Games on Posets

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Based on the paper "On the Complexity of Computing Winning Strategies for Finite Poset Games" by Michael Soltys and Craig Wilson

Outline

Context of Poset Games

- Simple to describe
- Computing winning strategies appears intractable in all but simplest cases
- Proof of 1st player having a winning strategy does not immediately yield a feasible algorithm for computing the strategy

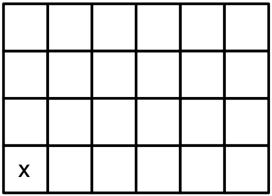
Partially Ordered Sets (Posets)

- Set U together with ordering relation $\preceq$
- $\preceq$ satisfies following properties:
  - **Anti-Symmetry:** If $a \preceq b$, then $b \npreceq a$
  - **Transitivity:** If $a \preceq b$ and $b \preceq c$, then $a \preceq c$
- If $a \npreceq b$ and $b \npreceq a$, then $a || b$ (incomparable)

(Finite) Poset Games [1]

- Games between 2 players
- From finite Poset $(U, \preceq)$, Poset Game $(A, \preceq)$ is played as follows:
  1. Initialize $A = U$
  2. Select an $x \in A$, remove all $y \in A$ such that $x \preceq y$
  3. Game ends when $A = \emptyset$, player unable to select an element loses
- A **round** is a sequence of two consecutive moves (first player, then second)

Chomp[2]



- Special case of poset game.
- Ordering relation can be thought of as a chocolate bar.
- Don't eat the poisoned square!

More Formally...

- Set of pairs ("cells") $\{(i,j)\,|1 \leq i \leq n, 1 \leq j \leq m\}$
- Select pair (i_0, j_0)
- Remove all (i,j) such that $i \geq i_0$ and $j \geq j_0$
- Player left with only $(1,1)$ loses

Chomp Configurations

- Represent as binary strings $\mathbb{X}$:

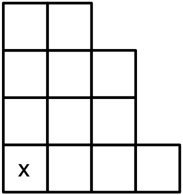


Figure: Chomp configuration with $\mathbb{X} = 00101101$

- 1s delimit rows, 0s indicate difference in rows lengths.

Translating Chomp to Geography

- Simple, direct polynomial-time translation from Poset Games to Geography, which shows Poset Games $\in$ PSPACE
- Translation from restricted form of Geography to Chomp also attempted
- Limitations of poset games revealed - not PSPACE-complete?

Mathematically...

GeneralizedGeography(GG):

$GG = \{\langle G, s\rangle |$ Player 1 has a winning strategy for the Generalized Geography game played on graph G starting at node $s\}$

Graph Construction: Part 1

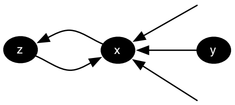


Figure: Base Construction of a poset game in Geography

Complications...

- Once a node $x \in G$ has been visited, must not be able to visit any y such that $x \preceq y$
- Need system of checks and balances

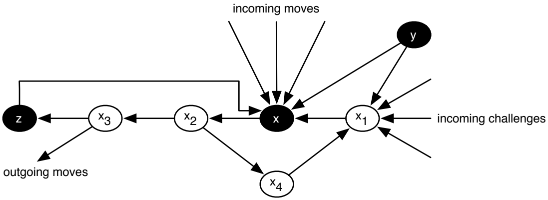


Figure: 5-node gadget

Gadget Construction

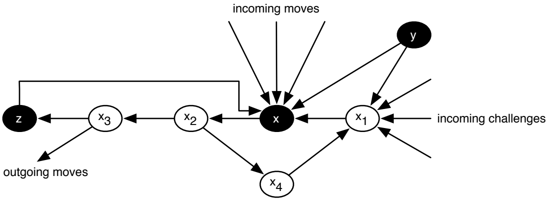


Figure: Construction of the 5-node gadget

Preventing Illegal Moves

- Additional nodes prevent players from making illegal moves.
- If player moves to node that "shouldn't exist", they will lose.
- Challenges to legal moves also result in a loss.

Example: Challenging an Illegal Move

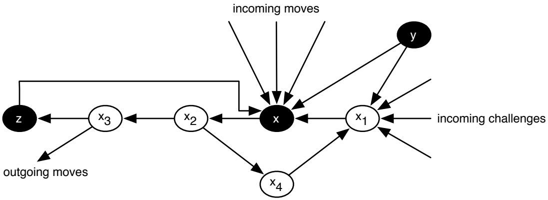


Figure: Challenging an illegal move

Chomp and Proof Complexity

- Second proof of Chomp $\in$ PSPACE
- Use theorems of $\mathbf{W}_1^1$ to show the existence of a winning strategy $\in$ PSPACE
- Introduce **segments** $\mathbb{X}^{[i]}$ of a configuration string $\mathbb{X}$:

$$\mathbb{X} = \overbrace{01}^{\mathbb{X}^{[1]}}\overbrace{10}^{\mathbb{X}^{[2]}}\overbrace{11}^{\mathbb{X}^{[3]}}\overbrace{00}^{\mathbb{X}^{[4]}}$$

Proof Complexity

- Main idea:

$$\Gamma_C \vdash \forall x \exists y \alpha(x, y)$$
$$\Updownarrow$$
$$\exists f \in C \text{ such that } \alpha(x, f(x))$$

- Different bounded arithmetic theories capture different classes

Variables of a Different "Sort"

- Three types of variables called **sorts**:
  1. Natural numbers $(x, y, z, \ldots)$
  2. Strings $(X, Y, Z, \ldots)$
  3. Sets of strings $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots)$
- Concerned with formulas from class $\Sigma_1^{\mathcal{B}}$:

$$(\exists \mathcal{X})(\forall Y)(\exists Z)\ldots(\forall y)\, A(\mathcal{X}, Y, Z, \ldots, y)$$

Description of $\mathbf{W}_1^1$ [3]

- Third-sorted theory for reasoning in PSPACE
- Symbols taken from set $\mathcal{L}_A^3 = \{0, 1, +, \times, ||_2, \in_2, \in_3, \leq, =\}$
- Axioms [3]:

B1. $x + 1 \neq 0$	**B7.** $(x \leq y \wedge y \leq x) \rightarrow x = y$
B2. $(x + 1 = y + 1) \rightarrow x = y$	**B8.** $x \leq x + y$
B3. $x + 0 = x$	**B9.** $0 \leq x$
B4. $x + (y + 1) = (x + y) + 1$	**B10.** $x \leq y \vee y \leq x$
B5. $x \times 0 = 0$	**B11.** $x \leq y \leftrightarrow x < y + 1$
B6. $x \times (y + 1) = (x \times y) + x$	**B12.** $x \neq 0 \rightarrow \exists y \leq x(y + 1 = x)$
L1. $X(y) \rightarrow y < \lvert X \rvert$	**L2.** $y + 1 = \lvert X \rvert \rightarrow X(y)$
SE. $[\lvert X \rvert = \lvert Y \rvert \wedge \forall i < \lvert X \rvert (X(i) \leftrightarrow Y(i))] \rightarrow X = Y$	

Achieving the Goal - Step 1

- Want to give formula $\Phi(\mathbb{X}, n, m)$ which decides whether $\mathbb{X}$ is valid Chomp game of size $n \times m$
- $\mathbb{X}$ is string of length $(n \times m)(n + m)$, with $(n \times m)$ segments
- Φ is conjunction of three formulas: $\phi_{\text{init}}, \phi_{\text{final}}$, and ϕ_{move}

Formulas ϕ_{init} and ϕ_{final}

- ϕ_{init} asserts $\mathbb{X}^{[1]}$ to be the initial configuration of the game:

$$\phi_{\text{init}}(\mathbb{X}^{[1]}, n, m) = \forall i \leq (n+m)((i > m) \rightarrow X^{[1]}(i) = 1)$$

- ϕ_{final} asserts $\mathbb{X}^{[n\times m]}$ to be the final configuration of the game:

$$\phi_{\text{final}}(\mathbb{X}^{[n\times m]}, n, m) = \forall i \leq (n+m)((i > n) \rightarrow X^{[n\times m]}(i) = 0)$$

The ϕ_{yields} formula

- ϕ_{yields} asserts that each segment of X can be obtained from one legal move on the previous one:

$$\phi_{\text{move}}(\mathbb{X}, n, m) = \forall i < (n \times m)(X^{[i]} \text{ “yields” } X^{[i+1]})$$

- Defining “yields” is where the fun begins!
- Attempt to discern what square(s) could have been played between $\mathbb{X}^{[i]}$ and $\mathbb{X}^{[i+1]}$
- Ensure differences between configs correspond to played square

Complete Yields Formula

$$(\exists j \leq NumOnes)(\exists k \leq NumZeros)[\mathrm{F}_0(1, k, \mathbb{X}) < \mathrm{F}_1(1, j, \mathbb{X})$$
$$\wedge(\exists p < |\mathbb{X}^{[i]}|)(\exists q \leq |\mathbb{X}^{[i]}|)(p = \mathrm{F}_0(1, k-1, \mathbb{X}^{[i]}) + 1 \wedge q = \mathrm{F}_1(p, j, \mathbb{X}^{[i]}))$$
$$\wedge(\forall r \leq |\mathbb{X}|)[(r < p \vee r > q \rightarrow \mathbb{X}^{[i+1]}(r) = \mathbb{X}^{[i]}(r))$$
$$\wedge(r < p + j \rightarrow \mathbb{X}^{[i+1]}(r) = 1)$$
$$\wedge(r \geq p + j \rightarrow \mathbb{X}^{[i+1]}(r) = 0)]]$$

Where

$$NumOnes = |\mathbb{X}^{[i]}| - numones(1, \mathbb{X}^{[i]})$$
$$NumZeroes = numones(1, \mathbb{X}^{[i]})$$

Achieving the Goal, Step 2

- Strategy function $\mathbb{S}$ from configurations to configurations
- Formula for either player having winning strategy:

$$\forall C \exists \mathbb{S} \left[\mathrm{Win}_{P_1}(\mathbb{S}, C) \vee \mathrm{Win}_{P_2}(\mathbb{S}, C) \right]$$

  where $\mathrm{Win}_{P_1}(\mathbb{S}, C)$ is a $\Sigma_1^{\mathcal{B}}$ formula
- Manipulate this to get formula for first player having a winning strategy:

$$(C = 0^m 1^n) \rightarrow \exists \mathbb{S} \left[\mathrm{Win}_{P_1}(\mathbb{S}, C) \right]$$

Applying the Witnessing Theorem

With

$$\mathbf{W}_1^1 \vdash (C = 0^m 1^n) \rightarrow \exists \mathbb{S}\, [\mathrm{Win}_{P_1}(\mathbb{S}, C)]$$

we have that the function for computing the winning strategy is in PSPACE.

Concluding Remarks

Is the problem of computing winning strategies for Poset Games PSPACE-complete?

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