Linearly Constrained NLO : Consider the following linearly constrained non-linear optimization (NLO) problem:

$$\text{(NLO)} \qquad \min \quad x_1 + x_2 + x_1x_2$$
$$\text{s.t.} \quad 3x_1 + 2x_2 = 5.$$

a. Reduce this problem to an unconstrained optimization problem.

b. Let us assume that the variables x_1 and x_2 are nonnegative. Make one step of the Reduced Gradient algorithm from the point $(1,1)^T$ when x_2 is the basis variable.

Solution

a. Substituting $x_2 = (5 - 3x_1)/2$ into the objective function, we get the reduced unconstrained problem

$$\min \quad f_N(x_1) = \frac{5}{2} + 2x_1 - \frac{3}{2}x_1^2.$$

b. Here, x_2 is the basis variable, so $B = 2$ and $N = 3$; $x^1 = (1,1)^T$.

Compute the reduced gradient:

$\nabla f(x^1) = \begin{pmatrix} 1 + x_2^1 \\ 1 + x_1^1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\nabla f_N(x_N^1) = (I, -B^{-1}N)\nabla f(x^1) = (1, -3/2)\nabla f(x^1) = -1.$

The search directions are:

$s_N^1 = -\nabla f_N(x_N^1) = 1;\;\; s_B^1 = -B^{-1}Ns_N^1 = -3/2s_N^1 = -3/2;\;\; s^1 = (s_N^1, s_B^1)^T = (1, -3/2)^T.$

The new values of the variables are, depending on the step-length λ:

$$\begin{aligned} x_N^2 = x_1^2 = x_N^1 + \lambda s_N^1 = x_1^1 + \lambda s_1^1 &= 1 + \lambda \\ x_B^2 = x_2^2 = x_B^1 + \lambda s_B^1 = x_2^1 + \lambda s_2^1 &= 1 - 3/2\lambda. \end{aligned}$$

which stays non-negative if $\lambda \leq \bar{\lambda}$, where

$$\bar{\lambda} = \begin{cases} \min\limits_{s_i^1 < 0}\{\frac{x_i^1}{-s_i^1}\} & \text{if } s^1 \not\geq 0 \\ \infty & \text{if } s^1 \geq 0. \end{cases}$$

So, $\bar{\lambda} = \min_{s_i^1<0}\{\frac{x_i^1}{-s_i^1}\} = 2/3.$

Now, we have to solve one-dimensional problem (line search):

$$\min_{0<\lambda\leq 2/3} f(1+\lambda, 1 - 3\lambda/2) = \min_{0<\lambda\leq 2/3} -\frac{3}{2}\lambda^2 - \lambda + 3.$$

The minimum is attained at $\lambda = \bar{\lambda} = 2/3$.

So, $x^2 = x^1 + \lambda s^1 = (5/3, 0)^T,\;\; f(x^2) = 5/3,\;\; \nabla f(x^2) = (8/3, 1)^T.$