The assignment is now due on November 13
In class, at the beginning of the lecture

We are going to provide some hints for question 5(b) in Assignment 4.

Suppose that P is a resolution refutation of $\alpha(\vec{p},\vec{q})\cup\beta(\vec{p},\vec{r})$, where the p_i's occur only positively in α and only negatively in β.

Let σ be a truth value assignment to $\vec{p}$. We apply a transformation to P to obtain a refutation $P|_\sigma$ and at the same time construct a feasible monotone interpolant $\mathbf{I}(\sigma)$. If C was a clause in the original refutation, it will be denoted C' after applying the procedure, and $\mathbf{I}_C(\sigma)$ will be the interpolant associated with clause C.

We say that a clause C' is an α-clause if it contains variables from among $\vec{p},\vec{q}$ only, and if it only contains variables from $\vec{p}$, it is an α-clause if all its ancestors are α-clauses. Similarily, we define a β-clause symmetrically, with $\vec{r}$ and negations of $\vec{p}$.

We let $\mathbf{I}_C(\sigma)$ be 0 if C' is an α-clause, and 1 if it is a β-clause. Initially, every clause C in α and β stays the same, i.e., $C'=C$, and clauses in α are declared to be α-clauses, and all the clauses in β are declared to be β-clauses. We now describe the rest of the procedure.

Suppose a clause C in P is obtained by:

$$\frac{C_1\cup\{x\} \qquad C_2\cup\{\bar{x}\}}{C} \tag{1}$$

Assume inductively that we have already transformed the premises, and we have obtained $(C_1\cup\{x\})', (C_2\cup\{\bar{x}\})'$, and we have also declared their sides.

The task is to define C' and $\mathbf{I}_C(\sigma)$. We do it separately for x being a variable in one of the three groups: p_i, q_i, r_i.

For example, in the case that $x=p_i$, then you might want to let C' be $(C_1\cup\{p_i\})'$ if $p_i=0$, and $(C_2\cup\{\bar{p}_i\})'$ otherwise. The temptation is now to define

$$\mathbf{I}_C := (p_i\vee\mathbf{I}_{C_1'})\wedge(\bar{p}_i\vee\mathbf{I}_{C_2'}) \tag{2}$$

This would work if we were not constructing a monotones circuit for the interpolant, but we are, so $\bar{p}_i$ is not possible.

(i) Show how to define C' and $\mathbf{I}_C$ so that we have monotonicity in the case that $x=p_i$. Keep in mind that one of the goals of the definition of C' is to ensure that it is either an α-clause or a β-clause. The second goal is to be able to do question **(iii)**.

To warm up, consider the truth table for $\mathbf{I}_C$ when defined the wrong way as in the paragraph above (by (2)):

p_i	$I_{C_1 \cup \{p_i\}}$	$I_{C_2 \cup \{\bar{p}_i\}}$	I_C
0	0	0	0
1	0	0	0
0	1	0	$\boxed{1}$
1	1	0	0
0	0	1	0
1	0	1	1
0	1	1	1
1	1	1	1

Note the two "problem" rows, where I_C goes from 1 to 0, despite the fact that p_i changes from 0 to 1. This is the only case where monotonicity is spoiled. But, using the values of $\mathbf{I}$ on the premises, we can make this problem vanish. The point is that we can turn the 1 in the box into a 0, while defining C' consistently. Show how/why. (Note also, that a symmetric answer is possible; turn the 0 below the boxed 1 into a 1.)

(ii) The cases for $x = q_i$ and $x = r_i$ are simpler than the case for p_i, so you might want to do this question before **(ii)**. (Note, if your answer is very complicated, something is not right.)

(iii) Now show that if C' is an α-clause, then $\alpha|_\sigma \models C'|_\sigma$, and same for β-clauses. This can be done with an inductive argument on the depth of the clause.

(iv) Define the interpolant for P to be $\mathbf{I}_{\{\}}$, i.e., the value of the interpolant at the final empty clause. Show, using **(iii)**, that it works correctly, it is feasible, and monotone.