Appendix B

Relations

In this section we present the basics of relations. Given two sets X, Y, $X \times Y$ denotes the set of (ordered) pairs $\{(x, y) | x \in X \wedge y \in Y\}$, and a *relation* R is just a subset of $X \times Y$, i.e., $R \subseteq X \times Y$. Thus, the elements of R are of the form (x, y) and we write $(x, y) \in R$ (we can also write xRy, Rxy or $R(x, y)$). In what follows we assume that we quantify over the set X and that $R \subseteq X \times X$; we say that

- R is *reflexive* if $\forall x$, $(x, x) \in R$,
- R is *symmetric* if $\forall x \forall y$, $(x, y) \in R$ if and only if $(y, x) \in R$,
- R is *antisymmetric* if $\forall x \forall y$, if $(x, y) \in R$ and $(y, x) \in R$ then $x = y$,
- R is *transitive* if $\forall x \forall y \forall z$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

Suppose that $R \subseteq X \times Y$ and $S \subseteq Y \times Z$. The *composition* of R and S is defined as follows:

$$R \circ S = \{(x, y) | \exists z, xRz \wedge zSy\}. \tag{B.1}$$

Let $R \subseteq X \times X$; we can define $R^n := R \circ R \circ \cdots \circ R$ recursively as follows:

$$R^0 = \mathrm{id}_X := \{(x, x) | x \in X\}, \tag{B.2}$$

and $R^{i+1} = R^i \circ R$. Note that there are two different equalities in (B.2); "$=$" is the usual equality, and "$:=$" is a definition.

Theorem B.1. *The following three are equivalent:*

(1) R *is transitive,*
(2) $R^2 \subseteq R$,
(3) $\forall n \geq 1$, $R^n \subseteq R$.

Problem B.2. *Prove theorem B.1.*

There are two standard ways of representing *finite* relations, that is, relations on $X \times Y$ where X and Y are finite sets. Let $X = \{a_1, \ldots, a_n\}$ and $Y = \{b_1, \ldots, b_m\}$, then we can represent a relation $R \subseteq X \times Y$:

(1) as a matrix $M_R = (m_{ij})$ where:

$$m_{ij} = \begin{cases} 1 & (a_i, b_j) \in R \\ 0 & (a_i, b_j) \notin R \end{cases},$$

(2) and as a directed graph $G_R = (V_R, E_R)$: $V_R = X \cup Y$ and ${}_{a_i}\bullet \longrightarrow \bullet_{b_j}$ is an edge in E_R iff $(a_i, b_j) \in R$.

B.1 Closure

Let P be a property[3] of relations, for example transitivity or symmetry. Let $R \subseteq X \times X$ be a relation, with or without the property P. The relation S given by

$$S := \bigcap_{Q \text{ has P, } R \subseteq Q} Q, \tag{B.3}$$

is called the *closure of R with respect to* P. Note that the closure may not exist, i.e., S could be $\emptyset$. Alternatively, S may be defined (equivalently) as the relation having the following three conditions:

(1) S has the property P,
(2) $R \subseteq S$, (B.4)
(3) $\forall Q \subseteq X \times X$, "$Q$ has P" and $R \subseteq Q$ implies that $S \subseteq Q$.

Problem B.3. *Prove that the two definitions are equivalent, that is, prove that S given by (B.3) and S given by (B.4) define the same relation.*

Theorem B.4. *For $R \subseteq X \times X$, $R \cup \mathrm{id}_X$ is the reflexive closure of R.*

Problem B.5. *Prove theorem B.4.*

Theorem B.6. *Given a relation $R \subseteq X \times Y$, the relation $R^{-1} \subseteq Y \times X$ is defined as $\{(x,y)|(y,x) \in R\}$. For $R \subseteq X \times X$, $R \cup R^{-1}$ is the symmetric closure of R.*

Problem B.7. *Prove theorem B.6.*

Theorem B.8. $R^+ := \bigcup_{i=1}^{\infty} R^i$ *is the transitive closure of R.*

[3] We have seen the concept of an abstract property in section 1.1.

Proof. We check that R^+ has the three conditions given in (B.4). First that R^+ itself has the property (i.e., it is transitive):

$$\begin{aligned} xR^+y \wedge yR^+z &\iff \exists m,n \geq 1, xR^m y \wedge yR^m z \\ &\Longrightarrow \exists m,n \geq 1, x(R^m \circ R^n)z \\ &\iff \exists m,n \geq 1, xR^{m+n}z \\ &\iff xR^+z \end{aligned}$$

so R^+ *is* transitive.

Second we check that $R \subseteq R^+$, but this follows by the definition of R^+.

We check now the last condition. Suppose S is transitive and $R \subseteq S$. Since S is transitive then $S^n \subseteq S$, for $n \geq 1$, i.e., $S^+ \subseteq S$, and since $R \subseteq S$, $R^+ \subseteq S^+$, so $R^+ \subseteq S$. □

Theorem B.9. $R^* = \bigcup_{i=0}^{\infty} R^i$ *is the reflexive and transitive closure of* R.

Proof. $R^* = R^+ \cup \mathrm{id}_X$. □

B.2 Equivalence relation

Let X be a set, and let I be an index set. The family of sets $\{A_i | i \in I\}$ is called a *partition* of X iff

(1) $\forall i$, $A_i \neq \emptyset$,
(2) $\forall i \neq j$, $A_i \cap A_j = \emptyset$,
(3) $X = \bigcup_{i \in I} A_i$.

Note that $X = \bigcup_{x \in X} \{x\}$ is the *finest* partition possible, i.e., the set of all singletons. A relation $R \subseteq X \times X$ is called an *equivalence relation* iff

(1) R is reflexive,
(2) R is symmetric,
(3) R is transitive.

For example, if x, y are strings over $\{0,1\}^*$, then the relation given by $R = \{(x,y) | \mathrm{length}(x) = \mathrm{length}(y)\}$ is an equivalence relation. Another example is $xRy \iff x = y$, i.e., the equality relation is the equivalence relation *par excellence*. Yet another example: $R = \{(a,b) | a \equiv b \pmod{m}\}$ is an equivalence relation (where "≡" is the congruence relation defined on page 114).

Theorem B.10. *Let $F : X \longrightarrow X$ be any* total *function (i.e., a function defined on all its inputs). Then the relation R on X defined as: $xRy \iff F(x) = F(y)$, is an equivalence relation.*

Problem B.11. *Prove theorem B.10.*

Let R be an equivalence relation on X. For every $x \in X$, the set $[x]_R = \{y|xRy\}$ is the *equivalence class of x with respect to R.*

Theorem B.12. *Let $R \subseteq X \times X$ be an equivalence relation. The following are equivalent:*

(1) aRb
(2) $[a] = [b]$
(3) $[a] \cap [b] \neq \emptyset$

Proof. (1) $\Rightarrow$ (2) Suppose that aRb, and let $c \in [a]$. Then aRc, so cRa (by symmetry). Since $cRa \wedge aRb$, cRb (transitivity), so bRc (symmetry), so $c \in [b]$. Hence $[a] \subseteq [b]$, and similarly $[b] \subseteq [a]$.

(2) $\Rightarrow$ (3) Obvious, since $[a]$ is non-empty as $a \in [a]$.

(3) $\Rightarrow$ (1) Let $c \in [a] \cap [b]$, so aRc and bRc, so by symmetry $aRc \wedge cRb$, so by transitivity aRb. □

Corollary B.13. *If R is an equivalence, then $(a, b) \notin R$ iff $[a] \cap [b] = \emptyset$.*

For every equivalence relation $R \subseteq X \times X$, let X/R denote the set of all equivalence classes of R.

Theorem B.14. *X/R is a partition of X.*

Proof. Given theorem B.12, the only thing that remains to be proven is that $X = \bigcup_{A \in X/R} A$. Since every $A = [a]$ for some $a \in X$, it follows that $\bigcup_{A \in X/R} A = \bigcup_{a \in X} [a] = X$. □

Let R_1, R_2 be equivalence relations. If $R_1 \subseteq R_2$, then we say that R_1 is a *refinement* of R_2.

Lemma B.15. *If $R_1 \subseteq R_2$, then for all $a \in X$, $[a]_{R_1} \subseteq [a]_{R_2}$.*

If X/R is finite then $\text{index}(R) := |X/R|$, i.e., the *index* of R (in X) is the size of X/R.

Theorem B.16. *If $R_1 \subseteq R_2$, then* $\text{index}(R_1) \geq \text{index}(R_2)$.

Problem B.17. *Prove theorem B.16.*