

putting f_{pm} are necessarily of exponential size (thus, removing the negation gate increases dramatically the circuit size necessary to compute f_{pm}). This also shows that giving an exponential lower bound for a monotone circuit family deciding a language in NP is not enough to show the separation of P and NP (see theorem 6.4.8).

One final observation is that perfect matching is not known to be complete for any natural complexity class.

7.1.2 Primality testing

We present the Rabin-Miller randomized algorithm for primality testing. Although a polytime (deterministic) algorithm for primality is now known (see [MA04]), randomized algorithms¹ are simpler and more efficient, and therefore still used in practice.

Algorithm 7.1.2 (Rabin-Miller)

On input p :

1. If p is even, accept if $p = 2$; otherwise, reject.
2. Select randomly a non-zero a in \mathbb{Z}_p .
3. Compute $a^{(p-1)} \pmod{p}$ and reject if $\neq 1$.
4. Let $(p-1) = st$ where s is odd and $t = 2^h$.
5. Compute the sequence $a^{s \cdot 2^0}, a^{s \cdot 2^1}, a^{s \cdot 2^2}, \dots, a^{s \cdot 2^h} \pmod{p}$.
6. If some element of this sequence is not 1, find the last element that is not 1, and reject if that element is not -1 .
7. Accept.

We say that a is a *witness* (of compositeness) for p if the algorithm rejects on a .

Theorem 7.1.3 *If p is a prime then the Rabin-Miller algorithm accepts it; if p is composite, then the algorithm rejects it with probability $\geq \frac{1}{2}$.*

PROOF: We show first that if p is prime, no witness exists, and so no branch of the algorithm rejects. If a were a stage 3 witness, $a^{(p-1)} \not\equiv 1 \pmod{p}$ then Fermat's little theorem would imply that p is composite. If a were a stage 6 witness, some b exists in \mathbb{Z}_p , where $b \not\equiv \pm 1 \pmod{p}$ and $b^2 = 1$

¹In fact it was the randomized test for primality that stirred interest in randomized computation in the late 1970's. Historically, the first randomized algorithm for primality was given by [SS77]; a nice exposition of this algorithm can be found in [Pap94, §11.1].

(mod p). Therefore, $(b^2 - 1) = 0 \pmod{p}$. Factoring, we obtain that

$$(b - 1)(b + 1) = 0 \pmod{p}$$

which implies that p divides $(b - 1)(b + 1)$. But because $b \not\equiv \pm 1 \pmod{p}$, both $(b - 1)$ and $(b + 1)$ are strictly between 0 and p . But that contradicts that $p \mid (b - 1)(b + 1)$, because p is a prime, so to divide the RHS it has to be a factor of the RHS, but both numbers are smaller than it.

We now show that if p is an odd composite number and a non-zero a is selected randomly in \mathbb{Z}_p , then $\Pr[a \text{ is a witness}] \geq \frac{1}{2}$.

For every nonwitness, the sequence computed in stage 5 is either all 1s or contains a -1 at some position, followed by 1s. So, for example, 1 is a nonwitness of the first kind and -1 is a nonwitness of the second kind because s is odd and $(-1)^{s \cdot 2^0} = -1 \pmod{p}$ and $(-1)^{s \cdot 2^1} = 1 \pmod{p}$.

Among all nonwitnesses of the second kind, find a nonwitness for which the -1 appears in the largest position in the sequence. Let x be such a nonwitness and let j be the position of -1 in its sequence, where the positions are numbered starting at 0. Hence $x^{s \cdot 2^j} = -1 \pmod{p}$ (and $x^{s \cdot 2^{j+1}} = 1 \pmod{p}$).

Because p is composite, either p is the power of a prime or we can write p as the product of q and r , two numbers that are co-prime. This yields two cases.

Case 1. Suppose that $p = q^e$ where q is prime and $e > 1$. Let $t = 1 + q^{e-1}$. From the binomial expansion of t^p we obtain:

$$t^p = (1 + q^{e-1})^p = 1 + pq^{e-1} + \sum_{l=2}^p \binom{p}{l} (q^{e-1})^l \quad (7.1)$$

which is congruent to 1 (mod p). Hence t is a stage 3 witness because, if $t^{p-1} = 1 \pmod{p}$, then $t^p = t \pmod{p}$, which from the observation about (7.1) is not possible. We use this one witness to get many others. If d is a (stage 3) nonwitness, we have $d^{p-1} = 1 \pmod{p}$, but then $dt \pmod{p}$ is a witness. Moreover, if d_1, d_2 are distinct nonwitnesses, then $d_1 t \not\equiv d_2 t \pmod{p}$. Otherwise,

$$d_1 = d_1 \cdot t \cdot t^{p-1} = d_2 \cdot t \cdot t^{p-1} = d_2 \pmod{p}.$$

Thus the number of (stage 3) witnesses must be at least as large as the number of nonwitnesses.

Case 2. By the CRT there exists $t \in \mathbb{Z}_p$ such that

$$\begin{aligned} t &\equiv x \pmod{q} & \Rightarrow & t^{s \cdot 2^j} \equiv -1 \pmod{q} \\ t &\equiv 1 \pmod{r} & & t^{s \cdot 2^j} \equiv 1 \pmod{r} \end{aligned}$$

Hence t is a witness because $t^{s \cdot 2^j} \neq \pm 1 \pmod{p}$ (see footnote ²) but $t^{s \cdot 2^{j+1}} = 1 \pmod{p}$. Now that we have one witness, we can get many more, and show that $dt \pmod{p}$ is a unique witness for each nonwitness d by making two observations.

First, $d^{s \cdot 2^j} = \pm 1 \pmod{p}$ and $d^{s \cdot 2^{j+1}} = 1 \pmod{p}$ owing to the way that j was chosen. Therefore $dt \pmod{p}$ is a witness because $(dt)^{s \cdot 2^j} \neq \pm 1 \pmod{p}$ and $(dt)^{s \cdot 2^{j+1}} = 1 \pmod{p}$.

Second, if d_1 and d_2 are distinct nonwitnesses, $d_1 t \neq d_2 t \pmod{p}$. The reason is that $t^{s \cdot 2^{j+1}} = 1 \pmod{p}$. Hence $t \cdot t^{s \cdot 2^{j+1} - 1} = 1 \pmod{p}$. Therefore, if $d_1 t = d_2 t \pmod{p}$, then

$$d_1 = d_1 t \cdot t^{s \cdot 2^{j+1} - 1} = d_2 t \cdot t^{s \cdot 2^{j+1} - 1} = d_2 \pmod{p}$$

Thus in case 2., as well, the number of witnesses must be at least as large as the number of nonwitnesses. \square

Note that by running the algorithm k times on independently chosen a , we can make sure that it rejects a composite with probability $\geq (1 - \frac{1}{2^k})$ (it will always accept a prime with probability 1). Thus, for $k = 100$ the probability of error, i.e., of a false positive, is negligible.

Thus we have a Monte Carlo algorithm for composites, and therefore $\text{PRIMES} = \{(n)_b | n \text{ is prime}\} \in \text{co-RP}$. Here $(n)_b$ denotes the binary encoding of the number n ; see section 7.2 for a definition of co-RP.

7.1.3 Pattern matching

In this section we design a randomized algorithm for pattern matching. Consider the set of strings over $\{0, 1\}$, and let $M : \{0, 1\} \rightarrow M_{2 \times 2}(\mathbb{Z})$, that is, M is a map from strings to 2×2 matrices over the integers (\mathbb{Z}) defined as follows:

$$\begin{aligned} M(\varepsilon) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ M(0) &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ M(1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

²To see why $t^{s \cdot 2^j} \neq \pm 1 \pmod{p}$ observe the following: suppose that $a = -1 \pmod{q}$ and $a = 1 \pmod{r}$, where $\gcd(q, r) = 1$. Suppose that $p = qr | (a + 1)$, then $q | (a + 1)$ and $r | (a + 1)$, and since $r | (a - 1)$ as well, it follows that $r | [(a + 1) - (a - 1)]$, so $r | 2$, so $r = 2$, so p must be even, which is not possible since we deal with even p 's in line 1 of the algorithm.