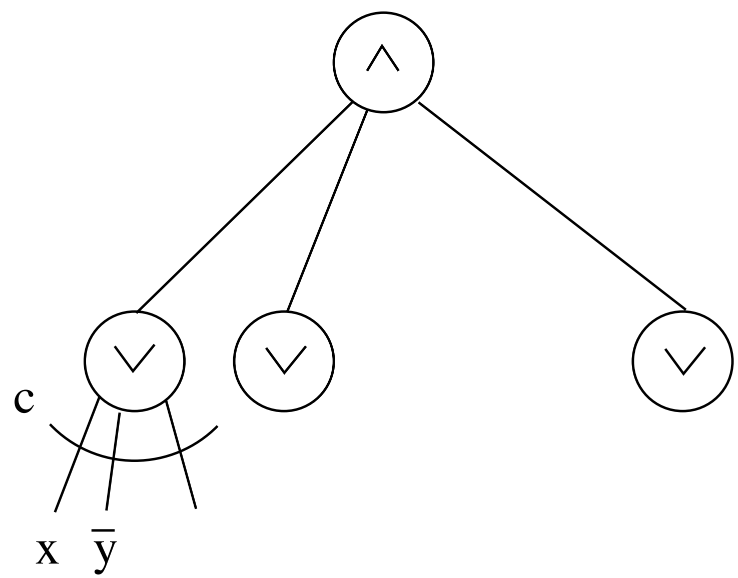
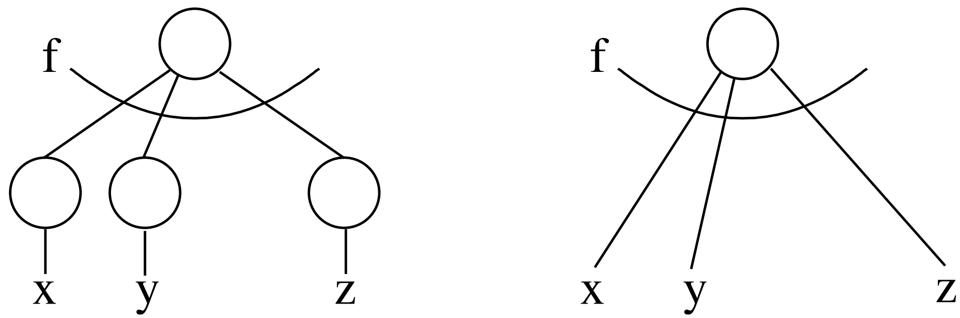
After random restriction r, the AND gate *depends* on a constant number of variables.



AND depends on $x, y, \ldots$; the variables in the literals reachable from the AND gate.

Prove by induction on c that the AND gates depends on "too many variables" with probability $O(\frac{1}{n^k})$.

Basis Case: $c = 1$.



Two cases:

1. AND has a large fan-in ($f \geq 4 \cdot k \cdot \ln(n)$)
2. AND has a small fan-in

In the first case (large fan-in), we have:

$$\Pr[\text{ AND-gate is } \textit{not } 0\ ] = O\left(\frac{1}{n^k}\right)$$

This is because it is very likely that a random r would have set one of the AND's inputs to 0.

So with high probability, the AND-gate depends on no variables.

In the second case (small fan-in), we have:

$$\Pr[\text{ AND-gate depends on } \textit{more} \text{ than } 18k \text{ inputs }] = O\left(\frac{1}{n^k}\right)$$

Induction Step: Assume the result holds for $(c-1)$.

Again, there are two cases:

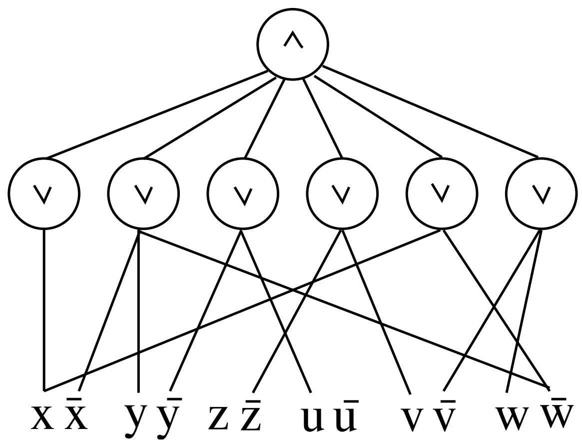
1. Before the random restriction, the AND-gate has many OR-gates below it with *disjoint input variables* $(d \cdot \ln(n),\ d = k \cdot 4^c)$
2. Before the random restriction, the AND-gate has few OR-gates below it with *disjoint input variables*

In the first case (many OR-gates below AND), it is very likely that after the random restriction one of the OR-gates will have all of its inputs set to 0, and so the AND-gate is 0, and so it depends on no variables.

$$\Pr[\text{ AND-gate is } \textit{not} = 0\ ] = O\left(\frac{1}{n^k}\right)$$

In the second case (few OR-gates below AND), choose a *maximal* set of OR-gates with *disjoint input variables*.

Let H be the set of the variables that occur in these OR gates.



In this example, a maximal set of ORs would consist of gates (numbered from left) $\{1, 3, 4\}$ and $H = \{x, y, u, z, v\}$.

$|H| \leq c \cdot d \cdot \ln(n)$.

Each of the OR-gates has a variable in H. $(*)$

Let $\{\tau_1, \tau_2, \ldots, \tau_l\}$, $l = 2^{|H|}$, be all the truth assignments to variables in H.

Let $A_i = C^{\tau_i}$, after simplifying, where C represents the whole AND-OR circuit.

By $(*)$, each OR gate in A_i either disappears or "looses" an input.

So the input fan-in in each A_i is at most $c - 1$.

By the induction hypothesis, the probability that A_i^r depends on more than e_{c-1} variables is bounded above by $O(\frac{1}{n^k})$.

If $H = \{x_1, x_2\}$, then

$$C = (\bar{x}_1 \wedge \bar{x}_2 \wedge A_1) \vee (\bar{x}_1 \wedge x_2 \wedge A_2) \vee (x_1 \wedge \bar{x}_2 \wedge A_3) \vee (x_1 \wedge x_2 \wedge A_4)$$

Generalize,

$$C = \bigvee_{i=1}^{l} (\ldots x\text{'s} \ldots) \wedge A_i \qquad (**)$$

Let h be the number of variables in H after a random restriction.

$$\Pr[\ h > 4cd + 2k\ ] = O\left(\frac{1}{n^k}\right)$$

Thus, with high probability

$$2^h \leq 2^{4cd+2k} =: m$$

Hence, there are at most m terms in $(**)$, so

$$\boxed{e_c = m \cdot e_{c-1} + (4cd + 2k)}$$

and $\Pr[\ C$ depends on more than e_c variables $]$

$$\leq \Pr[h > 4cd + 2k] + m \cdot \Pr[\ A_j \text{ depends on more than } e_{c-1} \text{ variables }]$$

$$\leq O\left(\frac{1}{n^k}\right) + m \cdot O\left(\frac{1}{n^k}\right) = O\left(\frac{1}{n^k}\right)$$