Software Engineering 4I03 Michael Soltys

DAY CLASS
DURATION OF EXAMINATION: 3 Hours
MCMASTER UNIVERSITY FINAL EXAMINATION December 2005

THIS EXAMINATION PAPER INCLUDES 4 PAGES AND 10 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

No aids allowed.

All questions worth 10 marks, for a total of 100.

1. Let x, y be strings, and let L be any language. We say that x and y are *distinguishable* by L if some string z exists such that exactly one of xz and yz is in L. Otherwise, x, y are *indistinguishable* ($x \in L \iff y \in L$). If x, y are indistinguishable by L we write $x \equiv_L y$. Show that $\equiv_L$ is an equivalence relation (i.e., show that it is reflexive, symmetric, and transitive).

2. In this question (which is a continuation of the previous question) you will prove the **Myhill-Nerode theorem**. Let L be a language, and let X be a set of strings; we say that X is *pairwise distinguishable* by L if every pair of strings in X are distinguishable by L.

   Define the *index* of L to be the maximum number of elements in any set that is pairwise distinguishable by L. The index of L may be finite or infinite.

   (a) Show that if L is recognized by a DFA with k states, then the index of L is at most k.

   (b) Show that if the index of L is a finite number k, then L is recognized by a DFA with k states.

   (c) Conclude that L is regular iff it has finite index.

3. Remember that a string x is a palindrome if $x = x^R$ (assume $\Sigma = \{0, 1\}$). Let L_{pal} be the language of palindromes. Show that L_{pal} is not regular. Then show that it is a context-free languages. Finally, consider L_{eqpal}, the language of palindromes with an equal number of 0s and 1s. Show that L_{eqpal} is not a context-free language.

4. Let G be a context-free grammar in Chomsky normal form that contains b variables (that is, $|V| = b$). Show that if G generates some string with a derivation having at least 2^b many steps, then $L(G)$ is infinite.

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5. Let INF $= \{\langle M\rangle |\ L(M)$ is infinite $\}$. Show that INF is not Turing-recognizable.

6. Let A and B be two disjoint languages over some alphabet Σ, i.e. $A, B \subseteq \Sigma^*$ and $A \cap B = \emptyset$. We say that a language C *separates* A and B if $A \subseteq C$ and $B \subseteq \overline{C}$, where $\overline{C}$ = complement of $C = \Sigma^* - C$. Show that any two disjoint co-Turing-recognizable languages A and B are separable by some decidable language C, by following the outline below.

   (a) Show that $\overline{A} \cup \overline{B} = \Sigma^*$, and that there are Turing machines M_1 and M_2 such that $\overline{A} = L(M_1)$ and $\overline{B} = L(M_2)$.

   (b) Let M be a Turing machine defined as follows: on input $x \in \Sigma^*$ M simulates M_1 and M_2 in parallel, and if M_1 is the first to accept, then M halts and accepts, and if M_2 is the first to accept, then M halts and rejects. Show that $L(M)$ is a decidable language.

   (c) Let $C = \overline{L(M)}$, and show that C is a decidable language that separates A and B.

7. Define $L_{\text{SQ}} = \{\langle M\rangle |\ M$ accepts $\langle M\rangle$ within n^2 steps, $n = |\langle M\rangle|$ = length of the encoding of $M\}$.

   (a) Show that L_{SQ} is decidable.

   (b) Let M_{SQ} be a TM that decides L_{SQ}. Show that there is at least one value of n and at least one input w of length n such that M_{SQ} on input w does not halt within n^2 steps.

8. Suppose that $\mathbf{P} = \mathbf{NP}$. Give a polytime algorithm that given a boolean formula α, it first tests for satisfiability, and if α is satisfiable, it outputs a satisfying assignment of α. (**Hint:** if $\mathbf{P} = \mathbf{NP}$, then testing for satisfiability is in $\mathbf{P}$; query the testing algorithm repeatedly to compute the satisfying assignment.)

9. Call a regular expression *star-free* if it does not contain any star operations. Let EQ_{SFREX} be the language of pairs of star-free regular expressions $\langle R, S\rangle$ such that $L(R) = L(S)$. Show that EQ_{SFREX} is in **coNP**.

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10. Each of the following decision problems either is in **P** or it is **NP**-complete. State which is the case, and justify your answer.

To justify "in **P**" outline a polytime algorithm.

To justify "**NP**-complete" state which of the standard **NP**-complete problems (listed on the last page of the exam) can be reduced to the given problem.

(a) **EXACT-4-SAT**
Instance: A set of clauses such that each clause has *exactly* 4 literals.
Question: Is there a truth assignment that satisfies all of the given clauses?

(b) **PERFECT-FIFTH-DEGREE**
Instance: A positive integer N presented in binary notation.
Question: Is there an integer k such that $N = k^5$?

(c) **DEGREE-2-HC**
Instance: An undirected graph $G = (V, E)$ such that every node v in V has degree 2. (The degree of a node is the number of edges touching it.)
Question: Does G have a Hamiltonian cycle? (A Hamiltonian cycle is a simple cycle that contains every vertex.)

(d) **DOUBLE-PARTITION**
Instance: $w_1, \ldots, w_d$ where $d, w_i \in \mathbb{N}$ are presented in binary.
Question: Is there a subset $S \subseteq \{1, \ldots, d\}$ such that $\sum_{i \in S} w_i = 2 \sum_{j \notin S} w_j$?

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List of **NP**-complete problems
(to be used for question 10)

EXACT-3-SAT
Instance: A set of clauses where each clause has exactly 3 literals.
Question: Is there a truth assignment that satisfies all the given clauses?

k-**COL** (for $k = 3, 4, 5, \ldots$)
Instance: An undirected graph $G = (V, E)$.
Question: Is there a proper k-colouring of G, i.e., an assignment of colours to the vertices of G such that no two vertices connected by an edge get the same colour and no more than 3 colours are used?

HamCycle
Instance: An undirected graph $G = (V, E)$.
Question: Does G have a Hamiltonian cycle, i.e., a simple cycle that contains every vertex in V?

IS
Instance: An undirected graph $G = (V, E)$ and a nonnegative integer k (presented in binary notation.)
Question: Does G have an independent set of size at least k, i.e., a subset $V' \subseteq V$ of size at least k such that no two vertices in V' are connected by an edge in G?

Partition
Instance: A sequence of positive integers $\langle d, w_1, ..., w_d \rangle$ (presented in binary notation).
Question: Is there a subset $S \subseteq \{1, ..., d\}$ such that $\sum_{i \in S} w_i = \sum_{j \notin S} w_j$?

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