part ① definitions.

Problem 4

The Lawn Products Co. has 80 tons of nitrate and 50 tons of phosphate available for 3 types of fertilizer. The mixture ratios and profit figures are given in the accompanying table. We wish to determine how the current inventory should be used to maximize the profit.

	Nitrate (tons / 1000 bags)	Phosphate (tons / 1000 bags)	Profit ($/1000 bags)
Regular lawn	4	2	300
Super lawn	4	3	500
Garden	2	2	400

(a) Formulate the problem as an LP. **2p**

(b) Transform the LP into the *standard form* $\{\min c^T x, Ax = b, x \geq 0\}$. **2p**

4a. Let x_1, x_2 and x_3 be the amount of Regular lawn, Super lawn and Garden fertilizers produced respectively (in 1000 bags). The LP can be formulated as:

$$\begin{aligned} \max\ & 300x_1 + 500x_2 + 400x_3 \\ s.t.\ & 4x_1 + 4x_2 + 2x_3 \leq 80 \\ & 2x_1 + 3x_2 + 2x_3 \leq 50 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

4b. The standard form is:

$$\begin{aligned} \max\ & 300x_1 + 500x_2 + 400x_3 \\ s.t.\ & 4x_1 + 4x_2 + 2x_3 + s_1 = 80 \\ & 2x_1 + 3x_2 + 2x_3 + s_2 = 50 \\ & x_1, x_2, x_3, s_1, s_2 \geq 0 \end{aligned}$$

part ④ basic, non-basic ... definitions

⇒ Pivoting

The variable that we pick to pivot on is the entering variable. → the column where the objective function's coefficient is most positive

exiting variable is the basic variable in the row with the minimum positive ratio.

⇒ Unboundness

If the column coefficients (except for the z row) of the entering variable are nonpositive, then the objective value is unbounded from above.

While solving an LP maximization problem we obtain the following tableau. The basic variables are x_1, x_2 and x_3.

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7		
1	0	0	0	3	a	b	c	=	2
0	1	0	0	1	-1	2	1	=	1
0	0	1	0	-1	d	-2	e	=	3
0	0	0	1	2	0	3	2	=	4

$1/1 = 1$

$3/e < 1 \rightarrow e > 3$

$4/2 = 2$

Give conditions on the variables a, b, c, d and e required to make each of the following statements true:

(a) The current tableau is optimal. **2p**

(b) The problem is unbounded. **2p**

(c) The current basic solution is feasible, but the objective function value can be improved by bringing x_7 into the basis and pivoting x_2 out. **2p**

2a. $a, b, c \geq 0$.

2b. $a < 0$, $d \leq 0$.

$\rightarrow$ need the column coefficients of the entering variables to be non-negative

x_4, x_6, x_7 have positive coefficients,

so assume x_5 is the entering variable.

$\therefore$ $a < 0$, $d \leq 0$.

2c. $c < 0$, $e > 3$.

The problem

- max $z = 2x_1 + 3x_2 + x_3$

  s.t. $x_1 + x_2 + x_3 \leq 40$

  $2x_1 + x_2 - x_3 \geq 10$

  $-x_2 + x_3 \geq 10$

  $x_1, x_2, x_3 \geq 0$

Standard Form

- We can convert the problem into standard form by introducing slack variables into the constraints

$$
\begin{array}{ll}
\max z = 2x_1 + 3x_2 + x_3 & \\
\text{s.t.} & x_1 + x_2 + x_3 + s1 \qquad\qquad = 40 \\
& 2x_1 + x_2 - x_3 \qquad - s2 \qquad = 10 \\
& -x_2 + x_3 \qquad\qquad - s3 = 10 \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0
\end{array}
$$

Finding a BFS

- We normally like to take the origin as an initial basic feasible solution to the problem. However, we can trivially see that taking $x_1=x_2=x_3=0$ cannot be satisfied by the bottom 2 constraints, as there is no positive s_2 or s_3 that we can choose to satisfy those equations.
- Instead, we will form a new problem for which we can use the origin as our BFS, and to which the optimal point is a feasible solution to our original problem.

Forming the Phase 1 problem

- We will introduce artificial variables y_1 and y_2 to constraints 2 and 3 such that we can trivially find a BFS to this problem. Our new objective function, w, will be to minimize $y_1 + y_2$, which is equivalent to maximizing $-y_1 - y_2$
- If we can find a solution to this problem such that $w = 0$, then we have found a solution where $y_1 = y_2 = 0$, which is then a feasible solution to the original problem. If we find an optimal point where $w \leq 0$, then there is no feasible solution for the original problem.

The Phase 1 problem

- max $w = - y_1 - y_2$
  s.t.
  $$\begin{aligned} x_1 + x_2 + x_3 + s1 &= 40 \\ 2x_1 + x_2 - x_3 - s2 + y_1 &= 10 \\ -x_2 + x_3 - s3 + y_2 &= 10 \\ x_1, x_2, x_3, s_1, s_2, s_3, y_1, y_2 &\geq 0 \end{aligned}$$
- Rearranging the objective function to put it into the tableau, we get $w + y_1 + y_2 = 0$

Initial Tableau

w	x_1	x_2	x_3	s_1	s_2	s_3	y_1	y_2	RHS
1	0	0	0	0	0	0	1	1	0
0	1	1	1	1	0	0	0	0	40
0	2	1	-1	0	-1	0	1	0	10
0	0	-1	1	0	0	-1	0	1	10

- The tableau is not yet in canonical form. We will make y_1 and y_2 basic variables by subtracting row 3 and row 4 from row 1.

1 - 3

$R_1 \rightarrow$ 1 -2 -1 1 0 1 0 0 1 -10

1 - 4

$R_1 \rightarrow$ 1 -2 0 0 0 1 1 0 0 -20

Canonical Form

w	x_1	x_2	x_3	s_1	s_2	s_3	y_1	y_2	RHS
1	-2	0	0	0	1	1	0	0	-20
0	1	1	1	1	0	0	0	0	40
0	2	1	-1	0	-1	0	1	0	10
0	0	-1	1	0	0	-1	0	1	10

- Our initial BFS contains w, s_1, y_1, and y_2 as basic variables, and all others as non-basic. This solution gives $w=-20$, $s_1=40$, $y_1=10$, $y_2=10$, with all other variables equal to 0.
- We can see that the objective function will increase if we give x_1 some positive value ($w - 2x_1 + s_2 + s_3 = -20 \rightarrow w = 2x_1 - s_2 - s_3 - 20$), so we will make x_1 a basic variable.

Deciding the exiting variable

- We need to determine which basic variable will exit if we make x_1 a basic variable. As all variables must be non-negative, we must determine which basic variable will decrease to 0 the fastest as we increase x_1 from 0. If we increase x_1 by some Δ, then we can derive the following equations from the tableau:
  $s_1 = 40 – \Delta$
  $y_1 = 10 – 2*\Delta$
  $y_2 = 10 – 0*\Delta$

  From this we can see that y_1 will reach 0 the fastest, and must be the exiting variable.

The first pivot

- As x_1 is entering and y_1 is leaving, we must pivot on x_1 on the row that depends on y_1 (row 3). This results in the following tableau.

3 × ½ , 2 − 3 , 1 − 2 × 3

w	x_1	x_2	x_3	s_1	s_2	s_3	y_1	y_2	RHS
1	0	1	-1	0	0	1	1	0	-10
0	0	0.5	1.5	1	0.5	0	-0.5	0	35
0	1	0.5	-0.5	0	-0.5	0	0.5	0	5
0	0	-1	1	0	0	-1	0	1	10

$w + x_2 - x_3 + s_1 + y_1 = -10$

$w_1 = -x_2 + x_3 - s_1 - y_1 - 10$

$\uparrow x_3$, $\downarrow w_1$

The second pivot

w	x_1	x_2	x_3	s_1	s_2	s_3	y_1	y_2	RHS
1	0	1	-1	0	0	1	1	0	-10
0	0	0.5	1.5	1	0.5	0	-0.5	0	35
0	1	0.5	-0.5	0	-0.5	0	0.5	0	5
0	0	-1	1	0	0	-1	0	1	10

- We can see that we can still improve in the direction of x_3, so we will make x_3 the next basic variable.
- Using the same decision rule from before, we can see that y_2 will be reduced to 0 first, and therefore must be our exiting variable.

$35/1.5 = 23.3$ $5/0.5 = 10$ $10/1 = 10$ → y_2

Finishing the Phase 1 problem

② -1.5 ④ ③ $+ \frac{1}{2}$ ④

- Pivoting on x_3 from row 4, we obtain the following tableau:

w	x_1	x_2	x_3	s_1	s_2	s_3	y_1	y_2	RHS
1	0	1	0	0	0	0	1	1	0
0	0	2	0	1	0.5	1.5	-0.5	-1.5	20
0	1	0	0	0	-0.5	-0.5	0.5	0.5	10
0	0	-1	1	0	0	-1	0	1	10

- We can see that there is no direction we can move in to further improve w, and also that the RHS is 0. From this, we can conclude that we have reduced the artificial variables to 0 and found a feasible solution to the original problem.

BFS to original problem

- From the final tableau, we obtain the solution x_1=10, x_3=10, s_1=20. Applying this to our original constraints to check our work, we can see that they are satisfied.
- max $z = 2x_1 + 3x_2 + x_3$
  s.t. $x_1 + x_2 + x_3 + s1 = 40$ (10 + 0 + 10 + 20 = 40)
  $2x_1 + x_2 – x_3 – s2 = 10$ (20 + 0 – 10 – 0 = 10)
  $-x_2 + x_3 – s3 = 10$ (-0 + 10 – 0 = 10)
  $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$
- Thus we have found a BFS to the original problem

Forming the Phase 2 problem

- To set up the Phase 2 problem, we take the final tableau from the Phase 1 problem and replace our first row with the original objective, z. As we have found a solution where the artificial variables are 0, and any non-zero value for these is an infeasible solution to this problem, we will remove these from the tableau to avoid giving them any value again.

Canonical Form

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
1	-2 0	-3	-1	0	0	0	0
0	0	2	0	1	0.5	1.5	20
0	1	0	0	0	-0.5	-0.5	10
0	0	-1	1	0	0	-1	10

- We can see that this tableau is not yet in canonical form, as x_1 and x_3 now have coefficients in the z-row. To convert to standard form, we add 2*(row 3) and 1*(row 4) to row 1 so that x_1 and x_3 are basic variables again.

The first pivot

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
1	0	-4	0	0	-1	-2	30
0	0	2	0	1	0.5	1.5	20
0	1	0	0	0	-0.5	-0.5	10
0	0	-1	1	0	0	-1	10

$20/2 = 10$

$10/-1 = -10$

- We can see that our objective value improves in any of the directions x_2, s_2, or s_3. We can pick any of these directions. In this case, we will let x_2 be the entering variable. We can see that s_1 is the only basic variable that will decrease with an increase in x_2, so it is the exiting variable.

(2) $\times \frac{1}{2}$, (1) + 4(2) (4) + (2)

The optimal solution

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
1	0	0	0	2	0	1	70
0	0	1	0	0.5	0.25	0.75	10
0	1	0	0	0	-0.5	-0.5	10
0	0	0	1	0.5	0.25	-0.25	20

- We can see that there is no direction in which the objective function improves from this point, so we have found an optimal solution. The optimal value is 70, and the optimal solution is x_1=10, x_2=10, x_3=20, s_1=0, s_2=0, s_3=0.