Exercise 3.3, parts i, ii and v

We want to design a box with dimensions $\ell \times b \times h$ such that the volume of the box is at least V, and the total surface area is minimal.

$$\min_{l,b,h} 2(\ell b + bh + \ell h), \quad \ell bh \geq V, \quad \ell, b, h > 0.$$

Non-convex problem! Replace ℓ by e^{x_1}, etc:

$$\min_{x_1,x_2,x_3} 2(e^{x_1+x_2} + e^{x_2+x_3} + e^{x_1+x_3}),$$

such that

$$x_1 + x_2 + x_3 \geq \ln(V), \quad x_1, x_2, x_3 \in \mathbf{R}.$$

Let

$$f(x_1, x_2, x_3) = 2(e^{x_1+x_2} + e^{x_2+x_3} + e^{x_1+x_3})$$

and

$$g(x_1, x_2, x_3) = \ln V - (x_1 + x_2 + x_3).$$

The transformed problem is of the form (CO) and satisfies Slater's regularity condition (why?).

KKT conditions: we are looking for $x \in \mathcal{F}$ such that

$$\nabla f(x) = -y \nabla g(x)$$

for some $y \geq 0$, and such that $yg(x) = 0$. This means

$$2 \begin{bmatrix} e^{x_1+x_2} + e^{x_1+x_3} \\ e^{x_1+x_2} + e^{x_2+x_3} \\ e^{x_1+x_3} + e^{x_2+x_3} \end{bmatrix} = -y \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad y\,(\ln V - (x_1 + x_2 + x_3)) = 0, \quad y \geq 0.$$

By solving this system we find the following KKT point:

$$x_1 = x_2 = x_3 = \frac{1}{3}\ln(V), \; y = 4V^{2/3}.$$

This gives for the original variables:

$$\ell = e^{x_1} = V^{1/3}, \quad b = e^{x_2} = V^{1/3}, \quad h = e^{x_3} = V^{1/3}.$$

So the optimal solution is the cube ($\ell = b = h$)!