SWE 3C03

Discrete Optimization ... Hungarian method Assignment problem

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The assignment problem

Primal and Dual problems

**The primal problem:**

\[
\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\sum_{j=1}^{n} x_{ij} = 1 \quad \forall i = 1 : n \\
\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j = 1 : n \\
x_{ij} \geq 0 \quad \forall i, j = 1 : n
\]

allocation/matching/(partial) assignment

**The dual problem:**

\[
\max \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j \\
u_i + v_j \leq c_{ij} \quad \forall i, j = 1 : n
\]

potential \((u,v)\) dual slack \(c_{ij} = c_{ij} - u_i - v_j\)

**Complementarity:**

\[
x_{ij}(c_{ij} - u_i - v_j) = x_{ij} c_{ij} = 0 \quad \forall i, j = 1 : n
\]
Primal-dual algorithms

At each iteration:

1. The algorithm has a dual feasible solution and a complementary partial primal vector which is feasible for a weaker version (relaxation) of the primal problem.

2. An iteration is either a primal or dual step:

   **Primal step:** The dual solution is fixed, while keeping complementarity the partial primal solution is brought closer to feasibility.

   **Dual step:** The primal vector is fixed. Improve feasible solution while keeping complementarity. The change is made so that at the next iteration the primal vector can be improved again.

3. There is a measure of primal infeasibility what decreases monotonically.
Dual feasible solution

**admissible cells**

Easy to make a dual feasible solution. Let $u_i = \min_{0 \leq j \leq n} c_{ij}$ and $v_j = \min_{0 \leq i \leq n} c_{ij} - u_i$. This is a dual feasible solution.

We keep the complementarity conditions, thus for the primal solution $(i, j)$ is an

- **admissible cell** if $\bar{c}_{ij} = c_{ij} - u_i - v_j = 0$
- **inadmissible cell** if $\bar{c}_{ij} = c_{ij} - u_i - v_j > 0$.

Admissible cells define a **Marriage problem**.

Try to solve the marriage problem by the labeling algorithm.

When a primal feasible solution is found using only admissible cells then an optimal solution is found!

*Recall the duality theorems.*
The Hungarian method

The algorithm

1. **Initialization.** If a row or column has only inadmissible cells stop, the problem is infeasible.

2. **Identify a dual feasible solution.**

3. **Try to solve the marriage problem.**
   - If the result is an assignment – done.
   - This is done by using the labeling routine.
   - If augmentation is found update \( x \).
     (scan and label!).
   - Augmentation improves the partial assignment by one element.
   - If a complete assignment is found, we have an optimal pair.

4. If no augmentation is possible, a cover with less than \( n \) lines of all admissible cells is found.

5. **Change the dual solution.**
   - Let \( \delta = \min \{ c_{ij} : (i, j) \text{ is not covered} \} \).
   - Let \( u_i := \begin{cases} u_i & \text{if row } i \text{ is covered} \\ u_i + \delta & \text{if row } i \text{ is not covered} \end{cases} \)
   - and \( v_j := \begin{cases} v_j & \text{if column } j \text{ is not covered} \\ v_j - \delta & \text{if column } j \text{ is covered} \end{cases} \)
   - The dual objective strictly increases.