Linear Optimization - Tutorial 3

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Week of September 27, 2020

Introduction to Simplex Method

$$
\begin{aligned}
\text{max} \quad & z = 3x_1 + 2x_2 + 5x_3 \\
\text{subject to} \quad & 2x_1 + x_2 \leq 4 \\
& x_3 \leq 5 \\
& x_j \geq 0 \text{ for } j = 1 \text{ to } 3
\end{aligned}
$$

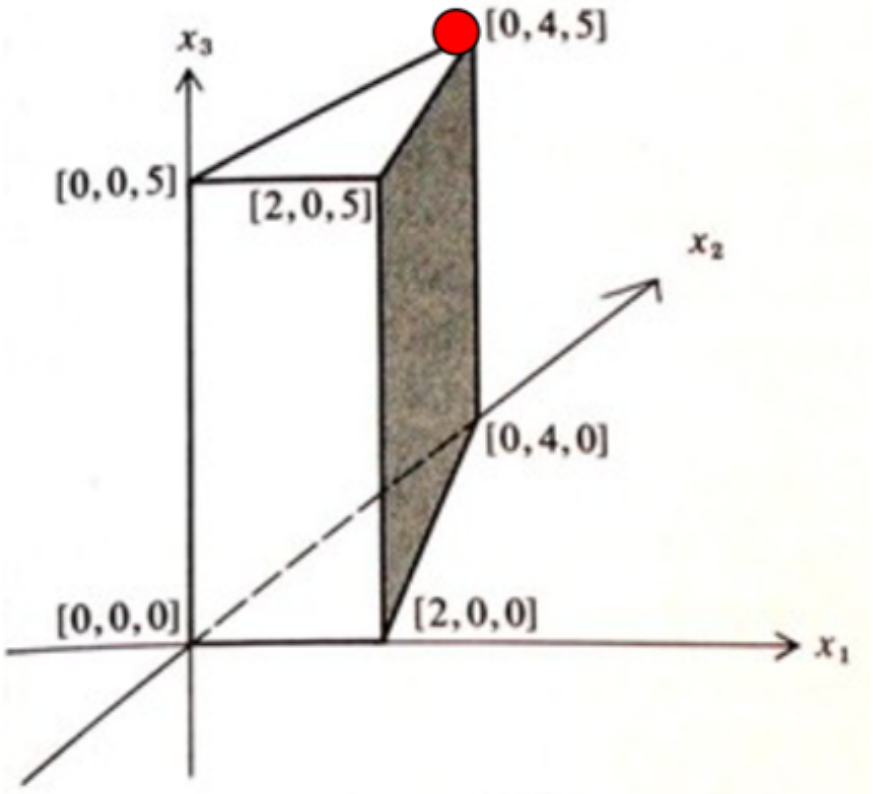
Polyhedron

The corresponding polyhedron has the following vertices:

Vertex	z
$(0, 0, 0)$	0
$(2, 0, 0)$	6
$(0, 0, 5)$	25
$(2, 0, 5)$	31
$(0, 4, 0)$	8
$(0, 4, 5)$	33

Optimal solution is $(0, 4, 5)$ and optimal value $z = 33$.

Polyhedron



(a) Polyhedron

We can start at the vertex $(0, 0, 0)$, then to travel to $(0, 0, 5)$ and finally at $(0, 4, 5)$ the maximum is achieved.

Remarks

- During the lectures You will learn all details about Simplex Method. Some of them in the previous example are omitted. It does not represent the entire Simplex Algorithm.
- Pivoting operations represents traveling along the polyhedron edges.
- Initial basic feasible solution is not always evident. To find the initial solution, we need the Phase I procedure.

Phase I

- Goal: To find an initial basic feasible solution(vertex) to start simplex algorithm
- The constraints of a linear programming problem can be expressed in the form

$$\text{s.t. } Ax = b$$
$$\text{and } x \geq 0$$

Phase I

In order to find a solution(vertex) to start simplex algorithm, consider the artificial minimization problem

$$\begin{gathered}\text{Min } \sum_{i=1}^{m} v_i \text{ or Max } \sum_{i=1}^{m} -v_i \\ \text{s.t. } Ax + v = b \\ \text{and } x \geq 0,\ v \geq 0\end{gathered}$$

If there is a feasible solution for this problem, it must be $v = 0$. If the original problem has no feasible solution, then $\sum_{i=1}^{m} v_i$ is greater than zero.

Example

Hamilton Bank makes four kinds of loans to its customers and these loans yield the following annual interest rates to the bank:
First mortgage 14%
Second mortgage 20%
Home improvement 20%
Personal overdraft 10%
The bank has a maximum foreseeable lending capability of $250 million and is further constrained by the policies: first mortgages must be at least 55% of all mortgages issued and at least 25% of all loans issued (in $ terms) second mortgages cannot exceed 25% of all loans issued (in $ terms) to avoid public displeasure and the introduction of a new windfall tax the average interest rate on all loans must not exceed 15%.

Example

Formulate the bank's loan problem as an LP so as to maximise interest income whilst satisfying the policy limitations.
Note here that these policy conditions, whilst potentially limiting the profit that the bank can make, also limit its exposure to risk in a particular area. It is a fundamental principle of risk reduction that risk is reduced by spreading money (appropriately) across different areas.

Next Week

- Simplex Method exercise.