Name ______________________ Student No. ______________________

No aids allowed. Answer all questions on test paper. Use backs of sheets for scratch work.

Total Marks: 40

[10] 1. Define the predicate

$$\mathrm{GE}(x,y) = \begin{cases} 0 & \text{if } x < y \\ 1 & \text{if } x \geq y \end{cases}$$

Give a Register Machine program for computing $\mathrm{GE}(x,y)$. You may use the "copy macro" $R_i \leftarrow R_j$ as though it were an actual RM instruction, but otherwise use only correct RM instructions.

Make sure that you respect the input/output conventions.

Solution

c_0	goto 10 if $R_1 = R_2$	output 1 if $x = y$
c_1	$R_3 \leftarrow R_1$	
c_2	$R_4 \leftarrow R_2$	
c_3	$R_3 \leftarrow R_3 + 1$	increment x
c_4	$R_4 \leftarrow R_4 + 1$	increment y
c_5	goto 8 if $R_3 = R_2$	we know $x < y$
c_6	goto 10 if $R_4 = R_1$	we know $x > y$
c_7	goto 3 if $R_1 = R_1$	unconditional goto
c_8	$R_1 \leftarrow 0$	output 0
c_9	goto 12 if $R_1 = R_1$	unconditional goto
c_{10}	$R_1 \leftarrow 0$	
c_{11}	$R_1 \leftarrow R_1 + 1$	output 1
c_{12}		

[20] 2. In this exercise we are going to show that the function $\max(x, y)$ is primitive recursive, where

$$\max(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } x < y \end{cases}$$

Do this question by following the outline below:

(a) First show that the predecessor function $\text{pd}(x)$ is primitive recursive, where

$$\text{pd}(x) = \begin{cases} x - 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution: We define $\text{pd}(x)$ by primitive recursion as follows:

$$\text{pd}(0) = Z$$
$$\text{pd}(y + 1) = \pi_{2,1}(y, \text{pd}(y))$$

(b) Show that limited subtraction, $x \dot{-} y$, is primitive recursive:

$$x \dot{-} y = \begin{cases} x - y & \text{if } y \leq x \\ 0 & \text{if } y > x \end{cases}$$

Solution: We define $x \dot{-} y$ by primitive recursion as follows:

$$x \dot{-} 0 = \pi_{1,1}(x)$$
$$x \dot{-} (y + 1) = \pi_{3,3}(x, y, \text{pd}(x \dot{-} y))$$

(c) Show that $x + y$ is primitive recursive.

Solution: This is done in the course slides.

(d) Finally, show that $\max(x, y)$ is primitive recursive by expressing it in terms of limited subtraction and addition.

Solution: $\max(x, y) = (x \dot{-} y) + y$

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[10] 3. Show that not every total computable function is primitive recursive; use the diagonal argument presented in class.

Solution: Let $f_1, f_2, f_3, \ldots$ be the list of all pr functions, where f_i is the pr fn whose encoding is i.

We are only interested in unary fns, so if f_i has arity greater than one, we replace it by S (the unary successor function). Let the new list be $g_1, g_2, g_3, \ldots$, where $g_i = f_i$ if f_i was unary, and $g_i = S$ otherwise.

Let $U(x, y) = g_x(y)$, so U is a total computable fn. However, U is not pr; for suppose that it is. Then so is $D(x) = S(U(x, x))$. If U were pr, so would be D.

But if D is pr, then $D = g_e$ for some e. This gives us a contradiction, since $g_e(e) = D(e) = g_e(e) + 1$.