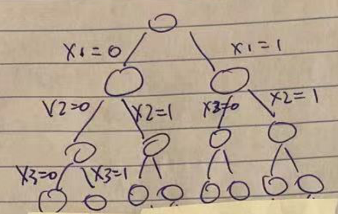
Tut 6 example

① max $15x_1 + 12x_2 + 4x_3 + 2x_4$

Subject to $8x_1 + 5x_2 + 3x_3 + 2x_4 \le 10$

x_k binary for $k=1$ to 4

Sol: originally, we can have



As the problems get bigger, complete enumeration would take too long ∵ the number of possible solutions explodes exponentially.

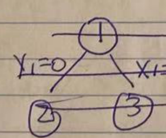
The trick is to use LP relaxation to bound the optimal integer solutions in a subtree of the enumeration tree, which allows us to eliminate many of the enumeration tree. This is called branch and bound algorithm.

Algorithm:

To start off, we first find a feasible solution x^*. At each iteration, we will refer to x^* as the incumbent solution and its objective value z^* as the incumbent objective. Here, incumbent means "best so far". Next, mark the root node as active.

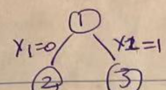
as mentioned in lecture:

While there is an active node do
- select an active node i
- mark i as inactive.
- Solve $LP(i)$: denote solution as $x(i)$;
- case 1: $Z_{LP(i)} \leq Z_*$:
  - prune node i;
- case 2: $Z_{LP(i)} > Z_*$ and $x(i)$ is feasible for $IP(i)$:
  - incumbant solution $\leftarrow x(i)$ and $Z_* \leftarrow Z_{LP(i)}$
- case 3: $Z_{LP(i)} > Z_*$ and $x(i)$ is infeasible for $IP(i)$:
  - mark children of node i as active.



$x_{LP}(1) = (5/8, 1, 0, 0)$

$z_{LP}(1) = 21\ 3/8$



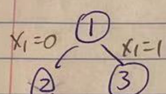
Incumbent
$x^* = (0,0,0,0)$
$z^* = 0$

maximize $x_1 x_2$ ← $x_{LP}(1) = (5/8, 1, 0, 0)$

$x_2=0$, $x_1 = 10/8 \rightarrow z^* = 18.75$ $\quad Z_{LP}(1) = 21\ 3/8$

$x_2=1$, $x_1 = 5/8 \rightarrow z^* = 21\ 3/8$

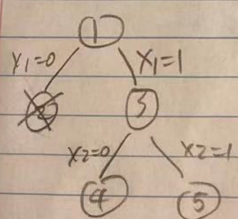
$Z_{LP}(1) > Z^*$, and $X(1)$ not feasible for IP (1) (not integer), ∴ we are in case 3, ∴ branch on the node and mark its children as active



Incumbent
$x^* = (0,0,0,0)$
$z^* = 0$

$X_{LP}(2) = (0,1,1,1)$
$Z_{LP}(2) = 18$

we are in case (2), ∴ replace the incumbent with this better solution, and prune node 2

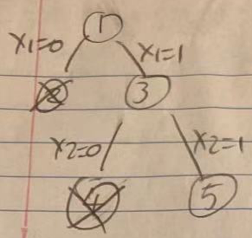


Incumbent
$x^* = (0,1,1,1)$
$z^* = 18$

$X_{LP}(3) = (5/8, 1, 0, 0)$

$Z_{LP}(3) = 21\ 3/8$

in case 3, branch on the node and mark its children as active.

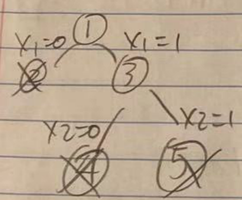


$$\boxed{\begin{array}{l}\text{Incumbent}\\ x^* = (0,1,1,1)\\ z^* = 18\end{array}}$$

$x_{LP}(4) = (1, 0, 2/3, 0)$

$z_{LP}(4) = 17\frac{2}{3}$

Case 1, prune node 4.



$x_{LP}(5)$ = Infeasible

$z_{LP}(5)$ = infeasible

Case 1, prune node 5.

∴ incumbent:

$x^* = (0,1,1,1)$

$z^* = 18$

⑫ max $23x_1 + 19x_2 + 28x_3 + 14x_4 + 44x_5$

s.t. $8x_1 + 7x_2 + 11x_3 + 6x_4 + 19x_5 \le 25$

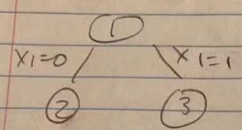
$x_{1-5} = 0$ or 1

↳ relax it, $0 \le x_{1-5} \le 1$

Incumbent

$x^* = (0,0,0,0,0)$

$z^* = 0$

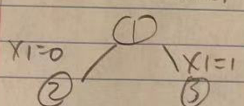


$x_{LP(2)} = (0, 1, 1, 1/6, 0)$

$z_{LP(2)} = 63.33$

~~Case 3~~

not a feasible solution, ∴ look at node 3, compare their z^* value, choose the larger one to branch on.

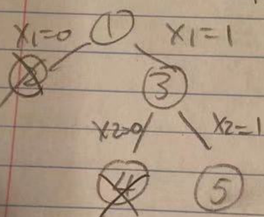


$x_{LP(3)} = (1, 1, 10/11, 0, 0)$

$z_{LP(3)} = 67.45$

Case 3

$z_{LP(3)} > z_{LP(2)}$, ∴ branch on node 3



$x_{LP(4)} = (1, 0, 1, 1, 0)$

$z_{LP(4)} = 65$

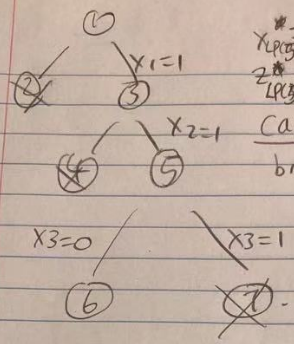
Case 2.

Incumbent:

$x^* = (1, 0, 1, 1, 0)$

$z^* = 65$.

Case 2



$x^*_{LP(5)} = (1, 1, {}^{10}/_{11}, 0, 0)$

$z^*_{LP(5)} = 67.45$.

Case 3

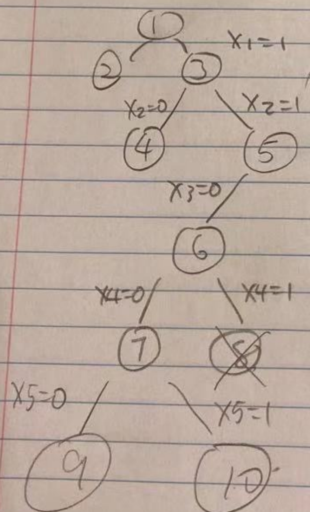
branch on node 5.

$x_{LP(6)} = (1, 1, 0, 1, {}^{4}/_{19})$

$z_{LP(6)} = 65.26$.

$x_{LP(7)}$ = infeasible.

$z_{LP(7)}$ = infeasible



$x_{LP(7)} = (1, 1, 0, 0, {}^{10}/_{19})$

$z_{LP(7)} = 65.16$

$x_{LP(8)} = (1, 1, 0, 1, {}^{2}/_{19})$

$z_{LP(8)} = 60.63$.

$x_{LP(9)} = (1, 1, 0, 0, 0)$

$z_{LP(9)} = 42$.

Case 1

$x_{LP(10)}$ = infeasible.

∴ Incumbent solution:

$x^* = 1, 0, 1, 1, 0$

$z^* = \cancel{65}\ 65$