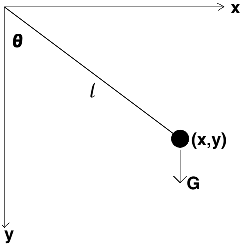
Conversion methods for improving structural analysis of differential-algebraic equation systems

Guangning Tan

PhD defense, 3 August 2016

Simple pendulum

ODE formulation



$$\theta'' = -\frac{G}{\ell}\sin\theta$$

- Need initial values (IVs) θ, θ'
- 2 degrees of freedom (DOF)

DAE formulation

$$\begin{aligned} 0 &= f_1 = x'' + x\lambda \\ 0 &= f_2 = y'' + y\lambda - G \\ 0 &= f_3 = x^2 + y^2 - \ell^2 \qquad\qquad 0 = f_3' = 2(xx' + yy') \end{aligned}$$

- IVs (x, y) should satisfy f_3; then (x', y') satisfy f_3'

Structural analysis (SA) of DAEs

The Σ-method by John Pryce

- determines
  - index
  - DOF
  - constraints
  - variables/derivatives that need consistent IVs
- equivalent to Pantelides's algorithm
- succeeds on many practical DAE problems of interest

$$
\begin{aligned}
0 &= f_1 = x'' + x\lambda \\
0 &= f_2 = y'' + y\lambda - G \\
0 &= f_3 = x^2 + y^2 - \ell^2
\end{aligned}
$$

$$
\Sigma = \begin{array}{c|ccc|c}
 & x & y & \lambda & c_i \\
f_1 & 2 & & 0^\bullet & 0 \\
f_2 & & 2^\bullet & 0 & 0 \\
f_3 & 0^\bullet & 0 & & 2 \\
d_j & 2 & 2 & 0 &
\end{array}
$$

Structural analysis says

- solve f_3 for x, y,
- solve f_3' for x', y', and
- solve with nonsingular Jacobian linear systems

$$
\mathbf{J} = \begin{array}{c|ccc|}
 & x'' & y'' & \lambda \\
f_1 & 1 & & x \\
f_2 & & 1 & y \\
f_3'' & 2x & 2y &
\end{array}
$$

$$
\left(f_1^{(k)}, f_2^{(k)}, f_3^{(2+k)}\right) \quad \text{for} \quad \left(x^{(2+k)}, y^{(2+k)}, \lambda^{(k)}\right)
$$

counting up from $k = 0$

Limitations of structural analysis

- Structural analysis can fail on simple, solvable DAEs with singular Jacobian matrices

  "In such cases the algorithm terminates without detecting all the equation subsets which ought to be differentiated." [Pantelides, 1988]

- Even in success case, it may overestimate index at arbitrarily high number

  "Pantelides' algorithm applied to DAEs of index 1 may perform an arbitrarily high number of iterations and differentiations." [Reissig, Martinson, Barton, 2000]

Motivation

- I want to dissolve DAE theorists' concerns and doubt toward structural analysis
- Make this useful tool more reliable

This is not a trivial task

- *"Wrong structural analysis. ... They pose another interesting and difficult task, more in computer science than in numerical analysis. ... What are the principles and methods involved, and can they be at least partially automated?"*

[Nedialkov & Pryce, 2007]

Contributions

"A DAE should not be formulated to exhibit more degrees of freedom than the underlying problem has."

- Give insight into structural analysis's failures that were not well understood before
- Develop two conversion methods that systematically reformulate DAEs on which structural analysis fails into equivalent success cases
- Show how to improve efficiency by exploiting block triangular form of a DAE
- Automate a conversion process in a computer algebra system

Modified pendulum I

- Multiply pendulum by nonsingular $\begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ 1 & & 1 \end{bmatrix}$

$$
\begin{aligned}
0 = f_1 &= (x'' + x\lambda) + (y'' + y\lambda - G) \\
0 = f_2 &= \phantom{(x'' + x\lambda) + {}} (y'' + y\lambda - G) + (x^2 + y^2 - \ell^2) \\
0 = f_3 &= (x'' + x\lambda) \phantom{{} + (y'' + y\lambda - G)} + (x^2 + y^2 - \ell^2)
\end{aligned}
$$

$$
\Sigma = \begin{array}{c|ccc|c}
 & x & y & \lambda & c_i \\
f_1 & 2^\bullet & 2 & 0 & 0 \\
f_2 & \boxed{0} & 2^\bullet & 0 & 0 \\
f_3 & 2 & \boxed{0} & 0^\bullet & 0 \\
d_j & 2 & 2 & 0 &
\end{array}
\qquad
\mathbf{J} = \begin{array}{c|ccc|}
 & x'' & y'' & \lambda \\
f_1 & 1 & 1 & x + y \\
f_2 & & 1 & y \\
f_3 & 1 & & x
\end{array}
$$

$$
d_j - c_i > \boxed{\sigma_{ij}} \geq 0 \qquad\qquad \det \mathbf{J} \equiv 0
$$

- $\operatorname{Val}\Sigma = 4 > 2 =$ DOF of pendulum
- SA says "give arbitrary x, x', y, y' to start integration", wrong!

Conversion method I: linear combination (LC)

Replace an f_i by a combination of equations

$$\mathbf{J} = \frac{\partial(f_1, f_2, f_3)}{\partial(x'', y'', \lambda)} = \begin{array}{c|ccc} & x'' & y'' & \lambda \\ \hline f_1 & 1 & 1 & x+y \\ f_2 & & 1 & y \\ f_3 & 1 & & x \end{array}$$

- Find $u = [-1, 1, 1]^T \in \operatorname{coker} \mathbf{J}$
- u does not depend on x'', y'', λ, hence nor does
  $\overline{f} \leftarrow u_1 f_1 + u_2 f_2 + u_3 f_3 = 2(x^2 + y^2 - \ell^2)$
- $\partial \overline{f} / \partial(x'', y'', \lambda) = u^T \mathbf{J} = \mathbf{0}^T$
- Replace an f_i by $\overline{f} \Rightarrow$ decrease $\operatorname{Val} \Sigma$

$$\begin{aligned} \text{If we replace} \quad f_1 &= (x'' + x\lambda) + (y'' + y\lambda - G) \\ \text{by} \quad \overline{f} &= 2(x^2 + y^2 - \ell^2) \end{aligned}$$

$$\overline{\Sigma} = \begin{array}{c|ccc|c} & x & y & \lambda & c_i \\ \hline \overline{f} & 0^\bullet & 0 & & 2 \\ f_2 & 0 & 2^\bullet & 0 & 0 \\ f_3 & 2 & 0 & 0^\bullet & 0 \\ \hline d_j & 2 & 2 & 0 & \end{array} \qquad \overline{\mathbf{J}} = \begin{array}{c|ccc|} & x'' & y'' & \lambda \\ \hline \overline{f}'' & 4x & 4y & \\ f_2 & & 1 & y \\ f_3 & 1 & & x \\ \hline \end{array}$$

- $\operatorname{Val}\overline{\Sigma} = 2$, $\det \overline{\mathbf{J}} = 4\ell^2 \neq 0 \Rightarrow$ success
- $u_1 = -1 \Rightarrow$ we can recover $f_1 = \dfrac{1}{u_1}\left(\overline{f} - u_2 f_2 - u_3 f_3\right)$
- Now structural analysis says
  "to start integration, give x, x', y, y' subject to $\overline{f}, \overline{f}' = 0$"

Ideas behind conversion methods

- In a success case, structural analysis indicates how to

  “determine N derivatives that satisfy M constraints”

  where DOF $= N - M$ of a DAE
- Failing on a solvable DAE, structural analysis gives wrong structural information, obviously
- Provided conditions satisfied for a conversion method, we can reduce $\mathrm{Val}\,\Sigma$
- If the Σ-method succeeds on a converted DAE, then DOF of the original problem must have been overestimated—it appears *freer* than it should

Modified pendulum II

Perform a linear transformation on $x,\ y,\ \lambda$

$$\begin{bmatrix} x \\ y \\ \lambda \end{bmatrix} = \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\begin{aligned} 0 = f_1 &= (z_1 + z_2)'' + (z_1 + z_2)(z_3 + z_1) \\ 0 = f_2 &= (z_2 + z_3)'' + (z_2 + z_3)(z_3 + z_1) - G \\ 0 = f_3 &= (z_1 + z_2)^2 + (z_2 + z_3)^2 - \ell^2 \end{aligned}$$

$$\Sigma = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ d_j \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \left[\begin{array}{ccc} 2^{\bullet} & 2 & \colorbox{lightgray}{0} \\ \colorbox{lightgray}{0} & 2^{\bullet} & 2 \\ 0 & 0 & 0^{\bullet} \end{array}\right] \\ \begin{array}{ccc} 2 & 2 & 2 \end{array} \end{array} \begin{array}{c} c_i \\ 0 \\ 0 \\ 2 \\ \\ \end{array} \qquad \mathbf{J} = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3'' \end{array} \begin{array}{c} \begin{array}{ccc} z_1'' & z_2'' & z_3'' \end{array} \\ \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ 2x & 2(x+y) & 2y \end{bmatrix} \end{array}$$

$$\det(\mathbf{J}) \equiv 0$$

- $\operatorname{Val}\Sigma = 4 > 2 =$ DOF of pendulum

Conversion method II: expression substitution (ES)

Substitute new variables for common subexpressions

$$\mathbf{J} = \begin{array}{cc} & \begin{array}{ccc} z_1'' & z_2'' & z_3'' \end{array} \\ \begin{array}{c} f_1 \\ f_2 \\ f_3'' \end{array} & \left[\begin{array}{ccc} 1 & 1 & \\ & 1 & 1 \\ 2(z_1 + z_2) & 2(z_1 + 2z_2 + z_3) & 2(z_2 + z_3) \end{array}\right] \end{array}$$

- Find $v = [1, -1, 1]^T \in \ker \mathbf{J}$
- <u>Common subexpressions</u> are replaced, according to

$$0 = g_1 = -w_1 + \left(z_1 - \frac{v_1}{v_3} z_3\right) = -w_1 + (\underline{z_1 - z_3})$$

$$0 = g_2 = -w_2 + \left(z_2 - \frac{v_2}{v_3} z_3\right) = -w_2 + (\underline{z_2 + z_3})$$

- $z_1 = w_1 + z_3$, $z_2 = w_2 - z_3$

- Replace z_1 by $w_1 + z_3$, replace z_2 by $w_2 - z_3$

$$
\begin{aligned}
0 &= \overline{f}_1 = (w_1 + w_2)'' + (w_1 + w_2)(w_1 + 2z_3) \\
0 &= \overline{f}_2 = w_2'' + w_2(w_1 + 2z_3) - G \\
0 &= \overline{f}_3 = (w_1 + w_2)^2 + w_2^2 - \ell^2 \\
0 &= g_1 = -w_1 + (z_1 - z_3) \\
0 &= g_2 = -w_2 + (z_2 + z_3)
\end{aligned}
$$

$$
\overline{\Sigma} = \begin{array}{c} \\ \overline{f}_1 \\ \overline{f}_2 \\ \overline{f}_3 \\ g_1 \\ g_2 \\ d_j \end{array}
\begin{array}{c}
\begin{array}{ccccc} z_1 & z_2 & z_3 & w_1 & w_2 \end{array} \\
\left[\begin{array}{ccccc}
 & & 0 & 2^\bullet & 2 \\
 & & 0^\bullet & \colorbox{lightgray}{0} & 2 \\
 & & & 0 & 0^\bullet \\
0^\bullet & & 0 & \colorbox{lightgray}{0} & \\
 & 0^\bullet & 0 & & \colorbox{lightgray}{0}
\end{array}\right] \\
\begin{array}{ccccc} 0 & 0 & 0 & 2 & 2 \end{array}
\end{array}
\begin{array}{c} c_i \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ \\ \end{array}
\qquad
\overline{\mathbf{J}} = \begin{array}{c} \\ \overline{f}_1 \\ \overline{f}_2 \\ \overline{f}_3'' \\ g_1 \\ g_2 \end{array}
\begin{array}{c}
\begin{array}{ccccc} z_1 & z_2 & z_3 & w_1'' & w_2'' \end{array} \\
\left[\begin{array}{ccccc}
 & & 2\alpha & 1 & 1 \\
 & & 2w_2 & & 1 \\
 & & & 2\alpha & 2(\alpha + w_2) \\
1 & & -1 & & \\
 & 1 & 1 & &
\end{array}\right]
\end{array}
$$

$$\alpha = w_1 + w_2$$

- $\operatorname{Val} \overline{\Sigma} = 2 =$ DOF of pendulum, $\det(\overline{\mathbf{J}}) = -4\ell^2 \neq 0 \Rightarrow$ success

Index overestimation problem on Reissig's DAEs

- Despite nonsingular $\mathbf{J}$, structural analysis can overestimate index
- Mass matrix has nonzero linearly dependent rows

$$
\begin{aligned}
0 = f_1 &= x_1 + x_2' + x_3' && + q_1(t)\\
0 = f_2 &= x_2 + x_2' + x_3' && + q_2(t)\\
0 = f_3 &= x_3 \qquad\quad + x_4' + x_5' && + q_3(t)\\
0 = f_4 &= x_4 \qquad\quad + x_4' + x_5' && + q_4(t)\\
0 = f_5 &= x_5 && + q_5(t)
\end{aligned}
$$

$$
\Sigma = \begin{array}{c|ccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & c_i\\
\hline
f_1 & 0^\bullet & 1 & 1 & & & 0\\
f_2 & & 1^\bullet & 1 & & & 0\\
f_3 & & & 0^\bullet & 1 & 1 & 1\\
f_4 & & & & 1^\bullet & 1 & 1\\
f_5 & & & & & 0^\bullet & 2\\
\hline
d_j & 0 & 1 & 1 & 2 & 2 &
\end{array}
\qquad
\mathbf{J} = \begin{array}{c|ccccc|}
 & x_1 & x_2' & x_3' & x_4'' & x_5''\\
\hline
f_1 & 1 & 1 & 1 & & \\
f_2 & & 1 & 1 & & \\
f_3' & & & 1 & 1 & 1\\
f_4' & & & & 1 & 1\\
f_5'' & & & & & 1\\
\hline
\end{array}
$$

$$\det(\mathbf{J}) = 1$$

- Size $n = 2k - 1$, differentiation index 1, structural index k

- Replace f_1 by $\overline{f}_1 = f_1 - f_2$, replace f_3 by $\overline{f}_3 = f_3 - f_4$

$$\begin{aligned}
0 = \overline{f}_1 &= x_1 - x_2 & &+ q_1(t) - q_2(t)\\
0 = f_2 \; &= x_2 & + x_2' + x_3' &+ q_2(t)\\
0 = \overline{f}_3 &= x_3 - x_4 & &+ q_3(t) - q_4(t)\\
0 = f_4 \; &= x_4 & + x_4' + x_5' &+ q_4(t)\\
0 = f_5 \; &= x_5 & &+ q_5(t)
\end{aligned}$$

$$\overline{\Sigma} = \begin{array}{c|ccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & c_i\\
\overline{f}_1 & 0^\bullet & 0 & & & & 1\\
f_2 & & 1^\bullet & 1 & & & 0\\
\overline{f}_3 & & & 0^\bullet & 0 & & 1\\
f_4 & & & & 1^\bullet & 1 & 0\\
f_5 & & & & & 0^\bullet & 1\\
d_j & 1 & 1 & 1 & 1 & 1 &
\end{array}
\qquad
\overline{\mathbf{J}} = \begin{array}{c|ccccc|}
 & x_1' & x_2' & x_3' & x_4' & x_5'\\
\overline{f}_1' & 1 & -1 & & & \\
f_2' & & 1 & 1 & & \\
\overline{f}_3' & & & 1 & -1 & \\
f_4' & & & & 1 & 1\\
f_5' & & & & & 1
\end{array}$$

$$\det \overline{\mathbf{J}} = 1$$

- correct structural index 1

Wrong structural analysis remedied so far

- DAEs arbitrarily constructed to be failure cases
- DAEs in literature

DAE	Reference	Method	Result
Robot arm	Campbell et al. (1995)	LC, ES	correct index & nonsingular **J**
Transistor amplifier	Mazzia et al. (2003)	LC, ES	
Ring modulator		LC, ES	
Coupled DAE	Scholz et al. (2013)	LC	
Reissig's DAEs[1]	Reissig et al. (1999)	LC, ES	

---

[1] Structural analysis succeeds but overestimates index

Conclusions

- Provide insight into structural analysis's failure
  - Identically (but not structurally) singular Jacobian
  - Overestimation of DOF of the underlying problem
- Convert to an equivalent problem by reducing $\mathrm{Val}\,\Sigma$
  - Linear combination of equations
  - Expression substitutions by new variables
- Improve efficiency by exploiting a block triangular form
- Can automate conversions in a computer algebra system

References

G. Tan[†], N. Nedialkov, J. Pryce

- [†]Conversion methods for improving structural analysis of DAEs. Submitted to BIT Numerical Mathematics arXiv:1608.06691
- [†]Conversion methods, block triangularization, and structural analysis of DAEs. Submitted to BIT Numerical Mathematics arXiv:1608.06693
- [†]Symbolic-numeric methods for improving structural analysis of DAEs. The 2015 AMMCS conference proceedings DOI 10.1007/978-3-319-30379-6 68
- DAESA—a Matlab tool for structural analysis of DAEs: Theory & Algorithm. ACM TOMS, 41 (2015)

More examples

Robot arm

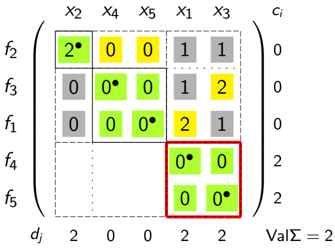
Ring modulator

Coupled system

Transistor amplifier

Campbell & Griepentrog's Robot Arm

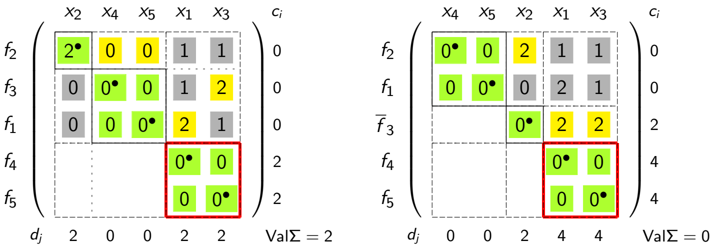
$$
\begin{aligned}
0 = f_3 = & -x_3'' - 2c(x_3)(x_1' + x_3')^2 - d(x_3)x_1'^2 \\
& + (2x_3 - x_2)\big(a(x_3) - 9b(x_3)\big) \\
& - 2x_1'^2 c(x_3) - d(x_3)\big(x_1' + x_3'\big)^2 \\
& {\color{red} -\big(a(x_3) + b(x_3)\big)(x_4 - x_5)} \\
0 = f_1 = & -x_1'' + 2c(x_3)(x_1' + x_3')^2 + d(x_3)x_1'^2 \\
& + (2x_3 - x_2)\big(a(x_3) + 2b(x_3)\big) \\
& {\color{red} +a(x_3)(x_4 - x_5)}
\end{aligned}
$$



$$
\begin{array}{c|ccccc|c}
 & x_2 & x_4 & x_5 & x_1 & x_3 & c_i \\
\hline
f_2 & 2^\bullet & 0 & 0 & 1 & 1 & 0 \\
f_3 & 0 & 0^\bullet & 0 & 1 & 2 & 0 \\
f_1 & 0 & 0 & 0^\bullet & 2 & 1 & 0 \\
f_4 & & & & 0^\bullet & 0 & 2 \\
f_5 & & & & 0 & 0^\bullet & 2 \\
\hline
d_j & 2 & 0 & 0 & 2 & 2 & \mathrm{Val}\Sigma = 2
\end{array}
$$

- $\det \mathbf{J} \equiv 0 \Rightarrow$ failure
- Singular $\mathbf{J}_{22} = \dfrac{1}{2 - \cos^2 x_3} \begin{bmatrix} -2 - \cos x_3 & 2 + \cos x_3 \\ 2 & -2 \end{bmatrix}$
- Block 2: equations f_3, f_1 in variables x_4, x_5

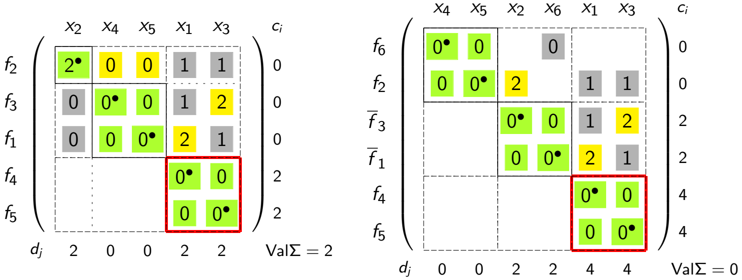
Block LC method on robot arm



$$\begin{array}{c|ccccc|c} & x_2 & x_4 & x_5 & x_1 & x_3 & c_i \\ \hline f_2 & 2^\bullet & 0 & 0 & 1 & 1 & 0 \\ f_3 & 0 & 0^\bullet & 0 & 1 & 2 & 0 \\ f_1 & 0 & 0 & 0^\bullet & 2 & 1 & 0 \\ f_4 & & & & 0^\bullet & 0 & 2 \\ f_5 & & & & 0 & 0^\bullet & 2 \\ \hline d_j & 2 & 0 & 0 & 2 & 2 & \mathrm{Val}\Sigma = 2 \end{array}
\qquad
\begin{array}{c|ccccc|c} & x_4 & x_5 & x_2 & x_1 & x_3 & c_i \\ \hline f_2 & 0^\bullet & 0 & 2 & 1 & 1 & 0 \\ f_1 & 0 & 0^\bullet & 0 & 2 & 1 & 0 \\ \overline{f}_3 & & & 0^\bullet & 2 & 2 & 2 \\ f_4 & & & & 0^\bullet & 0 & 4 \\ f_5 & & & & 0 & 0^\bullet & 4 \\ \hline d_j & 0 & 0 & 2 & 4 & 4 & \mathrm{Val}\Sigma = 0 \end{array}$$

- $u = (2, 2 + \cos x_3)^T \in \operatorname{coker} \mathbf{J}_{22}$
- x_4, x_5 not in u
- $\overline{f}_3 \leftarrow 2f_3 + (2 + \cos x_3)f_1$
- $\mathrm{Val}\,\overline{\Sigma} = 0 < 2 = \mathrm{Val}\,\Sigma$, nonsingular $\mathbf{J} \Rightarrow$ success

Block ES method on robot arm

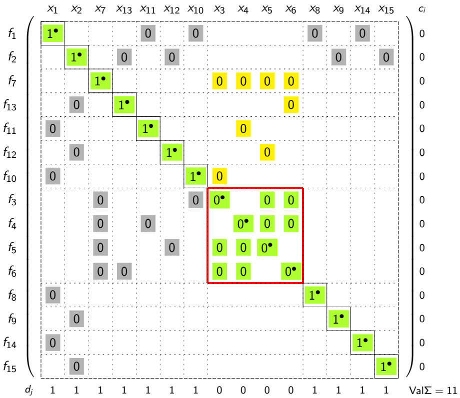


$$\begin{array}{c|ccccc|c} & x_2 & x_4 & x_5 & x_1 & x_3 & c_i \\ \hline f_2 & 2^\bullet & 0 & 0 & 1 & 1 & 0 \\ f_3 & 0 & 0^\bullet & 0 & 1 & 2 & 0 \\ f_1 & 0 & 0 & 0^\bullet & 2 & 1 & 0 \\ f_4 & & & & 0^\bullet & 0 & 2 \\ f_5 & & & & 0 & 0^\bullet & 2 \\ \hline d_j & 2 & 0 & 0 & 2 & 2 & \mathrm{Val}\,\Sigma = 2 \end{array}$$

$$\begin{array}{c|cccccc|c} & x_4 & x_5 & x_2 & x_6 & x_1 & x_3 & c_i \\ \hline f_6 & 0^\bullet & 0 & & 0 & & & 0 \\ f_2 & 0 & 0^\bullet & 2 & & 1 & 1 & 0 \\ \overline{f}_3 & & & 0^\bullet & 0 & 1 & 2 & 2 \\ \overline{f}_1 & & & 0 & 0^\bullet & 2 & 1 & 2 \\ f_4 & & & & & 0^\bullet & 0 & 4 \\ f_5 & & & & & 0 & 0^\bullet & 4 \\ \hline d_j & 0 & 0 & 2 & 2 & 4 & 4 & \mathrm{Val}\,\Sigma = 0 \end{array}$$

- Constant $v = (1,1)^T \in \ker \mathbf{J}_{22}$
- Introduce x_6 and $0 = f_6 = -x_6 + (x_4 - x_5)$
- Replace x_4 by $x_6 + x_5$ in $f_3,\ f_1$
- $\mathrm{Val}\,\overline{\Sigma} = 0 < 2 = \mathrm{Val}\,\Sigma$, nonsingular $\mathbf{J} \Rightarrow$ success

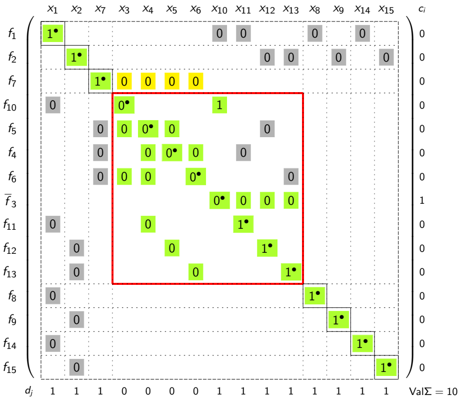
Ring modulator DAE



$$\begin{array}{c|ccccccccccccccc|c}
 & x_1 & x_2 & x_7 & x_{13} & x_{11} & x_{12} & x_{10} & x_3 & x_4 & x_5 & x_6 & x_8 & x_9 & x_{14} & x_{15} & c_i \\
\hline
f_1 & 1^\bullet & & & & 0 & & 0 & & & & & 0 & & 0 & & 0 \\
f_2 & & 1^\bullet & & 0 & & 0 & & & & & & & 0 & & 0 & 0 \\
f_7 & & & 1^\bullet & & & & & 0 & 0 & 0 & 0 & & & & & 0 \\
f_{13} & & 0 & & 1^\bullet & & & & & & & 0 & & & & & 0 \\
f_{11} & 0 & & & & 1^\bullet & & & & 0 & & & & & & & 0 \\
f_{12} & & 0 & & & & 1^\bullet & & & & 0 & & & & & & 0 \\
f_{10} & 0 & & & & & & 1^\bullet & 0 & & & & & & & & 0 \\
f_3 & & & 0 & & & & 0 & 0^\bullet & & 0 & 0 & & & & & 0 \\
f_4 & & & 0 & & 0 & & & & 0^\bullet & 0 & 0 & & & & & 0 \\
f_5 & & & 0 & & & 0 & & 0 & 0 & 0^\bullet & & & & & & 0 \\
f_6 & & & 0 & 0 & & & & 0 & 0 & & 0^\bullet & & & & & 0 \\
f_8 & 0 & & & & & & & & & & & 1^\bullet & & & & 0 \\
f_9 & & 0 & & & & & & & & & & & 1^\bullet & & & 0 \\
f_{14} & 0 & & & & & & & & & & & & & 1^\bullet & & 0 \\
f_{15} & & 0 & & & & & & & & & & & & & 1^\bullet & 0 \\
\hline
d_j & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \mathrm{Val}\Sigma = 11
\end{array}$$

- $\mathrm{Val}\,\Sigma = 11$, $\det \mathbf{J} = 0$
- 4×4 singular block

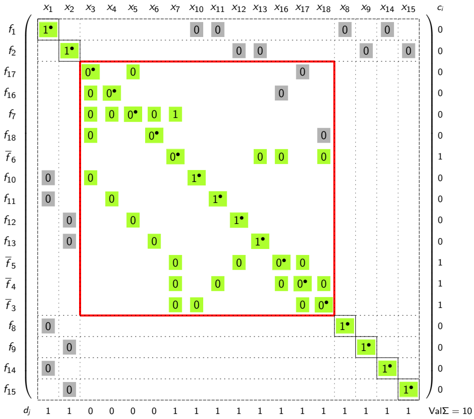
Block LC method on ring modulator



	x_1	x_2	x_7	x_3	x_4	x_5	x_6	x_{10}	x_{11}	x_{12}	x_{13}	x_8	x_9	x_{14}	x_{15}	c_i
f_1	$1^\bullet$							0	0			0		0		0
f_2		$1^\bullet$								0	0		0		0	0
f_7			$1^\bullet$	0	0	0	0									0
f_{10}	0			$0^\bullet$				1								0
f_5			0	0	$0^\bullet$	0				0						0
f_4			0		0	$0^\bullet$	0		0							0
f_6			0	0	0		$0^\bullet$				0					0
$\overline{f}_3$								$0^\bullet$	0	0	0					1
f_{11}	0				0				$1^\bullet$							0
f_{12}		0				0				$1^\bullet$						0
f_{13}		0					0				$1^\bullet$					0
f_8	0											$1^\bullet$				0
f_9		0											$1^\bullet$			0
f_{14}	0													$1^\bullet$		0
f_{15}		0													$1^\bullet$	0
d_j	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1	$\mathrm{Val}\Sigma = 10$

- $(1,-1,1,-1)^T \in \operatorname{coker} \mathbf{J}_{88}$
- $\overline{f}_3 \leftarrow f_3 - f_4 + f_5 - f_6$
- $\mathrm{Val}\,\overline{\Sigma} = 10 < 11 = \mathrm{Val}\,\Sigma$
- $\det \overline{\mathbf{J}} \neq 0 \Rightarrow$ success
- 4×4 singular block merges with four 1×1 blocks

Block ES method on ring modulator



$$
\begin{array}{c|cccccccccccccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_{10} & x_{11} & x_{12} & x_{13} & x_{16} & x_{17} & x_{18} & x_8 & x_9 & x_{14} & x_{15} & c_i \\
\hline
f_1 & 1^\bullet & & & & & & & 0 & 0 & & & & & & 0 & & 0 & & 0 \\
f_2 & & 1^\bullet & & & & & & & & 0 & 0 & & & & & 0 & & 0 & 0 \\
f_{17} & & & 0^\bullet & & 0 & & & & & & & & 0 & & & & & & 0 \\
f_{16} & & & 0 & 0^\bullet & & & & & & & & 0 & & & & & & & 0 \\
f_7 & & & 0 & 0 & 0^\bullet & 0 & 1 & & & & & & & & & & & & 0 \\
f_{18} & & & 0 & & & 0^\bullet & & & & & & & & 0 & & & & & 0 \\
\overline{f}_6 & & & & & & & 0^\bullet & & & & 0 & 0 & & 0 & & & & & 1 \\
f_{10} & 0 & & 0 & & & & & 1^\bullet & & & & & & & & & & & 0 \\
f_{11} & 0 & & & 0 & & & & & 1^\bullet & & & & & & & & & & 0 \\
f_{12} & & 0 & & & 0 & & & & & 1^\bullet & & & & & & & & & 0 \\
f_{13} & & 0 & & & & 0 & & & & & 1^\bullet & & & & & & & & 0 \\
\overline{f}_5 & & & & & & & 0 & & & 0 & & 0^\bullet & 0 & & & & & & 1 \\
\overline{f}_4 & & & & & & & 0 & & 0 & & & 0 & 0^\bullet & 0 & & & & & 1 \\
\overline{f}_3 & & & & & & & 0 & 0 & & & & & 0 & 0^\bullet & & & & & 1 \\
f_8 & 0 & & & & & & & & & & & & & & 1^\bullet & & & & 0 \\
f_9 & & 0 & & & & & & & & & & & & & & 1^\bullet & & & 0 \\
f_{14} & 0 & & & & & & & & & & & & & & & & 1^\bullet & & 0 \\
f_{15} & & 0 & & & & & & & & & & & & & & & & 1^\bullet & 0 \\
\hline
d_j & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \mathrm{Val}\Sigma = 10
\end{array}
$$

- $(-1, 1, -1, 1)^T \in \ker \mathbf{J}_{88}$
- $0 = f_{16} = -x_{16} + (x_4 + x_3)$

  $0 = f_{17} = -x_{17} + (x_5 - x_3)$

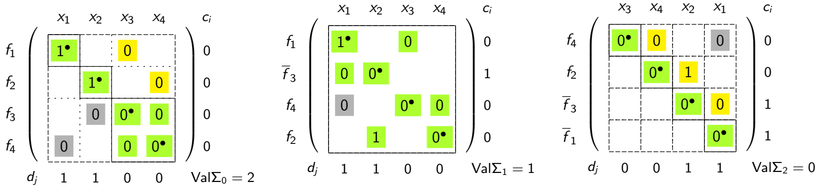
  $0 = f_{18} = -x_{18} + (x_6 + x_3)$
- $\mathrm{Val}\,\overline{\Sigma} = 10 < 11 = \mathrm{Val}\,\Sigma$
- $\det \overline{\mathbf{J}} \neq 0 \Rightarrow$ success
- One 12×12 block and six 1×1 blocks

Scholz and Steinbrecher's coupled system

- Linear constant coefficient DAE of d-index 3 (S&S, 2013)

$$\begin{aligned} 0 &= f_1 = -x_1' + x_3 - b_1(t) & \qquad 0 &= f_3 = x_2 + x_3 + x_4 - b_3(t) \\ 0 &= f_2 = -x_2' + x_4 - b_2(t) & \qquad 0 &= f_4 = -x_1 + x_3 + x_4 - b_4(t) \end{aligned}$$

- SA reports $\nu_S = 1$, obviously wrong
- Apply block LC method



$$\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & c_i \\ \hline f_1 & 1^\bullet & & 0 & & 0 \\ f_2 & & 1^\bullet & & 0 & 0 \\ f_3 & & 0 & 0^\bullet & 0 & 0 \\ f_4 & 0 & & 0 & 0^\bullet & 0 \\ \hline d_j & 1 & 1 & 0 & 0 & \mathrm{Val}\Sigma_0 = 2 \end{array}$$

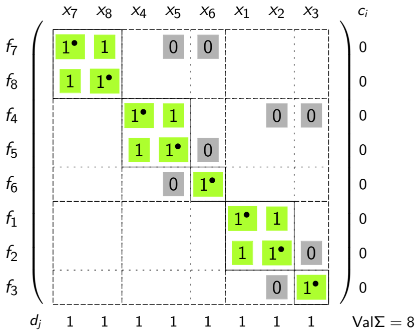
$$\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & c_i \\ \hline f_1 & 1^\bullet & & 0 & & 0 \\ \overline{f}_3 & 0 & 0^\bullet & & & 1 \\ f_4 & 0 & & 0^\bullet & 0 & 0 \\ f_2 & & 1 & & 0^\bullet & 0 \\ \hline d_j & 1 & 1 & 0 & 0 & \mathrm{Val}\Sigma_1 = 1 \end{array}$$

$$\begin{array}{c|cccc|c} & x_3 & x_4 & x_2 & x_1 & c_i \\ \hline f_4 & 0^\bullet & 0 & & 0 & 0 \\ f_2 & & 0^\bullet & 1 & & 0 \\ \overline{f}_3 & & & 0^\bullet & 0 & 1 \\ \overline{f}_1 & & & & 0^\bullet & 1 \\ \hline d_j & 0 & 0 & 1 & 1 & \mathrm{Val}\Sigma_2 = 0 \end{array}$$

$$\overline{f}_3 = f_4 - f_3 \qquad \overline{f}_1 = -f_1 - f_2 + \overline{f}_3' + f_4$$

- $\mathrm{Val}\,\Sigma_0 = 2 > \mathrm{Val}\,\Sigma_1 = 1 > \mathrm{Val}\,\Sigma_2 = 0$
- $\det \overline{\mathbf{J}} = 1 \Rightarrow$ success

Transistor amplifier DAE

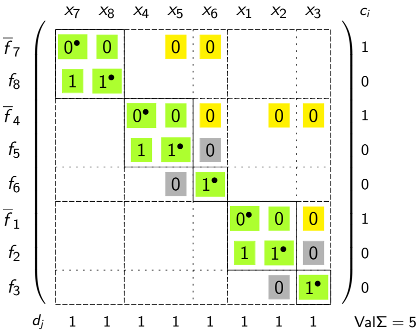


$$\begin{array}{c|cccccccc|c}
 & x_7 & x_8 & x_4 & x_5 & x_6 & x_1 & x_2 & x_3 & c_i \\
\hline
f_7 & 1^{\bullet} & 1 & & 0 & 0 & & & & 0 \\
f_8 & 1 & 1^{\bullet} & & & & & & & 0 \\
f_4 & & & 1^{\bullet} & 1 & & & 0 & 0 & 0 \\
f_5 & & & 1 & 1^{\bullet} & 0 & & & & 0 \\
f_6 & & & & 0 & 1^{\bullet} & & & & 0 \\
f_1 & & & & & & 1^{\bullet} & 1 & & 0 \\
f_2 & & & & & & 1 & 1^{\bullet} & 0 & 0 \\
f_3 & & & & & & & 0 & 1^{\bullet} & 0 \\
\hline
d_j & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \mathrm{Val}\Sigma = 8
\end{array}$$

- $\mathrm{Val}\,\Sigma = 8$, $\det \mathbf{J} = 0$
- Three 2×2 singular blocks

  $(1,1)^T \in \operatorname{coker} \mathbf{J}_{11}$,
  $\operatorname{coker} \mathbf{J}_{22}$,
  $\operatorname{coker} \mathbf{J}_{44}$

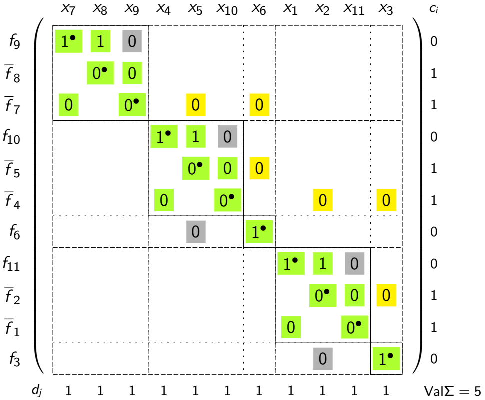
Block LC method on transistor amplifier



$$\begin{array}{c|cccccccc|c}
 & x_7 & x_8 & x_4 & x_5 & x_6 & x_1 & x_2 & x_3 & c_i \\
\hline
\overline{f}_7 & 0^{\bullet} & 0 & & 0 & 0 & & & & 1 \\
f_8 & 1 & 1^{\bullet} & & & & & & & 0 \\
\overline{f}_4 & & & 0^{\bullet} & 0 & 0 & & 0 & 0 & 1 \\
f_5 & & & 1 & 1^{\bullet} & 0 & & & & 0 \\
f_6 & & & & 0 & 1^{\bullet} & & & & 0 \\
\overline{f}_1 & & & & & & 0^{\bullet} & 0 & 0 & 1 \\
f_2 & & & & & & 1 & 1^{\bullet} & 0 & 0 \\
f_3 & & & & & & & 0 & 1^{\bullet} & 0 \\
\hline
d_j & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \mathrm{Val}\Sigma = 5
\end{array}$$

- $\overline{f}_7 \leftarrow f_7 + f_8$
  $\overline{f}_4 \leftarrow f_4 + f_5$
  $\overline{f}_1 \leftarrow f_1 + f_2$
- $\mathrm{Val}\,\overline{\Sigma} = 5 < 8 = \mathrm{Val}\,\Sigma$
- $\det \overline{\mathbf{J}} \neq 0 \Rightarrow$ success

Block ES method on transistor amplifier



$$\begin{array}{c|ccccccccccc|c}
 & x_7 & x_8 & x_9 & x_4 & x_5 & x_{10} & x_6 & x_1 & x_2 & x_{11} & x_3 & c_i \\
\hline
f_9 & 1^\bullet & 1 & 0 & & & & & & & & & 0 \\
\overline{f}_8 & & 0^\bullet & 0 & & & & & & & & & 1 \\
\overline{f}_7 & 0 & & 0^\bullet & & 0 & & 0 & & & & & 1 \\
f_{10} & & & & 1^\bullet & 1 & 0 & & & & & & 0 \\
\overline{f}_5 & & & & & 0^\bullet & 0 & 0 & & & & & 1 \\
\overline{f}_4 & & & & 0 & & 0^\bullet & & & 0 & & 0 & 1 \\
f_6 & & & & & 0 & & 1^\bullet & & & & & 0 \\
f_{11} & & & & & & & & 1^\bullet & 1 & 0 & & 0 \\
\overline{f}_2 & & & & & & & & & 0^\bullet & 0 & 0 & 1 \\
\overline{f}_1 & & & & & & & & 0 & & 0^\bullet & & 1 \\
f_3 & & & & & & & & & 0 & & 1^\bullet & 0 \\
\hline
d_j & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \mathrm{Val}\Sigma = 5
\end{array}$$

- $(1,1)^T \in \ker \mathbf{J}_{11}$,
  $\ker \mathbf{J}_{22}$,
  $\ker \mathbf{J}_{44}$
- $0 = f_9 = -x_9 + (x_7 - x_8)$
  $0 = f_{10} = -x_{10} + (x_4 - x_5)$
  $0 = f_{11} = -x_{11} + (x_1 - x_2)$
- $\mathrm{Val}\,\overline{\Sigma} = 5 < 8 = \mathrm{Val}\,\Sigma$
- $\det \overline{\mathbf{J}} \neq 0 \Rightarrow$ success