Linear Optimization - Tutorial 1

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About Me

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- 1st year M.Eng student working on research in the area of combinatorial optimization.
- Hold a B.Eng from McMaster University.

Linear Optimization Formulation

A linear function is of the form:

- $f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \dots + c_nx_n$
  $= \sum_{i=1}^{n} c_ix_i$
  e.g. $3x_1 + 2x_2 - 5x_3$

- Can also be rewritten as:
  min c^Tx T → transpose , c^T means c, but with its rows swapped for columns
  s.t. $Ax \le b$
  and $x \ge 0$ → ∵ non-negativity constraints of linear program

Example 1

- John is deciding which ridesharing service to drive for, Uber or Lyft. After some analysis, he decided that on average, driving for Uber will allow him to get an average of 1 rider every 10 minutes, and driving for Lyft will let him get 3 riders per hour. The average Uber passenger will pay him \$10, and the average Lyft passenger will pay him \$15.

  Assume John drives 4 hours a day.

Step 1:
determine the
decision variables

Let x_1 be the # of Uber rides
Let x_2 be the # of Lyft rides

Step 2:
Determine the objective
function

maximize $z = 10x_1 + 15x_2$

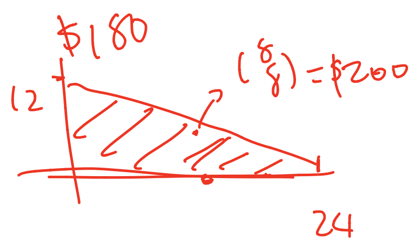
Step 3: Determine the constraints

$\frac{1}{6}x_1 + \frac{x_2}{3} \leq 4 \rightarrow x_1 + 2x_2 \leq 24$

Let's try to find an optimal solution by hand:

Uber: 1 $ / min

Lyft: 0.8 $ / min



Some possible points:

$(0, 12) \rightarrow \$180$

$(8, 8) \rightarrow \$200$

$(24, 0) \rightarrow \$240$

Example 1 cont.

$(20, 0) \to \$200$ $\quad 20 + 2x_2 \leq 24, \; x_2 = 2$

$(20, 2) \to \$220$ $\quad \therefore (20, 2)$ is an optimal solution

- Uber has recently updated their policy where each driver can only drive 20 passengers per day. How will this affect the formulation? This adds one more constraint: $x_1 \leq 20$
- John decides that he wants to drive full time (i.e. 8 hours per day). How will the revised formulation look like?

constraints become:

$x_1 \leq 20$

$\frac{1}{6}x_1 + \frac{1}{3}x_2 \leq 8 \to x_1 + 2x_2 \leq 48$

Example 2

- Jared is hungry and wants to buy lunch. He has three choices to choose from: Subway, Taco del Mar, or Popeyes. Jared is very health conscious and is looking to not exceed 1200 calories for his lunch, but he also knows that any meal with less than 1000 calories will leave him hungry in the afternoon. Subway sandwiches are $6 and contains 300 calories. Taco del Mar burritos are $8 and contains 450 calories and Popeyes fried chicken costs $10 and contain 600 calories. As a poor grad student, Jared is looking for the cheapest lunch possible that satisfies his constraints.

1. determine the decision variables

$x_1 \to$ # of subway sandwiches
$x_2 \to$ # of Taco del Mar burritos
$x_3 \to$ # of Popeyes fried chicken

Step 2:
Determine the objective function

min. $6x_1 + 8x_2 + 10x_3$

Step 3: Determine the constraints

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$
x_i : integer
$1000 \leq 300x_1 + 450x_2 + 600x_3 \leq 1200$

Find an optimal set of x :

From the constraints, we set:

$0 \leq x_1 \leq 4$ $\quad x_1 + x_2 + x_3 \geq 1$
$0 \leq x_2 \leq 2$
$0 \leq x_3 \leq 2$

$x_3 = 0$	$x_3 = 1$	$x_3 = 2$
$x_2 = (0, 1, 2)$	$x_1 + x_2 \leq 1$	$x_1 = x_2 = 0$
$x_2 = 0, x_1 = 4$ (4 0 0) \$24	$x_1 = 1$ or $x_2 = 1$	(0 0 2) \$20
$x_2 = 1$, (2 1 0) \$20		
$x_2 = 2$, (1 2 0) \$22	(0 1 1) (1 0 1)	
	\$18 \$16	

Although (1 0 1) (\$16) is the cheapest, it doesn't meet the calories constraints. So the optimal solution is (0 1 1), which is 1 burrito and 1 fried chicken.

Example 3

- Jamie is taking the O03 Linear Optimization class and wants to get the highest possible mark. She knows that the assignments are worth 30%, the midterm is worth 30%, and the final is worth 40% of her final grade. Jamie estimates that every hour she spends on the assignment can increase her mark by 1%, every hour she studies for the midterm increases her mark by 0.5%, and every 2 hours she studies for the final increases her mark by 2 %. Due to her heavy course load, she can only spend a total of 80 hours on this course.

t

1. determine the decision variables

$x_1 \to$ # hours spends on assignment

$x_2 \to$ # hours sp ds on midterm

$x_3 \to$ # hours on exam

Step 2: Determine the objective function

max $x_1 + 0.5x_2 + x_3$

Step 3: Determine the constraints

$x_1 \leq 30$, $\overset{x_2 \leq 60}{0.5x_2 \leq 30}$, $x_3 \leq 40$

$x_1 + x_2 + x_3 \leq 80$

Find solution:

assignment: 1 pt / hr

midterm : 0.5 pt / hr

exam : 1 pt / hr

the same.

$\therefore$ (30, 10, 40)

final mark = $30 + 10 \times 0.5 + 40 = 75$

What if we change the question to:

assignment: 1 pt / hr

midterm : 1 pt / hr

exam : 0.5 pt / 1 hr

objective function : max. $x_1 + x_2 + \frac{1}{4}x_3$

Constraints : $x_1 \leq 30$

$x_2 \leq 30$

$0.5x_3 \leq 40$, $x_3 \leq 80$

(30 30 20) is the optimal solution.

final mark = $30 + 30 + 20 \times 0.5 = 30 + 30 + 10 = 70$

Jamie will fail the final exam. But Dr. Deza said in order to pass the course, students need to get above 50% on the final exam. In order to pass the exam, Jamie needs to spent at least 40 hours on exam. So the solution is (a 40 − a 20) for $10 \leq a \leq 30$

Next Week

- How to solve LP's in Excel using Excel Solver.
- Bring your laptops to follow along.