangle\{p_1, p_2\}$, $t_3 t_2 t_4 t_1$ since $\{p_1, p_2\}[t_3\rangle\{p_1, p_4\}[t_2\rangle\{p_3, p_4\}[t_4\rangle\{p_5\}[t_1\rangle\{p_1, p_2\}$.
- A firing step-sequence: $\{t_2, t_3\}\{t_4\}\{t_1\}$ since $\{p_1, p_2\}[\{t_2, t_3\}\rangle\{p_3, p_4\}[\{t_4\}\rangle\{p_5\}[\{t_1\}\rangle\{p_1, p_2\}$.