# Slater points and Lagrange/Wolfe duals

## 1 Slater point and Ideal Slater point

- Consider a convex problem with two constraints (1) $x_1 \leq 0$ and (2) $x_1^2 + x_2^2 \leq 1$. Both of them are regular constraints, one linear and the other nonlinear. Point $(0, 0)^T$ is a Slater point of this problem, while it's not an Ideal Slater point.
- Consider a convex problem with two constraints (1) $x_1 \leq -0.5$ and (2) $x_1^2 + x_2^2 \leq 1$. Both of them are nonsingular constraints, one linear and the other nonlinear. Point $(-0.6, 0)^T$ is an Ideal Slater point of this problem (it is a Slater point as well).

## 2 Verification of the Slater condition

Check whether the following problem satisfies Slater condition:

$$
\begin{array}{ll}
\min\limits_{x} & f(x) = x^2 \\
\text{s.t.} & g_1(x) = x - x^2 \geq 0 \\
 & g_2(x) = x^2 \leq 0 \\
 & x \geq 0
\end{array}
\tag{1}
$$

Actually, there is only one feasible point $x = 0$, meaning all constraints are singular. However, constraint $g_2(x) \leq 0$ is nonlinear, which means that there is no point which satisfies Slater condition for this problem.

## 3 Wolfe and Lagrange Dual

Consider the following convex optimization problem:

$$
\begin{array}{lr}
\min & x_1 + e^{x_2} \\
\text{s.t.} & 3x_1 - 2e^{x_2} \geq \ 10 \\
 & x_2 \geq \ \ 0 \\
 & x \in \mathbb{R}^2.
\end{array}
$$

Write down the Wolfe dual and Lagrange dual of this problem and show that the optimal value is 5 with $x = (4, 0)^T$. Note that the Slater condition holds for this example.

*Solution*:

*Wolfe dual*: The Wolfe dual is given by

$$
\begin{array}{lr}
\sup & f(x) + \sum_{j=1}^m y_j g_j(x) \\
\text{s.t.} & \nabla f(x) + \sum_{j=1}^m y_j \nabla g_j(x) = \ 0 \\
 & y \geq 0.
\end{array}
$$

which in our case become

$$\begin{array}{ll} \sup & x_1 + e^{x_2} + y_1(10 - 3x_1 + 2e^{x_2}) - y_2 x_2 \\ \text{s.t.} & 1 - 3y_1 = 0 \\ & e^{x_2} + 2e^{x_2} y_1 - y_2 = 0 \\ & x \in \mathbb{R}^2,\ y \geq 0. \end{array}$$

Wolfe dual is a non-convex problem. The first constraint gives $y_1 = \frac{1}{3}$, and thus the second constraint becomes

$$\frac{5}{3} e^{x_2} - y_2 = 0.$$

Now, we can eliminate $y_1$ and $y_2$ from the objective function. We get the function

$$f(x_1, x_2) = x_1 - x_1 + \frac{5}{3} e^{x_2} - \frac{5}{3} x_2 e^{x_2} + \frac{10}{3} \implies f(x_2) = \frac{5}{3} e^{x_2} - \frac{5}{3} x_2 e^{x_2} + \frac{10}{3}.$$

This function has a maximum when

$$f'(x_2) = -\frac{5}{3} x_2 e^{x_2} = 0,$$

which is only true when $x_2 = 0$ and $f(0) = 5$. Hence the optimal value of the *Wolfe dual* problem is 5 when $(x, y) = (4, 0, \frac{1}{3}, \frac{5}{3})$.

*Lagrange dual*: We can double check this answer by using the *Lagrange dual*:

$$\begin{array}{ll} \sup & \psi(y) \\ \text{s.t.} & y \geq 0, \end{array}$$

where $\psi(y) = \inf_x \{ f(x) + \sum_{j=1}^m y_j g_j(x) \}$.

So,

$$\begin{aligned} \psi(y) &= \inf_{x \in \mathbb{R}^2} \{ x_1 + e^{x_2} + y_1(10 - 3x_1 + 2e^{x_2}) - y_2 x_2 \} \\ &= \inf_{x_1 \in \mathbb{R}} \{ x_1 - 3y_1 x_1 \} + \inf_{x_2 \in \mathbb{R}} \{ (1 + 2y_1) e^{x_2} - y_2 x_2 \} + 10 y_1. \end{aligned}$$

We have

$$\inf_{x_1 \in \mathbb{R}} \{ x_1 - 3y_1 x_1 \} = \begin{cases} 0 & \text{for } y_1 = \frac{1}{3} \\ -\infty & \text{otherwise.} \end{cases}$$

Now, for fixed $y_1$, $y_2$ let

$$h(x_2) = (1 + 2y_1) e^{x_2} - y_2 x_2.$$

Then $h$ has a minimum when

$$h'(x_2) = (1 + 2y_1) e^{x_2} - y_2,$$

i.e., when $x_2 = \log(\frac{y_2}{1+2y_1})$. Further, $h(x_2) = h(\log(\frac{y_2}{1+2y_1})) = y_2 - y_2 \log\left(\frac{y_2}{1+2y_1}\right)$. Hence, we have

$$\inf_{x_2 \in \mathbb{R}} \{ (1 + 2y_1) e^{x_2} - y_2 x_2 \} = y_2 - y_2 \log\left(\frac{y_2}{1 + 2y_1}\right)$$

Thus, the *Lagrange dual* becomes

$$\begin{array}{ll} \sup & \psi(y) = 10y_1 + y_2 - y_2 \log\left(\frac{y_2}{1+2y_1}\right) \\ \text{s.t.} & y_1 = \frac{1}{3} \\ & y_2 \geq 0. \end{array}$$

Now we have

$$\frac{d}{dy_2}\psi\left(\frac{1}{3}, y_2\right) = \log\left(\frac{3y_2}{5}\right) = 0,$$

when $y_2 = \frac{5}{3}$ and $\psi(\frac{1}{3}, \frac{5}{3}) = 5$.