Common Problems with Assignment 1
CS 2MJ3 Fall 2009

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I tried to mark Assignment 1 as leniently as I could, but it still didn't turn out very well, it appears (58% average). Not much we can do about that now, but there were a few common problems that appeared consistently on many of the assignments. Unless you got a mark above 90% on Assignment 1, chances are you made one of the following mistakes.

1 Subset and Set Membership Confusion

If A is a set, there's a huge difference between, "$x \in A$" and "$x \subseteq A$." The former notation means that x is one of the elements in the set, A, while the latter means that every element in the set, x, is also an element of the set, A.

A course–specific example I see is this sort of thing:

$$\varepsilon \subseteq L^* \text{ or } \{\varepsilon\} \in L^*$$

ε is a string, not a set (unless you're modeling strings as... you know what, never mind: ε isn't a set, period.) and so it contains no elements. Thus it isn't a subset of anything so it is incorrect to write, "$\varepsilon \subseteq L^*$". It does happen to be a string contained in L^*, however, so we can write "$\varepsilon \in L^*$"

L^* is a set containing only strings (not sets of strings) and $\{\varepsilon\}$ is a set containing one string. So unless braces ("{" and "}") are actually symbols in the alphabet (which would be awfully confusing!), it is not correct to write "$\{\varepsilon\} \in L^*$" but it would be fine to write "$\{\varepsilon\} \subseteq L^*$."

2 L^+ is not $L \setminus \{\varepsilon\}$

I don't know where people are getting this. I sure hope *I* didn't write or say anything to suggest this; you all have my sincere apologies if I was delirious enough to do so at any point. L^+ is most certainly not $L \setminus \{\varepsilon\}$. In fact, it isn't even $L^* \setminus \{\varepsilon\}$, UNLESS $\varepsilon \notin L$.

As it's been written many times,

$$L^0 = \{\varepsilon\}$$
$$L^{k+1} = L^k L \quad \text{for } k \geq 0$$
$$L^* = \bigcup_{i=0}^{\infty} L^i$$
$$L^+ = \bigcup_{i=1}^{\infty} L^i$$

For example,

$$\begin{aligned}
\{a,b\}^0 &= \{\varepsilon\} \\
\{a,b\}^1 &= \{a,b\} \\
\{a,b\}^2 &= \{aa, ab, ba, bb\} \\
\{a,b\}^3 &= \{aaa, aab, aba, abb, baa, bab, bba, bbb\} \\
&\vdots \\
\{a,b\}^+ &= \{a, b, aa, ab, ba, bb, aaa, aab, aba, \ldots\} \\
\{a,b\}^* &= \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, aba, \ldots\}
\end{aligned}$$

That example used only an alphabet, but the same applies to sets of strings:

$$\begin{aligned}
\{cat, dog\}^0 &= \{\varepsilon\} \\
\{cat, dog\}^1 &= \{cat, dog\} \\
\{cat, dog\}^2 &= \{catcat, catdog, dogcat, dogdog\} \\
\{cat, dog\}^3 &= \{catcatcat, catcatdog, catdogcat, catdogdog, \\
&\qquad dogcatcat, dogcatdog, dogdogcat, dogdogdog\} \\
&\vdots \\
\{cat, dog\}^+ &= \{cat, dog, catcat, catdog, dogcat, dogdog, \\
&\qquad catcatcat, catcatdog, catdogcat, catdogdog, \ldots\} \\
\{cat, dog\}^* &= \{\varepsilon, cat, dog, catcat, catdog, dogcat, dogdog, \\
&\qquad catcatcat, catcatdog, catdogcat, catdogdog, \ldots\}
\end{aligned}$$

Note that if $\varepsilon \in L$ then $L^+ = L^*$.

3 Using the Pumping Lemma to show that a language is not regular

The Pumping Lemma for regular languages says that every regular language has a certain property. Thus, if we can show that a given language does *not* have that property, it follows that it cannot be a regular language. By far the biggest problem people have with the pumping lemma is knowing which variables (among n, z, u, v, w and i) *can* and *cannot* be explicitly chosen.

At its root, this is really a problem with understanding universal and existential quantifiers and how to negate a proposition involving such quantifiers. There are two rules of negation involved here. Suppose P is a predicate. Then,

$$\neg(\forall x \quad P(x)) \Leftrightarrow (\exists x \quad \neg P(x))$$
$$\neg(\exists x \quad P(x)) \Leftrightarrow (\forall x \quad \neg P(x))$$

They look a bit cryptic, but they're actually quite intuitive. The first one says that if it's not the case that P is true of all things, then there must be some thing for which it's false. The second says if nothing exists for which P is true, then P must be false for all things.

Now the pumping lemma says that if L is a regular language then,

$$\exists n \geq 0 \ \forall z \in L \text{ s.t. } |z| > n \ \exists u, v, w \text{ s.t. } z = uvw, |uv| \leq n, v \neq \varepsilon \ \forall i \geq 0 \ uv^i w \in L$$

and as I mentioned before, we use this as a tool to prove that L is NOT regular by showing that this rather long and complicated proposition is false. If we want to show it's false, we must show its negation is true. So let's find its negation using those two rules for negating quantified predicates:

$$\neg\left(\exists n \geq 0 \ \forall z \in L \text{ s.t. } |z| > n \ \exists u, v, w \text{ s.t. } z = uvw, |uv| \leq n, v \neq \varepsilon \ \forall i \geq 0 \ uv^i w \in L\right)$$
$$\forall n \geq 0 \ \neg\left(\forall z \in L \text{ s.t. } |z| > n \ \exists u, v, w \text{ s.t. } z = uvw, |uv| \leq n, v \neq \varepsilon \ \forall i \geq 0 \ uv^i w \in L\right)$$
$$\forall n \geq 0 \ \exists z \in L \text{ s.t. } |z| > n \ \neg\left(\exists u, v, w \text{ s.t. } z = uvw, |uv| \leq n, v \neq \varepsilon \ \forall i \geq 0 \ uv^i w \in L\right)$$
$$\forall n \geq 0 \ \exists z \in L \text{ s.t. } |z| > n \ \forall u, v, w \text{ s.t. } z = uvw, |uv| \leq n, v \neq \varepsilon \ \neg\left(\forall i \geq 0 \ uv^i w \in L\right)$$
$$\forall n \geq 0 \ \exists z \in L \text{ s.t. } |z| > n \ \forall u, v, w \text{ s.t. } z = uvw, |uv| \leq n, v \neq \varepsilon \ \exists i \geq 0 \ \neg\left(uv^i w \in L\right)$$
$$\forall n \geq 0 \ \exists z \in L \text{ s.t. } |z| > n \ \forall u, v, w \text{ s.t. } z = uvw, |uv| \leq n, v \neq \varepsilon \ \exists i \geq 0 \ uv^i w \notin L$$

The last line is what we're actually trying to prove when we use the pumping lemma, so let's look at it up close. The first part ($\forall n \geq 0$) indicates that we need to show something is true for all $n \geq 0$. So we must begin our proof (that L is not regular) by saying, "**Let $\mathbf{n \geq 0}$**." This isn't a huge source of problems for most of you, but it's often forgotten.

Next we have, $\exists z \in L$ s.t. $|z| > n$. Here is where several of you ran into trouble. Many of you wrote things like, "let $z = 00011$" and tried to finish the proof that way. THIS WILL NOT WORK. It doesn't work because the negation of the lemma says there is a z that is *longer* than n which satisfies the rest of the statement, and this must be true of EVERY $n \geq 0$ (as we observed in the last paragraph). So choosing $z = 00011$ takes care of the cases, $n = 0, 1, 2, 3, 4$, but unless we have a z for all the other values of n, we haven't proved anything. The only way to handle this condition is by choosing a z that depends on the value of n. Any fixed–length z will be inadequate (because there will always be a longer one needed).

In my solution to 4(b), I (along with a few other students) chose $z = 0^{n+1}1^n$. In my solution to 4(a), I began with, let $n \geq 2$, which looks like it contradicts what I said about n in the beginning (but it doesn't). In fact, I could have chosen $n \geq 0$, as usual, but then my z would have been $1^{(n+2)^2}$ instead of 1^{n^2}.

Since z merely has to be sufficiently *long*, this is a reasonable shortcut to have taken for the sake of simpler expressions.

The next section deals with the decomposition of z (u, v, w). This is, perhaps, the greatest source of trouble for people. A substantial number of you pointed out that with a particular decomposition of z, you could pump the v portion and get a string outside the language, thus, "proving" the language is not regular. Unfortunately it proves no such thing. Why doesn't it? Well, look at that part of the statement: "$\forall u, v, w$ such that..."

It doesn't matter whether THERE EXIST strings u, v, w that can be pumped to form a string outside the language; what must be shown is that ANY decomposition (that satisfies the three constraints) can be pumped to form such a string. This is probably the most common problem with proofs that attempt to use the pumping lemma: they explicitly choose the string, v, to be pumped. You can't choose v; you must simply deduce the possible strings v could be given that $z = uvw, |uv| \leq n$, and $v \neq \varepsilon$. You must then show that in every one of theses cases, $\exists i \geq 0$ s.t. $uv^iw \notin L$.

To summarize:

1. "Let $n \geq 0$." Just write that. DO NOT CHOOSE a specific value for n.
2. CHOOSE a string, $z \in L$, which is guaranteed to be longer than n. This means, of course, that it must be based on n (e.g. $z = a^nb^n, z = 1^{n!}, z = 0^n1^{2n}$, etc.).
3. "Let u, v, w be strings s.t. $z = uvw, |uv| \leq n$, and $v \neq \varepsilon$." Just write that. DO NOT CHOOSE these strings yourself.
4. Use the awesome power of logic to figure out all the possible strings v could be, given what you know so far.
5. CHOOSE a number, $i \geq 0$ (obviously not 1) such that $uv^iw \notin L$.
6. In some cases, it will be painfully obvious that $uv^iw \notin L$, in some cases it will be easy to see after a very brief explanation, and in some cases showing that $uv^iw \notin L$ will be the hardest part of the whole proof.

As always, please mail me (jamesnd2@mcmaster.ca) if you have questions or comments about this.