Continuity of Operators on Continuous and Discrete Time Streams

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Dedication

Jan A. Bergstra's research in theoretical computer science is extraordinary in its diversity and scale and, especially, in its conceptual and technical depth. His achievement stands out and defies the simple conventions of contemporary judgment and praise. As a scientist and intellectual, and a longstanding friend and collaborator of ours, he is unique. For this birthday celebration, we offer congratulations and encouragement, and record our admiration and gratitude.

Abstract

We consider the semantics of networks processing streams of data from a complete metric space. We consider two types of data streams: those based on continuous time (used in networks of physical components and analog devices), and those based on discrete time (used in concurrent algorithms). The networks are both governed by global clocks and together model a huge range of systems. Previously, we have investigated these two types of networks separately. Here we combine their study in a unified theory of stream transformers, given as fixed points of equations. We begin to develop this theory by using the standard mathematical techniques of topology to prove certain computationally desirable properties of these semantic functions, notably continuity, which is significant for models of a physical system, according to Hadamard's principle.

Key words and phrases: analog computing, analog networks, compact open topology, continuous stream operations, continuous time streams, discrete time streams, fixed points, Hadamard's principle, synchronous concurrent algorithms, topological algebras.

1 Introduction

Computation is a general phenomenon that involves data, specifications, programs, systems and devices. Whilst the diversity of these components seems unlimited there are common factors that can characterise computation, for example: data representation and coding; levels of abstraction defined by operations; semantic models and logics for reasoning about behaviour; subcomponents and architecture; modularity and compositionality; and physical properties such as time and space. Some taxonomic order can be attempted by first classifying the nature of the data.

In digital computation, at the heart of our theoretical understanding are countable sets of discrete data that can be faithfully coded by strings and natural numbers, since the classical theory of computability and complexity is founded upon the data types of strings and natural numbers. However, from earliest times, many computations concern analog processes involving physical quantities; streams of messages and signals in time; and objects and scenes in 3-dimensional space. Physical models of data, distributed in time and space, can be found in hybrid embedded systems, analog computers of the first half of the twentieth century [Sma01, Cla10], and new and unconventional technologies for computation that involve (for example) quantum or DNA systems. We must compute on uncountable sets of continuous data, modelled and represented by constructions with real and complex numbers, on scalar and vector fields. Computations with continuous data require special computability theories, involving the approximation of functions on topological, metric, normed and ordered spaces of various kinds.

The generalisations of computability theory to arbitrary data aims at models analysing the computability of functions $f\colon A \to B$ on any sets A and B. In general, models of computation fall into one of two classes: *concrete models*, which involve building a representation of the data type; and *abstract models*, which involve programming directly with the primitive operations of the data type [TZ04]. Some very general theories are possible that can deal essentially with any sets A and B [TZ00, TZ05, TZ06] and they provide a starting point and mathematical tools for analysing any computational phenomena. However, we have found that certain general classes of data types also require a specialised analysis and a customised theory.

In many computations, one finds that continuous or analog data are represented by functions of the form $u\colon X \to A$, where X is a set of points in time or space, and A is a set of data. More specifically, in some cases the sets X and A have a topology (possibly discrete) and the functions of interest are those in the set

$$\mathcal{C}[X, A] \;=\; \{u\colon X \to A \mid u \text{ is a continuous total function}\}.$$

Thus, we are interested in models of computation for functions of the form

$$\Phi\colon A^r \times \mathcal{C}[X, A]^m \;\to\; \mathcal{C}[X, A]^n.$$

In this paper we will study how functions Φ are specified as fixed points and computed by topological methods. Typically we deal with fixed points of operators with contracting properties that are derived from equations describing a system. In a companion paper

[TZ11] we will study concrete and abstract computability models for functions Φ on the data type $\mathcal{C}[X, A]$, and compare them.

1.1 Examples of data in time and space

Each data type of the above form $\mathcal{C}[X, A]$ arises typically in some practical situation, and has its own special features. The algorithmic models that are characteristic of that situation determine, or at least suggest, a corresponding computability theory. For example, in the case that X is *time*, we have:

(i) *Analog streams*: For signal processing, X is continuous time $\mathbb{T} = \mathbb{R}^{\geq 0}$ (the non-negative reals), and the space A of data may be the reals $\mathbb{R}$ or $[0, 1]$, or even continuous mappings from a compact space to $\mathbb{R}$.

(ii) *Digital* (*bit*) *streams*: For bit processing, X is discrete time $\mathbb{T} = \{0, 1, 2, \dots\}$, and the data are bits $A = \{0, 1\}$.

Alternatively, in the case of *space*, we have:

(iii) *Graphic scenes*: In 3-dimensional volume graphics, X can be continuous space, $X = \mathbb{R}^3$, and data are attributes of spatial objects, such as colour or opacity, measured by $A = \{0, 1\}^k$ or $A = [0, 1]$.

(iv) *Machine states*: In machine states, X could be a 2-dimensional discrete address space, $X = \mathbb{Z}^2$, and data are k-bit words $A = \{0, 1\}^k$.

(v) *Analog fields*: Quite generally, X can be a continuous space modelled by a manifold, and data can be measurements from a normed vector space.

We have encountered computability theories for some of these data types before: *discrete streams* processed by digital networks [TZ94, TTZ09]; *continuous streams* processed by analog networks [TZ07], and *spatial objects* in volume graphics [CT00, BSHT98, Joh06]. The mathematical question arises: How much do these data types and their computability theories have in common?

1.2 Some general models for time

For a huge range of spaces X and A, we can equip $\mathcal{C}[X, A]$ with the compact-open topology and consider the partial functions on $\mathcal{C}[X, A]$ that are computable or approximably computable with respect to the topology.

First, we consider the general case of the data type $\mathcal{C}[X, A]$. In Section 2, we study the *local uniform topology* on $\mathcal{C}[X, A]$, which is generated by the family of pseudometrics

$$\mathsf{d}_K(u, v) \ =_{df} \ \sup \{ \, \mathsf{d}(u(t), v(t)) \mid t \in K \, \}$$

for all compact $K \subseteq X$. This is the same[1] as the topology generated by the *inverse limit* representation

$$\mathcal{C}[X, A] \ = \ \varprojlim \{ \, \mathcal{C}(K, A) \mid K \text{ compact } \subseteq X \, \}.$$

[1] Details in §2.1.

Next we consider the notion of *compact exhaustion* of X. We note how, with the assumption of *σ-compactness* of X, the above topology of $\mathcal{C}[X, A]$ is metrisable.

However, it seems to matter whether or not X is thought of as modelling time or space. From Section 3 onwards, we concentrate on the special case that X represents time, i.e., $X = \mathbb{T}$, where $\mathbb{T}$ is either $\mathbb{R}^{\geq 0}$ or $\mathbb{N}$, representing (respectively) continuous and discrete time. In these two cases the functions $u \in \mathcal{C}[\mathbb{T}, A]$ are called "streams", and $\mathcal{C}[\mathbb{T}, A]$ is called a "stream space". These cases will be supported by two *running examples*, taken from our earlier work:

(1) *analog networks*, with continuous time $\mathbb{T} = \mathbb{R}^{\geq 0}$, using the theory developed in [TZ07], and especially the case study of a mass/spring/damper system investigated there;

(2) *synchronous concurrent algorithms* (SCAs), with discrete time $\mathbb{T}=\mathbb{N}$, using the theory developed in [TTZ09].

One of the main aims of this paper is to develop the two theories that analyse properties of networks processing digital and analog streams, introduced in [TZ07, TTZ09], from a common standpoint.

1.3 Results on operations on streams

The study of networks of processors lead us to stream transformers of the form

$$\Phi\colon A^r \times A^s \times \mathcal{C}[\mathbb{T}, A]^p \;\to\; \mathcal{C}[\mathbb{T}, A]^m$$

where for tuples of system parameters $\boldsymbol{c} \in A^r$, initial values $\boldsymbol{a} \in A^s$ and input streams $\boldsymbol{x} \in \mathcal{C}[\mathbb{T}, A]^p$,

$$\Phi(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in \mathcal{C}[\mathbb{T}, A]^m$$

is obtained as the *fixed point* of a *contracting operator*

$$F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}} :\; \mathcal{C}[\mathbb{T}, A]^m \;\to \mathcal{C}[\mathbb{T}, A]^m \tag{1.1}$$

where

$$F\colon A^r \times A^s \times \mathcal{C}[\mathbb{T}, A]^p \;\to\; (\mathcal{C}[\mathbb{T}, A]^m \;\to\; \mathcal{C}[\mathbb{T}, A]^m)$$

is represented more conveniently in the uncurried form:

$$F\colon A^r \times A^s \times \mathcal{C}[\mathbb{T}, A]^p \times \mathcal{C}[\mathbb{T}, A]^m \;\to\; \mathcal{C}[\mathbb{T}, A]^m,$$

so that $F(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}, \;\cdot\;) = F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}$ in (1.1).

We assume that F satisfies a *causality* condition (discussed in Section 3), which is natural in the context of stream processing and turns out to be crucial in the proofs of the following theorems.[2] First, we establish:

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[2] Interestingly, it is not clear how to define (or even make sense of) the concept of causality in the general case for X (taking, for example, $X = \mathbb{Z}^2$ or $\mathbb{R}^3$).

Theorem 1 (Existence and uniqueness): *If F is contracting and causal, then Φ exists and is unique.*

Next, in Section 4, we verify

Theorem 2 (Continuity): *If F is contracting, causal and continuous, then Φ is continuous.*

1.4 Physical interpretation of the results

The significance of Theorems 1 and 2 is that *continuity* implies the *stability* of the fixed point solution Φ to the specification given by F with respect to the system parameters, initial values and input streams. This means that small changes in tuples of system parameters $\boldsymbol{c} \in A^r$, initial values $\boldsymbol{a} \in A^s$ and input streams $\boldsymbol{x} \in \mathcal{C}[\mathbb{T}, A]^p$ will result in small changes in the behaviour of the systems as defined by $\Phi(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in \mathcal{C}[\mathbb{T}, A]^m$. Here "small" is measured by any topology chosen for the task in hand.

The significance of continuity is expressed in *Hadamard's principle* [Had52] which, in the present context, can be (re-)formulated in the form [CH53, Had64]:

> *for a model of a physical system to be acceptable, the behaviour of the model must depend continuously on the data.*

This principle formalises the fact that if the system's behaviour depends significantly on small perturbations in its data, then it cannot behave in a stable fashion and its physical observation cannot be reliable. This is because, for example, repeating an experiment or computation will involve small variations of physical data, and for the system to be observable the corresponding variation in behaviour must also be small. Here observation is a form of classical measurement, of course. (See also Discussion 4.2.14.)

Simpler forms of Theorems 1 and 2 were proved in [TZ07] for a stronger notion of contraction of the operator F. The proofs here (especially of Theorem 2) are much more intricate. The notion used here (unlike the stronger one) is satisfied by the case study associated with our running example of analog networks.

Thus, Theorems 1 and 2 are the basis for a general method of giving semantics to interesting classes of analog networks. The freedom to choose topologies appropriate to the physics of the problem, and work with conventional approximation methods, is an attractive feature of this method, which, we feel, makes up for the previous apparent neglect of suitable semantics for analog networks.

This paper seeks to compare, and partially unify, theories of stream transformers on $\mathcal{C}[\mathbb{T}, A]$ for discrete and continuous time $\mathbb{T}$. It is motivated by models of network stream processing in [TZ07, TTZ09]. The methods are those of [TZ04, TZ05, TZ07, TTZ09]. We have tried to make this paper independent of these articles; however, the motivation and technicalities are best apprehended in the light of our entire work.

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2 Function spaces

One aim of this paper is to use basic topology to model stream processing. The standard ideas and methods of topology constrain our models to streams that are continuous, and also (for now) total and deterministic.[3]. However, even under these constraints, there is no shortage of interesting examples and applications.

Most of the definitions and results we need can be found in standard topology texts such as [HY61, Kel55, Eng89, Sim63].

2.1 Topology of uniform convergence

Let (X, d_X) and (A, d_A) be two metric spaces. Let $\mathcal{C}[X, A]$ be the set of continuous functions from X to A.

Examples 2.1.1 (Two running examples). The theory of this paper will be applied to the following two cases, which will form running examples throughout.

• ***Example 1: Analog networks*** [TZ07]. Here X is the set $\mathbb{R}^{\geq 0}$ of non-negative reals, modelling continuous time, and A is typically (though not necessarily) Euclidean n-space $\mathbb{R}^n$.

• ***Example 2: Synchronous concurrent algorithms*** (***SCAs***) [TT91, TTZ09]. Here X is the set $\mathbb{N}$ of non-negative integers, modelling discrete time, and A is typically $\mathbb{R}^n$ or $[0, 1]^n$.

Elements of $\mathcal{C}[X, A]$ will be denoted $u, v, \ldots$. From Section 3 onwards, where X is assumed to model either continuous or discrete time, these elements of $\mathcal{C}[X, A]$ will be called *streams*, and $\mathcal{C}[X, A]$ will be called a *stream space*.

Assume, first, that X is a compact set K. Then $\mathcal{C}[K, A]$ is easily metrisable, with the metric

$$\mathsf{d}_\mathsf{C}(u, v) \;=_{df}\; \sup_{x \in K} \mathsf{d}_A(u(x), v(x)).$$

The resulting topology is the *topology of uniform convergence* on $\mathcal{C}[K, A]$.

We will always use $K, K', \ldots$ for compact sets.

Lemma 2.1.2. *If K is compact and A is complete, then $\mathcal{C}[K, A]$ is complete.*

Proof: Let (u_n) be a Cauchy sequence in $\mathcal{C}[K, A]$, i.e., a uniform Cauchy sequence on K. By completeness of A, it has a pointwise limit u, i.e., for all $x \in K$, $u(x) = \lim_n u_n(x)$. By

[3] But see §5.2(2) below.

a standard argument [Rud76], u is continuous, and also the uniform limit of the sequence (u_n), i.e., the limit in $\mathcal{C}[K, A]$. $\square$

Without the assumption that X is compact, $\mathcal{C}[X, A]$ is still "locally metrisable", with the topology of *uniform convergence on compacta*, or the *local uniform topology*, which is generated by neighbourhoods of points $u \in \mathcal{C}[X, A]$ of the form

$$\mathbf{N}_K(u, \epsilon) =_{df} \{ v \in \mathcal{C}[X, A] \mid \mathsf{d}_K(u, v) < \epsilon \} \tag{2.1}$$

for all compact $K \subseteq X$ and $\epsilon > 0$, where d_K is the pseudometric defined by

$$\mathsf{d}_K(u, v) =_{df} \sup_{x \in K} \mathsf{d}_A(u(x), v(x)). \tag{2.2}$$

In fact, in the case that X is σ-compact (see §2.3), the space $\mathcal{C}[X, A]$, with the local uniform topology, is metrisable (Metrisability Lemma 2.3.11).

Another characterisation of this topology on $\mathcal{C}[X, A]$ can be given using the notions of an *inverse system* of topological spaces, and the *inverse limit* of such a system [HY61, Eng89]. So consider the inverse system consisting of the family of topological spaces

$$\langle \mathcal{C}[K, A] \mid K \subseteq X \rangle \tag{2.3a}$$

each with the topology of uniform convergence, directed by the partial order

$$\mathcal{C}[K, A] \le \mathcal{C}[K', A] \quad \Longleftrightarrow_{df} \quad K \subseteq K' \tag{2.3b}$$

with maps

$$\iota_{K',K} : \mathcal{C}[K', A] \to \mathcal{C}[K, A] \qquad (K \subseteq K') \tag{2.3c}$$

where $\iota_{K',K}$ is the restriction: $\iota_{K',K}(u) = u\restriction_K$.

The *inverse limit* of this family is the space $\mathcal{C}[X, A]$, with the topology generated by the family of maps

$$\iota_K : \mathcal{C}[X, A] \to \mathcal{C}[K, A] \qquad (K \subseteq X),$$

i.e., the "least" topology on $\mathcal{C}[X, A]$ which makes these maps continuous, where ι_K is the restriction: $\iota_K(u) = u\restriction_K$.

The following lemma can be easily checked.

Lemma 2.1.3. *The inverse limit topology on $\mathcal{C}[X, A]$, as defined above, is the same as the local uniform topology.*

A third characterisation of the topology on $\mathcal{C}[X, A]$ can be given, as the *compact-open topology* [Eng89, §3.4], which is defined as having *subbasic open sets* of the form

$$\mathbf{M}(K, U) =_{df} \{ u \in \mathcal{C}[X, A] \mid \forall x \in K : u(x) \in U \} \tag{2.4}$$

for all compact subsets K of X and open subsets U of A.

For a point x in any metric space, let $\mathbf{N}(x, r)$ and $\mathbf{N}[x, r]$ denote, respectively, the open and closed neighbourhoods of x with radius r.

Lemma 2.1.4. *On $\mathcal{C}[X, A]$, the compact-open topology is the same as the local uniform topology.*

Proof: For the sake of completeness we give a proof here. (Proofs for spaces more general than metric spaces can be found in [Kel55, Ch. 7], [Eng89, 8.2.6].) Let $U, \ldots$ range over open subsets of A.

(i) To show the local uniform topology is at least as fine as the compact open topology: Let $u \in \mathbf{M}(K, U)$. We must find some $\epsilon > 0$ such that

$$\mathbf{N}_K(u, \epsilon) \;\subseteq\; \mathbf{M}(K, U). \tag{2.5}$$

For all $x \in K$, $u(x) \in U$, and so, since U is open, there exists $\epsilon_x > 0$ such that $\mathbf{N}(u(x), \epsilon_x) \subseteq U$. Further, by continuity of u, for all $x \in K$ there exists $\delta_x > 0$ such that for all $x' \in \mathbf{N}(x, \delta_x)$, $\mathsf{d}(u(x), u(x')) < \epsilon_x/2$, and hence $\mathbf{N}(u(x'), \epsilon_x/2) \subseteq U$.

By compactness of K, there exists a finite number of points $x_1, x_2, \ldots, x_N$ in K such that $K \subseteq \bigcup_{i=1}^{N} \mathbf{N}(x_i, \delta_{x_i})$. Let

$$\epsilon \;=\; \min(\frac{\epsilon_{x_1}}{2}, \ldots, \frac{\epsilon_{x_N}}{2}).$$

Then ϵ satisfies (2.5).

(ii) To show that the compact open topology is at least as fine as the local uniform topology: Given K and u, consider a neighbourhood $\mathbf{N}_K(u, \epsilon)$ of u. By continuity of u, for all $x \in K$ there exists $\delta_x > 0$ such that

$$\forall y \in \mathbf{N}[x, \delta_x] : \mathsf{d}(u(x), u(y)) < \epsilon/2. \tag{2.6}$$

Hence

$$\forall x \in K : \; \forall y \in \mathbf{N}[x, \delta_x] : \quad \mathbf{N}(u(x), \epsilon/2) \;\subseteq\; \mathbf{N}(u(y), \epsilon). \tag{2.7}$$

By compactness of K, there exists a finite number of points $x_1, x_2, \ldots, x_N$ in K such that

$$K \;\subseteq\; \bigcup_{i=1}^{N} \mathbf{N}(x_i, \delta_{x_i}).$$

Let $C_i = \mathbf{N}[x_i, \delta_{x_i}] \cap K$ for $i = 1, \ldots, N$. Then each C_i is a closed subset of K, hence compact, and $K = \bigcup_{i=1}^{N} C_i$. Now put $U_i = \mathbf{N}(u(x_i), \epsilon/2)$. By (2.6) and (2.7), for $i = 1, \ldots, N$

$$u \in \mathbf{M}(C_i, U_i) \;\subseteq\; \mathbf{N}_{C_i}(u, \epsilon).$$

Hence

$$u \;\in\; \bigcap_{i=1}^{N} \mathbf{M}(C_i, U_i) \;\subseteq\; \mathbf{N}_K(u, \epsilon).$$

□

Corollary 2.1.5. *On $\mathcal{C}[X,A]$, the following three topologies are equivalent: the compact-open topology, the local uniform topology and the inverse limit topology.*

Proof: From Lemmas 2.1.3 and 2.1.4. □

Later we will see another equivalent formulation of this topology, in terms of a metric, in the case of σ-compactness of X (Metrisability Lemma 2.3.11).

2.2 Limits and Cauchy sequences in $\mathcal{C}[X,A]$

The space $\mathcal{C}[X,A]$ is "locally metrisable" by the pseudometrics d_K defined by (2.2). There are also "local" concepts of limit and Cauchy sequence.

Definition 2.2.1 (Local uniform convergence of a sequence in $\mathcal{C}[X,A]$).
A sequence (u_n) of elements of $\mathcal{C}[X,A]$ is said to *converge locally uniformly* to a limit $u \in \mathcal{C}[X,A]$ if

$$\forall K \subseteq X\, \forall \epsilon > 0\, \exists N\, \forall n \geq N : \mathsf{d}_K(u_n,\, u) \leq \epsilon.$$

Such a limit (if it exists) is easily seen to be unique:

Lemma 2.2.2. *If the sequence (u_n) converges locally uniformly to u and to v in $\mathcal{C}[X,A]$, then $u = v$.*

Lemma 2.2.3. *A point $u \in \mathcal{C}[X,A]$ is in the closure of a set $U \subseteq \mathcal{C}[X,A]$ if, and only if, there is a sequence of elements of U which converges locally uniformly to u.*

Definition 2.2.4 (Locally uniform Cauchy sequence). A sequence (u_n) of elements of $\mathcal{C}[X,A]$ is *locally uniformly Cauchy* if

$$\forall K\, \forall \epsilon > 0\, \exists N\, \forall m, n \geq N : \mathsf{d}_K(u_m,\, u_n) \leq \epsilon,$$

Lemma 2.2.5 (Local uniform completeness of $\mathcal{C}[X,A]$). *Suppose A is complete. Then $\mathcal{C}[X,A]$ is locally uniformly complete, in the sense that a locally uniform Cauchy sequence in $\mathcal{C}[X,A]$ converges locally uniformly to a limit.*

Proof: Let (u_n) be a locally uniform Cauchy sequence in $\mathcal{C}[X,A]$. For any K, the sequence $u_0{\restriction_K},\, u_1{\restriction_K}, \ldots$ is a uniform Cauchy sequence in the space $\mathcal{C}[K,A]$, and so, by completeness of $\mathcal{C}[K,A]$ (Lemma 2.1.2), has a (unique) limit $u^{(K)}$ in $\mathcal{C}[K,A]$. By uniqueness of limits (Lemma 2.2.2), these limits are *compatible*, in the sense that for any K, K' $u^{(K)}{\restriction_{K\cap K'}} = u^{(K')}{\restriction_{K\cap K'}}$. The desired limit u can then be defined as the common extension on X of all the $u^{(K)}$. □

2.3 σ-compact spaces; compact exhaustions.

Definition 2.3.1 (σ-compactness; compact exhaustions).

(a) A topological space X is *σ-compact* if it is a union of an increasing sequence of compact subsets (K_k):

$$X = \bigcup_{k=0}^{\infty} K_k \quad \text{where} \quad K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots .$$

(b) Further, this sequence (K_k) is called a *compact exhaustion* of X if for each compact $K \subseteq X$ there exists k such that $K \subseteq K_k$.

Remark 2.3.2. Not every σ-compact space has a compact exhaustion; i.e., condition (b) in Definition 2.3.1 is not redundant. For a counterexample, take $X = [0, 1]$, the unit real interval, and let $K_k = \{0\} \cup [\frac{1}{k}, 1]$ $(k = 1, 2, \dots)$. Then $\bigcup_k K_k = X$, but X itself, which is compact, is not contained in any K_k.

If we want an example in which the space X is not compact, we can modify the above example by taking $X = [0, \infty)$ and $K_k = \{0\} \cup [\frac{1}{k}, k]$. Then the compact subset $[0, 1] \subseteq X$ is not contained in any K_k.

Examples 2.3.3 (Compact exhaustions). Consider the space X in our two running examples (Examples 2.1.1).

• ***Example 1: Analog networks*** $(X = \mathbb{R}^{\geq 0})$. A compact exhaustion of X is given by $K_k = [0, k]$, or, more generally, $K_k = [0, k \cdot \tau]$ for some fixed $\tau > 0$.

• ***Example 2: SCAs*** $(X = \mathbb{N})$. A compact exhaustion of X is given by taking $K_k = \{0, 1, \dots, k\}$.

Remark 2.3.4. The fact that (K_k) forms a compact exhaustion of X in Example 2.3.3(1) follows from the Heine-Borel Theorem: a subset of $\mathbb{R}^{\geq 0}$ (or of $\mathbb{R}^n$) is compact if, and only if, it is closed and bounded [Rud76]. As for Example 2, a subset of $\mathbb{N}$ is compact if, and only if, it is finite.

For the rest of this paper we assume:

Assumption 2.3.5 (σ-compactness). *The space X is σ-compact, with compact exhaustion (K_k).*

As we have seen, this applies to our two running examples, with $X = \mathbb{R}^{\geq 0}$ and $X = \mathbb{N}$.

Now much of the work in §§2.1 and 2.2 remains valid when one restricts attention to the compact sets in the particular compact exhaustion (K_k). (See Lemmas 2.3.6, 2.3.7, 2.3.9 and 2.3.10 below. But see also Remark 2.3.8.)

Lemma 2.3.6. *The local uniform topology on $\mathcal{C}[X, A]$ (cf. equation (2.1)) is the same as that generated by the neighbourhoods*

$$\mathbf{N}_{K_k}(u, \epsilon) \quad = \quad \{ v \in \mathcal{C}[X, A] \mid \mathsf{d}_{K_k}(u, v) < \epsilon \}$$

for $k = 0, 1, 2, \ldots$ *and* $\epsilon > 0$.

The main step in proving this is to show that for any neighbourhood $\mathbf{N}_K(u, \epsilon)$ in the local uniform topology there is a k such that

$$u \;\in\; \mathbf{N}_{K_k}(u, \epsilon) \;\subseteq\; \mathbf{N}_K(u, \epsilon).$$

This follows by choosing k such that $K \subseteq K_k$, which is possible by property (b) in the definition (2.3.1) of σ-compactness.

From this follows also (cf. equations (2.3) and Lemma 2.1.3):

Lemma 2.3.7. *The inverse limit topology on* $\mathcal{C}[X, A]$ *is the same as that generated by the inverse system consisting of the family* $\langle \mathcal{C}[K_k, A] \mid k \in \mathbb{N} \rangle$ *directed by the (total) order*

$$\mathcal{C}[K_k, A] \;\leq \mathcal{C}[K_l, A] \quad \iff_{df} \quad k \leq l$$

with the restriction maps $\iota_{l,k} \colon \mathcal{C}[K_l, A] \to \mathcal{C}[K_k, A]$ $(k \leq l)$ *as before.*

Remark 2.3.8. However, the compact-open topology on $\mathcal{C}[X, A]$ *cannot* be re-defined by having subbasic open sets only of the form $\mathbf{M}(K_k, U)$ (cf. (2.4)).

Next, we give equivalent formulations of the concepts in §2.2 (limits, Cauchy sequences) which refer only to the compact sets (K_k) of the given exhaustion, using Lemma 2.3.6. (Compare Definitions 2.2.1 and 2.2.4).

Lemma 2.3.9. *Let* (u_n) *be a sequence of elements of* $\mathcal{C}[X, A]$ *and let* $u \in \mathcal{C}[X, A]$. *The following are equivalent:*

(a) *The sequence* (u_n) *converges locally uniformly to* u,

(b) $\forall k \, \forall \epsilon > 0 \, \exists N \, \forall n \geq N : \mathsf{d}_{K_k}(u_n, \, u) \leq \epsilon$,

(c) $\forall k \, \exists N \, \forall n \geq N : \mathsf{d}_{K_k}(u_n, \, u) \leq 2^{-k}$.

Lemma 2.3.10. *Let* (u_n) *be a sequence of elements of* $\mathcal{C}[X, A]$. *The following are equivalent:*

(a) *The sequence* (u_n) *is locally uniformly Cauchy,*

(b) $\forall k \, \forall \epsilon > 0 \, \exists N \, \forall m, n \geq N : \mathsf{d}_{K_k}(u_m, \, u_n) \leq \epsilon$,

(c) $\forall k \, \exists N \, \forall n \geq N : \mathsf{d}_{K_k}(u_n, \, u) \leq 2^{-k}$.

Next, under the σ-compactness assumption, $\mathcal{C}[X, A]$ is metrisable, as follows. Define

$$\mathsf{d}_{\mathsf{C}}(u, v) \;=_{df} \sum_{k=1}^{\infty} \min\big(\, \mathsf{d}_{K_k}(u, v), \; 2^{-k} \,\big). \tag{2.8}$$

In connection with this metric, we use the following notation. For $u \in \mathcal{C}[X,A]$, the *open ball* with centre u and radius ϵ is

$$\mathbf{B}(u,\epsilon) \;=_{df}\; \{\, v \in \mathcal{C}[X,A] \mid \mathsf{d}_{\mathsf{C}}(u,v) < \epsilon \,\}.$$

Lemma 2.3.11 (Metrisability). *d_{C} is a metric on $\mathcal{C}[X,A]$ which produces the inverse limit topology. Furthermore, if (u_n) is a sequence of elements of $\mathcal{C}[X,A]$, and $u \in \mathcal{C}[X,A]$, then*

(a) *(u_n) is locally uniformly Cauchy iff (u_n) is Cauchy w.r.t. d_{C},*

(b) *u_n converges locally uniformly to u iff u_n converges to u w.r.t. d_{C},*

(c) *if A is complete, then $\mathcal{C}[X,A]$ is complete w.r.t. d_{C}.*

Proof: First note the following (easily proved):

(1) for any k, if $\mathsf{d}_{K_k}(u,v) \le 2^{-k}$, then $\mathsf{d}_{\mathsf{C}}(u,v) \le (k+1)\cdot 2^{-k}$,

(2) for any k, if $\mathsf{d}_{K_k}(u,v) \ge 2^{-k}$, then $\mathsf{d}_{\mathsf{C}}(u,v) \ge 2^{-k+1}$.

From (1) follows

$$\mathbf{N}_{K_{2k}}(u, 2^{-2k}) \;\subseteq\; \mathbf{B}(u, 2^{-k})$$

(for $k > 2$), and from (2):

$$\mathbf{B}(u, 2^{-(k+1)}) \;\subseteq\; \mathbf{N}_{K_k}(u, 2^{-k}).$$

From these follows the equivalence of the two topologies, and also parts (a) and (b). Part (c) follows from (a) and (b). $\square$

2.4 Product spaces

The product space $\mathcal{C}[X,A]^m$ has the product topology, which, by definition, is the topology generated by the projections

$$\pi_i \colon \mathcal{C}[X,A]^m \;\to\; \mathcal{C}[X,A].$$

where

$$\pi_i(u_1,\dots,u_m) \;=\; u_i \qquad (i = 1,\dots,m).$$

We denote the members of $\mathcal{C}[X,A]^m$, or function tuples, by $\boldsymbol{u} \;=\; (u_1,\dots,u_m)$.

Lemma 2.4.1. *The product topology on $\mathcal{C}[X,A]^m$ can be characterised as any one of the following:*

(i) *the topology generated by the products of the basic open sets of $\mathcal{C}[X,A]$:*

$$\prod_{i=1}^{m} \mathbf{N}_{K_i}(u_i,\epsilon_i) \tag{2.9}$$

for all compact $K_i \subseteq X$, $\epsilon_i > 0$ *and* $u_i \in \mathcal{C}[X, A]$ $(i = 1, \ldots, m)$*;*

(ii) *the topology generated by the following:*

$$\prod_{i=1}^{m} \mathbf{N}_K(u_i, \epsilon) \tag{2.10}$$

for all compact $K \subseteq X$, $\epsilon > 0$ *and* $u_i \in \mathcal{C}[X, A]$*;*

(iii) *the topology given by the pseudometrics*

$$\mathsf{d}_K^m(\boldsymbol{u}, \boldsymbol{v}) \ =_{df} \ \left(\sum_{i=1}^{m} \mathsf{d}_K(u_i, v_i)^p \right)^{\frac{1}{p}} \tag{2.11}$$

(*where* $\boldsymbol{u} = (u_1, \ldots, u_m)$ *and* $\boldsymbol{v} = (v_1, \ldots, v_m)$) *for any fixed* p $(1 \leq p \leq \infty)$ *and all compact* $K \subseteq X$.

Note that two common special cases of (2.11) are formed by taking $p = 1$:

$$\mathsf{d}_K^m(\boldsymbol{u}, \boldsymbol{v}) \ = \ \sum_{i=1}^{m} \mathsf{d}_K(u_i, v_i)$$

and $p = \infty$:

$$\mathsf{d}_K^m(\boldsymbol{u}, \boldsymbol{v}) \ = \ \max_{i=1}^{m} \mathsf{d}_K(u_i, v_i). \tag{2.12}$$

which corresponds exactly to (ii), in the sense that the neighbourhood (2.10) is just $\{\, \boldsymbol{v} \mid \mathsf{d}_K^m(\boldsymbol{u}, \boldsymbol{v}) < \epsilon \,\}$.

We omit proofs, except to remark that the equivalence of the systems of open bases in (i) and (ii) can be seen by observing that any neighbourhood of a function tuple $\boldsymbol{u}$ of the form (2.9) contains a neighbourhood of $\boldsymbol{u}$ of the form (2.10), formed by defining

$$K \ = \ K_1 \cup \cdots \cup K_m \qquad \text{and} \qquad \epsilon \ = \ \min(\epsilon_1, \ldots, \epsilon_m),$$

and the equivalence of this topology with (iii) follows from the Hölder inequality [Rud76, Roy63].

We will usually drop the superscript 'm' from d_K^m.

Corollary 2.4.2. *$\mathcal{C}[X, A]^m$ is homeomorphic to $\mathcal{C}[X, A^m]$, under the mapping*

$$\boldsymbol{u} \ = \ (u_1, \ldots, u_m) \ \mapsto \ \widehat{\boldsymbol{u}} \tag{2.13}$$

where for all $x \in X$,

$$\widehat{\boldsymbol{u}}(x) \ = \ (u_1(x), \ldots, u_m(x)).$$

Proof: Under the mapping (2.13), and using the "$p = \infty$" pseudometric (2.12), the neighbourhood (2.10) can be rewritten as $\mathbf{N}_K(\widehat{\boldsymbol{u}}, \epsilon)$. $\square$

Hence $\mathcal{C}[X, A]^m$ can, for all practical purposes, be identified with $\mathcal{C}[X, A^m]$. In this way, many of our results for spaces $\mathcal{C}[X, A]$ can be easily seen to hold for $\mathcal{C}[X, A]^m$.

3 Stream spaces; Contracting operators and fixed points

3.1 Basic assumptions: continuous and discrete time

The work in this and the following sections applies to the cases where X represents *time*, either *continuous time* $X = \mathbb{R}^{\geq 0}$ or *discrete time* $X = \mathbb{N}$. Each of these cases includes one of our two running examples (Examples 2.1.1).

We will therefore henceforth write '$\mathbb{T}$' for X, with elements $t, t', \ldots, T, \ldots$. The space $\mathcal{C}[\mathbb{T}, A]$ is then the space of (respectively) *continuous* or *discrete A-valued streams*.

Note that in the discrete case $\mathbb{T} = \mathbb{N}$, *any* function from $\mathbb{T}$ to A is continuous, since $\mathbb{T}$ is discrete; hence $\mathcal{C}[\mathbb{T}, A] = [\mathbb{T} \to A]$, the set of all functions from $\mathbb{T}$ to A.

Note also that in both cases ($\mathbb{T} = \mathbb{R}^{\geq 0}$ and $\mathbb{T} = \mathbb{N}$), $\mathbb{T}$ is σ-compact (Assumption 2.3.5), with *standard exhaustions*, as we now specify:

Assumption 3.1.1 (Standard compact exhaustions).
As compact exhaustions of $\mathbb{T}$, we take, for $k = 0, 1, 2, \ldots$:

(1) *in the continuous case $\mathbb{T} = \mathbb{R}^{\geq 0}$, $K_k = [0,\ k \cdot \tau]$ for some fixed $\tau > 0$; and*

(2) *in the discrete case $\mathbb{T} = \mathbb{N}$, $K_k = \{0, 1, \ldots, k\}$.*

From now on, we use only the above standard exhaustions of $\mathbb{T}$.

By the metrisability lemma (2.3.11), $\mathcal{C}[\mathbb{T}, A]$ is metrisable. We also assume, from now on:

Assumption 3.1.2 (Completeness of A). *A is a complete metric space.*

Remark 3.1.3 (Completeness of $\mathcal{C}[\mathbb{T}, A]$). It follows, by Lemmas 2.2.5 and 2.3.11(c), that $\mathcal{C}[\mathbb{T}, A]$ is locally uniformly complete, and also (metrically) complete.

We will consider *operators* on the function space $\mathcal{C}[\mathbb{T}, A]$, mainly the form

$$F\colon A^q \times \mathcal{C}[\mathbb{T}, A]^p \ \to\ (\mathcal{C}[\mathbb{T}, A]^m \ \to\ \mathcal{C}[\mathbb{T}, A]^m) \tag{3.1}$$

For convenience, we usually representing F in the uncurried form

$$F\colon A^q \times \mathcal{C}[\mathbb{T}, A]^p \times \mathcal{C}[\mathbb{T}, A]^m \ \to\ \mathcal{C}[\mathbb{T}, A]^m. \tag{3.2}$$

Then for $\boldsymbol{a} \in A^q$ and $\boldsymbol{x} \in \mathcal{C}[\mathbb{T}, A]^p$, $F_{\boldsymbol{a},\boldsymbol{x}}$ is the operator

$$F_{\boldsymbol{a},\boldsymbol{x}} = F(\boldsymbol{a}, \boldsymbol{x},\ \cdot\ )\colon\ \mathcal{C}[\mathbb{T}, A]^m \ \to\ \mathcal{C}[\mathbb{T}, A]^m. \tag{3.3}$$

Examples 3.1.4 (Network stream transformers). Operators of the form (3.1) arise naturally in modelling *networks* of modules or processors, operating in either continuous or discrete time, as in our two running examples (2.1.1) of analog networks and SCAs, where A^q is the space of parameters from A, and $\mathcal{C}[\mathbb{T}, A]^p$ as the space of input streams. In fact these examples have a common form: the semantics of the network N is given by a *network stream transformer* Φ^N. This is obtained in two different ways in these two examples:

• ***Example 1: Analog networks***. Here $\Phi^N\colon A^q \times \mathcal{C}[\mathbb{T}, A]^p \to \mathcal{C}[\mathbb{T}, A]^m$ is obtained with $\Phi^N(\boldsymbol{a}, \boldsymbol{x})$ given as the *fixed point* of the *network state function* $F^N(= F$ in (3.1); cf. Theorem 1 below), under certain conditions (notably "contraction" and "causality" properties of F^N). The network state function F^N is formed from the *module functions* of N by simple vectorisation [TZ07].

• ***Example 2: SCAs***. Here Φ^N can be obtained directly from the module functions by *simultaneous primitive recursion* [TTZ09].

Both examples are discussed in greater detail in Examples 3.3.5 below.

Remark 3.1.5 (Network module functions in the two examples). The difference of approach between the two running examples in constructing Φ^N can be understood by noting that whereas the module functions of an analog network have the form (3.2), i.e., they are *stream transformers*, the module functions of an SCA have the form $\mathsf{F}\colon A^m \to A$, i.e., they operate from tuples of data to data.

Note also, however, that the network stream transformer Φ^N of an SCA network can *also* be obtained as a fixed point of a contracting operator! See Example 3.3.5(2) below.

Remark 3.1.6 (Network state function just vectorisation of module functions). In the case of an analog network N, the fact that the network state function F^N is formed from the *module functions* of N by simple vectorisation means that many interesting properties of the module functions, such as continuity or computability are easily seen to be inherited by F^N. (See Remark 4.2.11 and corresponding remarks in [TZ11].)

3.2 Causality of operators

We discuss two properties of operators that are important in application areas such as control theory [Son90, OW97].

An operator F as in (3.1) is said to satisfy *causality* if the output is "causally" related to the inputs, in the sense that *the output at any time depends only on the inputs up to that time*. We will give an exact definition below.

Discussion 3.2.1 (Causality and restriction to stream spaces). As we will see, *causality* of the contracting operator is a significant and natural assumption in our modelling, and in the proofs of our theorems. Interestingly, there does not seem to be an obvious generalisation of this property to functions with spatial domains (cf. Examples (iii)–(v) in §1.1). In the case $X = \mathbb{R}^3$, for example, it is not at all clear how one would define the concept of causality, or indeed what such a concept would mean here.

The same remarks apply to the concept of *shift invariance* (to be defined later, in §4.1), which is crucial in the proofs of Theorem 2.

Notation 3.2.2. For $0 \le a < b$ and $T > 0$, we write

(a) $\mathsf{d}_{a,b}(\boldsymbol{u}, \boldsymbol{v}) \ =_{df} \ \sup_{a \le t \le b} \mathsf{d}_A(\boldsymbol{u}(t), \boldsymbol{v}(t))$.

(b) $\mathsf{d}_T(\boldsymbol{u}, \boldsymbol{v}) \ =_{df} \ \mathsf{d}_{0,T}(\boldsymbol{u}, \boldsymbol{v}) \ = \ \sup_{0 \le t \le T} \mathsf{d}_A(\boldsymbol{u}(t), \boldsymbol{v}(t))$.

Note that $\mathsf{d}_0(\boldsymbol{u}, \boldsymbol{v}) \;=\; \mathsf{d}_A(\boldsymbol{u}(0), \boldsymbol{v}(0))$.

Notation 3.2.3. For $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}[\mathbb{T}, A]^m$ and $T \geq 0$, we write

(a) $\boldsymbol{u}{\restriction}_T =_{df} \boldsymbol{u}{\restriction}_{[0,T]}$

(b) $\boldsymbol{u}{\restriction}_{<T} =_{df} \boldsymbol{u}{\restriction}_{[0,T)}$

(c) With each $\boldsymbol{u} \in \mathcal{C}[[0,T], A]^m$ we associate the element $\boldsymbol{ext}_T(\boldsymbol{u})$ of $\mathcal{C}[\mathbb{T}, A]^m$ that extends $\boldsymbol{u}$ with a constant value equal to $\boldsymbol{u}(T)$, i.e.,

$$\boldsymbol{ext}_T(\boldsymbol{u})(t) \;=\; \begin{cases} \boldsymbol{u}(t) & \text{if } \; t \leq T \\ \\ \boldsymbol{u}(T) & \text{if } \; t > T. \end{cases}$$

(d) With each operator F as in (3.2), we can associate an operator

$$F{\restriction}_T \colon A^q \times \mathcal{C}[\mathbb{T}, A]^p \times \mathcal{C}[[0,T], A]^m \;\to\; \mathcal{C}[[0,T], A]^n,$$

by

$$F{\restriction}_T(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{u}) \;=\; F(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{ext}_T(\boldsymbol{u})){\restriction}_T \;.$$

Let F be as in (3.2).

Definition 3.2.4 (Causality). F is *causal*, or satisfies ***Caus***, if for all $(\boldsymbol{a}, \boldsymbol{x}) \in A^q \times \mathcal{C}[\mathbb{T}, A]^p$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}[\mathbb{T}, A]^m$

$$\forall T \geq 0, \qquad \boldsymbol{u}{\restriction}_{<T} = \boldsymbol{v}{\restriction}_{<T} \;\Longrightarrow\; F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{u})(T) \;=\; F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{v})(T). \tag{3.4}$$

Remarks 3.2.5. (a) Causality of F implies for all $\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}$:

$$F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{u})(0) \;=\; F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{v})(0)$$

by putting $T = 0$ in (3.4), since the antecedent of the implication is then trivially satisfied.

(b) Clearly, (3.4) is equivalent to the condition:

$$\forall T \geq 0, \qquad \boldsymbol{u}{\restriction}_{<T} = \boldsymbol{v}{\restriction}_{<T} \;\Longrightarrow\; F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{u}){\restriction}_T = F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{v}){\restriction}_T \;. \tag{3.5}$$

(c) If $\mathbb{T} = \mathbb{R}^{\geq 0}$ (analog network example), then this in turn is equivalent to the (apparently weaker) pair of conditions

$$\forall T > 0 \qquad \boldsymbol{u}{\restriction}_T = \boldsymbol{v}{\restriction}_T \;\Longrightarrow\; F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{u}){\restriction}_T \;=\; F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{v}){\restriction}_T, \tag{3.6a}$$

and

$$F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{u})(0) \;=\; F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{v})(0) \tag{3.6b}$$

since, by continuity, for $T > 0$

$$\boldsymbol{u}\restriction_{<T} = \boldsymbol{v}\restriction_{<T} \implies \boldsymbol{u}\restriction_T = \boldsymbol{v}\restriction_T .$$

This is not the case when $\mathbb{T} = \mathbb{N}$ (SCA example), where (3.6) is strictly weaker than (3.4) or (3.5).

Lemma 3.2.6. *If F is causal, then we can characterise the operator $F\restriction_T$ on $\mathcal{C}[[0,T],A]$ by*

$$F\restriction_T(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{u}\restriction_T) = F(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{u})\restriction_T .$$

Lemma 3.2.7. *For any $T > 0$,*

(a) *the injection $\iota_T^m \colon \mathcal{C}[[0,T],A]^m \to \mathcal{C}[\mathbb{T},A]^m$, defined by $\iota_T^m(\boldsymbol{u}) = \boldsymbol{ext}_T(\boldsymbol{u})$, is continuous;*

(b) *the projection $\pi_T^n \colon \mathcal{C}[\mathbb{T},A]^n \to \mathcal{C}[[0,T],A]^n$, defined by $\pi_T^n(\boldsymbol{u}) = \boldsymbol{u}\restriction_T$, is continuous.*

We need the following lemma for Theorem 2.

Lemma 3.2.8. (a) *If F is continuous, then so is $F\restriction_T$ for all $T > 0$.*

(b) *Conversely: Assume F is causal.*

(i) *If $F\restriction_T$ is continuous for all $T > 0$, then F is continuous.*

(ii) *Given any unbounded sequence $0 < T_1 < T_2 < \ldots$, if $F\restriction_{T_i}$ is continuous for $i = 1, 2, \ldots$, then F is continuous.*

Proof: For (a), use Lemmas 3.2.6 and 3.2.7, and the fact that $F\restriction_T = \pi_T^n \circ F \circ \iota_T^m$. For (b), use Lemma 3.2.6. $\square$

Remark 3.2.9 (Counterexample). Here is a counterexample to show that Lemma 3.2.8(*b*) needs the assumption ***Caus***. First we need some notation. For any $a \in A$, let ***const***(a) be the stream with constant value a. Also, for $u \in \mathcal{C}[\mathbb{T},\mathbb{R}]$ and $X \subseteq \mathbb{T}$, let $\sup_X(u) =_{df} \sup_{t \in X} u(t)$. Now let $A = \mathbb{R}$, and define $F \colon \mathcal{C}[\mathbb{T},A] \to \mathcal{C}[\mathbb{T},A]$ by

$$F(u) = \begin{cases} \boldsymbol{const}(\sup_{\mathbb{T}}(u)) & \text{if } u \text{ is bounded above on } \mathbb{T} \\ \\ \boldsymbol{const}(0) & \text{otherwise.} \end{cases}$$

Then clearly F does not satisfy causality. Also F is not continuous. For consider the sequence of streams

$$u_n(t) = \begin{cases} t & \text{if } t \leq n \\ n & \text{if } t > n. \end{cases}$$

Then (u_n) has a (pointwise, and locally uniform) limit v, where

$$v(t) = t \qquad \text{for all } t \geq 0.$$

Note that for all n, u_n is bounded, but v is unbounded. Further, $F(u_n) = \boldsymbol{const}(n)$, which does not have the limit $F(v) = \boldsymbol{const}(0)$. However, for all $T > 0$, $F{\restriction}_T$ is continuous, since for any stream u, $\sup_{[0,T]}(u)$ exists, and, as is easily seen, for any two streams u_1 and u_2,

$$|\sup_{[0,T]}(u_1) - \sup_{[0,T]}(u_2)| \;\le\; \mathsf{d}_T(u_1, u_2).$$

By checking the proof of Lemma 3.2.8(b), we can see that it also holds if "continuous" is replaced throughout by "uniformly continuous". Hence we have, for use in Section 6, the following

Lemma 3.2.10 (Test for uniform continuity). *Suppose F satisfies* ***Caus****, and for all $T > 0$:*

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}[\mathbb{T}, A]^m \left[\mathsf{d}_T(\boldsymbol{u}, \boldsymbol{v}) < \delta \implies \mathsf{d}_T(F(\boldsymbol{u}), F(\boldsymbol{v})) < \epsilon\right].$$

Then F is uniformly continuous.

Note that there is also a version of this lemma that uses discrete increments $T_1, T_2, \ldots$ instead of all $T > 0$, where we use part (ii) of Lemma 3.2.8(b).

3.3 Contracting operators

In this and the next few subsections we consider operators on $\mathcal{C}[\mathbb{T}, A]$ of the special form

$$F \colon \mathcal{C}[\mathbb{T}, A]^m \;\to\; \mathcal{C}[\mathbb{T}, A]^m. \tag{3.7}$$

This can be viewed either as a special case of (3.1), where $q = p = 0$, or (equivalently) as a case of (3.3), where we write 'F' instead of '$F_{\boldsymbol{a},\boldsymbol{x}}$' for simplicity.

Definition 3.3.1 (Contracting operator w.r.t. modulus and fixed increment). Let $0 < \lambda < 1$ and $\tau > 0$. F is *contracting w.r.t.* (λ, τ) if for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}[\mathbb{T}, A]^m$:

$$\text{for all } T \ge 0 \qquad \boldsymbol{u}{\restriction}_T = \boldsymbol{v}{\restriction}_T \implies \mathsf{d}_{T,\,T+\tau}(F(\boldsymbol{u}), F(\boldsymbol{v})) \;\le\; \lambda \cdot \mathsf{d}_{T,\,T+\tau}(\boldsymbol{u}, \boldsymbol{v}). \tag{3.8}$$

We then say that $F \in \boldsymbol{Contr}(\lambda, \tau)$, and call λ and τ the ***contraction modulus*** and ***contraction increment*** respectively.

Remark 3.3.2. Assuming F is causal, note that

$$\begin{aligned} \boldsymbol{u}{\restriction}_T = \boldsymbol{v}{\restriction}_T \quad &\implies \quad F(\boldsymbol{u}){\restriction}_T = F(\boldsymbol{v}){\restriction}_T \\ &\implies \quad \mathsf{d}_{T+\tau}(F(\boldsymbol{u}), F(\boldsymbol{v})) \;=\; \mathsf{d}_{T,\,T+\tau}(F(\boldsymbol{u}), F(\boldsymbol{v})). \end{aligned}$$

Hence (3.8) can be rewritten as

$$\text{for all } T \ge 0 \qquad \boldsymbol{u}{\restriction}_T = \boldsymbol{v}{\restriction}_T \implies \mathsf{d}_{T+\tau}(F(\boldsymbol{u}), F(\boldsymbol{v})) \;\le\; \lambda \cdot \mathsf{d}_{T+\tau}(\boldsymbol{u}, \boldsymbol{v}). \tag{3.8$'$}$$

Lemma 3.3.3. *If F is causal, $F \in \boldsymbol{Contr}(\lambda, \tau)$ and $0 < \tau' < \tau$, then $F \in \boldsymbol{Contr}(\lambda, \tau')$.*

Proof: Similar to the proof of [TZ07, Lemma 2.3.9], part (ii). (Warning! Note that "contracting" in [TZ07] means what we call "strongly contracting" in this paper: see Remark 3.3.7 below. Hence the proof is "similar", not identical.) $\square$

Remark 3.3.4 (Modulus of contraction as a family). A modulus of contraction could be given, more generally, as a *family of reals* $\langle \lambda_T \mid T \geq 0 \rangle$ with $0 < \lambda_T < 1$ for all $T \geq 0$, such that, for example, (3.8) would become

$$\text{for all } T \geq 0 \qquad \boldsymbol{u}\restriction_T = \boldsymbol{v}\restriction_T \quad \Longrightarrow \quad \mathsf{d}_{T,\,T+\tau}(F(\boldsymbol{u}), F(\boldsymbol{v})) \;\leq\; \lambda_T \cdot \mathsf{d}_{T,\,T+\tau}(\boldsymbol{u}, \boldsymbol{v}).$$

Thus we have a family of contraction moduli λ_T depending on time T (but a constant contraction modulus τ). Equivalently, we could consider a family $\langle \lambda_k \mid k \in \mathbb{N} \rangle$ such that

$$\text{for } k = 0, 1, 2, \ldots \quad \boldsymbol{u}\restriction_{k\tau} = \boldsymbol{v}\restriction_{k\tau} \quad \Longrightarrow \quad \mathsf{d}_{k\tau,\,(k+1)\tau}(F(\boldsymbol{u}), F(\boldsymbol{v})) \;\leq\; \lambda_k \cdot \mathsf{d}_{k\tau,\,(k+1)\tau}(\boldsymbol{u}, \boldsymbol{v}).$$

All the results obtained below still hold.

However we chose not to incorporate this generalisation in the exposition below, as it would lead to a surfeit of subscripts.[4]

Examples 3.3.5 (Contracting operators).

• ***Example 1: Two analog networks***.

Let us consider a practical example. In [TZ07] we analysed two case studies of mass/spring/damper systems (a simple and an iterated system, respectively). In this paper we reconsider the first of these (see Figure 1).

Here $A = \mathbb{R}$, and an analog network N_1 is constructed for this system [TZ07, Fig. 7] with the corresponding state function

$$F^{N_1}\colon\; \mathcal{C}[\mathbb{T}, \mathbb{R}]^3 \;\to\; \mathcal{C}[\mathbb{T}, \mathbb{R}]^3$$

where, putting

$$F^{N_1}(x, v, a) \;=\; (x', v', a')$$

we have, for $t \geq 0$:

$$x'(t) \;=\; \int_0^t v(s)ds \,+\, x_0 \tag{3.9a}$$

$$v'(t) \;=\; \int_0^t a(s)ds \,+\, v_0 \tag{3.9b}$$

$$a'(t) \;=\; \begin{cases} \frac{1}{M}(f(t) - Kx(t) - Dv(t)) & \text{if } \; t > 0 \\ \frac{1}{M}(f(t) - Kx_0 - Dv_0) & \text{if } \; t = 0 \end{cases} \tag{3.9c}$$

[4] As it is, the modulus of contraction for our network functions will be a *family* indexed by the stream inputs and network parameters (as in §4.2).

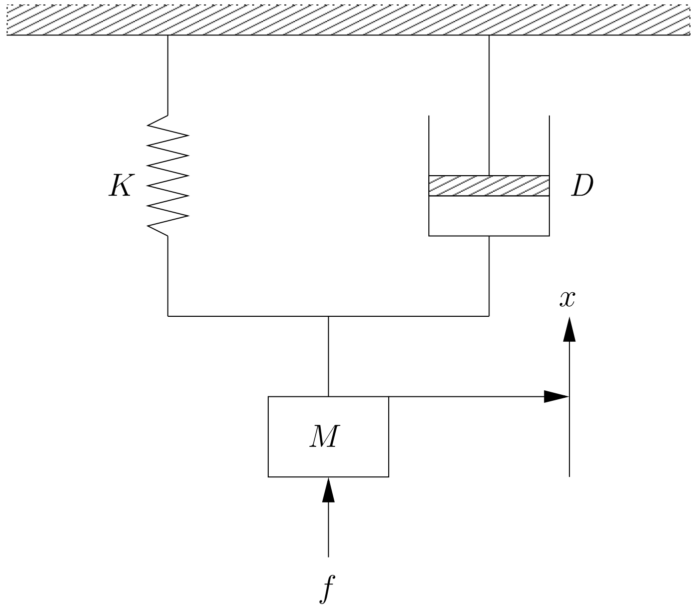

FIGURE 1: Case study: Mass/spring/damper system

with the *system parameters* M (mass), K (spring constant), D (damping constant); *initial values* x_0 (initial displacement), v_0 (initial velocity); *input stream* f (external force) and *remaining streams* x (displacement), v (velocity) and a (acceleration). (For now, the system parameters K, D, M and the initial values x_0, v_0 are taken as constant.)

The reason for form of the equational definition for $a'(t)$ given in (3.9c) instead of the simpler and (apparently) equivalent

$$a'(t) \;=\; \frac{1}{M}(f(t) - Kx(t) - Dv(t)) \qquad \text{for } \; t \geq 0 \tag{3.9c$'$}$$

is that the latter formulation *violates causality* for F^{N_1}. This is discussed further in [JZ11].

Now assume

$$M \;>\; \max(K, 2D). \tag{3.10}$$

Then, putting

$$\begin{aligned} \lambda \;&=_{df}\; \frac{\max(K, 2D)}{M}\,, \\ \tau \;&=_{df}\; \frac{D}{M} \end{aligned}$$

it is shown in [TZ07] that

$$F^{N_1} \in \boldsymbol{Contr}(\lambda, \tau).$$

Now in [JZ11] *another analog network* N_2 is constructed from the same mass/spring/ damper system (Figure 1) by eliminating the “acceleration” stream a from the network

N_1, using the fact that this stream can be defined as a linear combination of f, x and v (3.9a). So N_2 contains only two streams x, v (other than the input stream f) and network state function $F^{N_2}\colon\ \mathcal{C}[\mathbb{T},\mathbb{R}]^2 \rightarrow \mathcal{C}[\mathbb{T},\mathbb{R}]^2$ where, putting

$$F^{N_2}(x,v) \ = \ (x', v')$$

we have

$$x'(t) \ = \ \int_0^t v(s)ds \ + \ x_0, \tag{3.11a}$$

$$v'(t) \ = \ \frac{1}{M}\int_0^t (f(s) - Kx(s) - Dv(s))ds \ + \ v_0. \tag{3.11b}$$

This is a particular case of network state functions of the form

$$F\colon\ \mathcal{C}[\mathbb{T},\mathbb{R}]^m \ \rightarrow \ \mathcal{C}[\mathbb{T},\mathbb{R}]^m$$

$(m > 0)$ where

$$F(\boldsymbol{u})(t) \ = A\int_0^t \boldsymbol{u}(s)ds + \boldsymbol{x}(t) \tag{3.12}$$

where $A \in \mathbb{R}^{m\times m}$ (A nonzero) and $\boldsymbol{u}, \boldsymbol{x} \in \mathbb{R}^n$. In [JZ11] it is shown that for operators F of this kind,

$$F \in \boldsymbol{Contr}(\lambda,\tau) \ \text{ for } \ 0<\lambda<1 \ \text{ and } \ \tau \le \frac{\lambda}{\|A\|} \tag{3.13}$$

where $\|A\|$ is the matrix norm of A. (For convenience, we use the 'max' norm $\|\cdot\|_\infty$.) Applying this to the equations (3.11) for our first case study, we find that (3.13) holds, with

$$A \ = \ \begin{pmatrix} -\frac{K}{M} & -\frac{D}{M}, \\ 0 & 1 \end{pmatrix} \quad \text{and hence} \quad \|A\| \ = \ \max\big(\frac{K+D}{M},\ 1\big) \tag{3.14}$$

for all (positive) values of K, D, M, and all v_0, x_0 [JZ11].

A similar analysis applies to the second case study in [TZ07], involving an iterated mass/spring/damper system. We omit details.

• ***Example 2: SCAs***. Here the network stream transformer Φ^N is defined by *simultaneous primitive recursion*, [TTZ09, §7.1]. It can (hence) also be defined as the *fixed point* of a contracting operator, as we now show. Consider a function

$$\boldsymbol{f}\colon A^m \times [\mathbb{T} \rightarrow A]^p \times \mathbb{T} \ \rightarrow \ A^m$$

with the primitive recursive definition [TTZ09, §4.4]

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{a},\boldsymbol{x},0) \ &= \ \boldsymbol{g}(\boldsymbol{a},\boldsymbol{x}) \\ \boldsymbol{f}(\boldsymbol{a},\boldsymbol{x},t+1) \ &= \ \boldsymbol{h}(\boldsymbol{a},\boldsymbol{x},t,\boldsymbol{f}(\boldsymbol{a},\boldsymbol{x},t)) \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{g}&\colon A^m \times [\mathbb{T} \to A]^p \;\to\; A^m \\ \boldsymbol{h}&\colon A^m \times [\mathbb{T} \to A]^p \times \mathbb{T} \times A^m \;\to\; A^m \end{aligned}$$

Note that this is a simple recursion for an A^m-valued function, equivalent to the (m-fold) *simultaneous recursion* defining m A-valued functions given in [TTZ09, §4.4].

Now if we rewrite $\boldsymbol{f}$ in *curried form*

$$\boldsymbol{f}_{\boldsymbol{a},\boldsymbol{x}}(t) \;=_{df}\; \boldsymbol{f}(\boldsymbol{a},\boldsymbol{x},t)$$

then $\boldsymbol{f}_{\boldsymbol{a},\boldsymbol{x}}$ is a stream, or rather a stream m-tuple, and in fact, the *unique fixed point* of the *contracting operator*

$$F_{\boldsymbol{a},\boldsymbol{x}}\colon [\mathbb{T} \to A]^m \;\to\; [\mathbb{T} \to A]^m$$

defined by

$$F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{u}) \;=\; \boldsymbol{v}$$

where

$$\begin{aligned} \boldsymbol{v}(0) \;&=\; \boldsymbol{g}(\boldsymbol{a},\boldsymbol{x}) \\ \text{and for } t > 0 \qquad \boldsymbol{v}(t+1) \;&=\; \boldsymbol{h}(\boldsymbol{a},\boldsymbol{x},t,\boldsymbol{v}(t)). \end{aligned}$$

To show that $F_{\boldsymbol{a},\boldsymbol{x}}$ is contracting, we note that for all $T \in \mathbb{T}$ and $\boldsymbol{u},\boldsymbol{v} \in [\mathbb{T} \to A]^m$,

$$\boldsymbol{u}\restriction_T = \boldsymbol{v}\restriction_T \quad\Longrightarrow\quad F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{u})\restriction_{T+1} = F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{v})\restriction_{T+1}\,. \tag{3.15}$$

Hence by (3.15):

(i) $F_{\boldsymbol{a},\boldsymbol{x}} \in \boldsymbol{Contr}(\lambda, 1)$ for all $\lambda \in (0,1)$, and for all $(\boldsymbol{a},\boldsymbol{x}) \in A^m \times [\mathbb{T} \to A]^p$;

(ii) $F_{\boldsymbol{a},\boldsymbol{x}}$ is causal.

And so the existence of the network stream transformer for SCAs with unit delay [TTZ09, §4.4], which is defined there by a simple (simultaneous) primitive recursion, can be justified by, or reduced to, the theory of the present paper, using a fixed point construction based on contracting operators (see Theorem 1 below). However, this is not really necessary! The primitive recursive definition of the network stream transformer in [TTZ09, §4.4] is surely sufficient justification on its own for this function's existence.

Be that as it may, our fixed point construction applied to SCAs is along the lines of Kleene's construction in the proof of his first recursion theorem [Kle52, Thm XXVI], which in fact gives a justification of definition by recursion. Note, however, that this is obtained as the limit of a sequence of *partial streams*, starting with the empty stream, whereas the fixed point in our proof of Theorem 1 is obtained as a limit of a sequence of *total streams*, starting with an arbitrary stream. (At stage n, the approximations by these two methods give identical values at the first n places.) Thus, Kleene's framework involves *partial* functions, unlike the framework here and in [TTZ09]. See, however, Section 5.

Remark 3.3.6 (Strongly contracting operators). We can give a stronger contraction condition on operators, by removing the antecedent of (3.8) in Definition 3.3.1:

For $0 < \lambda < 1$ and $\tau > 0$, an operator F on $\mathcal{C}[\mathbb{T}, A]$ is said to be ***strongly contracting w.r.t.*** (λ, τ), or in $\boldsymbol{SContr}(\lambda, \tau)$, if for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}[\mathbb{T}, A]^m$:

$$\text{for all } T \geq 0 \qquad \mathsf{d}_{T,\,T+\tau}(F(\boldsymbol{u}), F(\boldsymbol{v})) \;\leq\; \lambda \cdot \mathsf{d}_{T,\,T+\tau}(\boldsymbol{u}, \boldsymbol{v}).$$

We could then develop a theory of *fixed points of strongly contracting operators*, which would in fact lead to much simpler proofs of Theorems 1 and 2 below (existence and continuity of fixed points).

However the concept of contracting operators, as we have defined it, seems more useful in practice. For instance, the stream transformers in the two case studies analysed in [TZ07, §4] for our analog network example (see [...]), as well as stream transformers in our SCA example (Example 3.3.5(2)), are all contracting, but (apparently) not strongly contracting.

Remark 3.3.7 (Different terminologies for contracting operators). In [TZ07] we used the terminology "weakly contracting" and "contracting", in place of (respectively) the present "contracting" and "strongly contracting". Our current terminology seems preferable, on the grounds of practical applicability, as explained in the previous remark.

3.4 Fixed point of contracting operator

We will prove the fixed point theorem (FPT) for contracting operators satisfying ***Caus***.

Remark 3.4.1. In fact, there is a simple direct proof of the FPT for *strongly contracting* operators (see Remark 3.3.6) without assuming causality [TZ07, Theorem 1].

Definitions 3.4.2. Let $T \geq 0$.

(a) $\boldsymbol{u}$ and $\boldsymbol{v}$ are *T-equivalent* if $\boldsymbol{u}{\restriction}_T = \boldsymbol{v}{\restriction}_T$.

(b) $\boldsymbol{u}$ is a *T-approximate fixed point* of F if $F(\boldsymbol{u}){\restriction}_T = \boldsymbol{u}{\restriction}_T$,

(c) $\boldsymbol{v}$ is a *T-approximate uniform limit* of a sequence $(\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \dots)$ if $\boldsymbol{v}{\restriction}_T$ is a uniform limit of the sequence $(\boldsymbol{u}_0{\restriction}_T,\ \boldsymbol{u}_1{\restriction}_T,\ \boldsymbol{u}_2{\restriction}_T, \dots)$ in $\mathcal{C}[[0,T], A]$.

Lemma 3.4.3. *If F is causal and $\boldsymbol{u}$ is a T-approximate fixed point of F, then $F(\boldsymbol{u})$ is also a T-approximate fixed point of F.*

Proof: Since $F(\boldsymbol{u}){\restriction}_T = \boldsymbol{u}{\restriction}_T$, by causality $F(F(\boldsymbol{u})){\restriction}_T = F(\boldsymbol{u}){\restriction}_T$. $\square$

Lemma 3.4.4. *If F is causal, then any stream in the range of F is a 0-approximate fixed point of F.*

Proof: Let $\boldsymbol{v} \in \boldsymbol{ran}(F)$[5], say $\boldsymbol{v} = F(\boldsymbol{u})$. Then

$$\begin{aligned} \boldsymbol{v}(0) &= F(\boldsymbol{u})(0) \\ &= F(\boldsymbol{v})(0) \end{aligned}$$

[5] i.e., the range of F

by Remark 3.2.5(a). □

Some intuition for Lemma 3.4.4 (in the case of our analog network example) is given by the following consideration. If $\boldsymbol{v}$ is a fixed point, or at least 0-approximate fixed point, of F, then $\boldsymbol{v}(0)$ is the "initial value" of the solution of the network equations. If, for example, F is a definite integral, or tuple of definite integrals, then $\boldsymbol{v}(0)$ is the tuple of constants of integration, which give this initial value. For an example of this, see Remark 4.1.7(f).

Theorem 1 (Fixed point of contracting and causal operator).
Given a stream transformer F as in (3.7), suppose $F \in \boldsymbol{Contr}(\lambda, \tau)$ for some $\lambda < 1$ and $\tau > 0$, and F is causal. Then F has a unique fixed point, i.e., there is a unique $\boldsymbol{u} \in \mathcal{C}[\mathbb{T}, A]^m$ such that $F(\boldsymbol{u}) = \boldsymbol{u}$.

Proof: 1. *Uniqueness*:

We will first prove: for all $k > 0$, a $k\tau$-approximate fixed point of F is unique up to $k\tau$-equivalence, i.e.,

$$F(\boldsymbol{u})\restriction_{k\tau} = \boldsymbol{u}\restriction_{k\tau} \;\wedge\; F(\boldsymbol{v})\restriction_{k\tau} = \boldsymbol{v}\restriction_{k\tau} \quad\Longrightarrow\quad \boldsymbol{u}\restriction_{k\tau} = \boldsymbol{v}\restriction_{k\tau} \tag{3.16}$$

by induction on k. For $k = 1$ this follows by noting that if $F(\boldsymbol{u})\restriction_{\tau} = \boldsymbol{u}\restriction_{\tau}$ and $F(\boldsymbol{v})\restriction_{\tau} = \boldsymbol{v}\restriction_{\tau}$ then

$$\begin{aligned} \mathsf{d}_\tau(\boldsymbol{u}, \boldsymbol{v}) &= \mathsf{d}_\tau(F(\boldsymbol{u}), F(\boldsymbol{v})) \\ &\leq \lambda \cdot \mathsf{d}_\tau(\boldsymbol{u}, \boldsymbol{v}) \end{aligned}$$

by (3.8) with $T = 0$ (since $F(u)(0) = F(v)(0)$, by causality of F and Remark 3.2.5(a)) and hence (since $\lambda < 1$) $\mathsf{d}_\tau(\boldsymbol{u}, \boldsymbol{v}) = 0$, i.e., $\boldsymbol{u}\restriction_{\tau} = \boldsymbol{v}\restriction_{\tau}$.

For the induction step, assume (3.16) holds for k, and suppose $F(\boldsymbol{u})\restriction_{(k+1)\tau} = \boldsymbol{u}\restriction_{(k+1)\tau}$ and $F(\boldsymbol{v})\restriction_{(k+1)\tau} = \boldsymbol{v}\restriction_{(k+1)\tau}$. Then by (3.16) $\boldsymbol{u}\restriction_{k\tau} = \boldsymbol{v}\restriction_{k\tau}$, and so

$$\begin{aligned} \mathsf{d}_{(k+1)\tau}(\boldsymbol{u}, \boldsymbol{v}) &= \mathsf{d}_{(k+1)\tau}(F(\boldsymbol{u}), F(\boldsymbol{v})) \\ &\leq \lambda \cdot \mathsf{d}_{(k+1)\tau}(\boldsymbol{u}, \boldsymbol{v}) \end{aligned}$$

by (3.8′), and hence (since $\lambda < 1$) $\mathsf{d}_{(k+1)\tau}(\boldsymbol{u}, \boldsymbol{v}) = 0$, i.e., $\boldsymbol{u}\restriction_{(k+1)\tau} = \boldsymbol{v}\restriction_{(k+1)\tau}$.

Interestingly, causality of F is not used in the inductive step.

This concludes the proof by induction of (3.16). Finally, if $F(\boldsymbol{u}) = \boldsymbol{u}$ and $F(\boldsymbol{v}) = \boldsymbol{v}$, then by (3.16) $\boldsymbol{u}\restriction_{k\tau} = \boldsymbol{v}\restriction_{k\tau}$ for all k, and so $\boldsymbol{u} = \boldsymbol{v}$.

2. *Existence*:

We use the notation $\mathcal{C}_k =_{df} \mathcal{C}[[0, k\tau], A]^m$ and

$$F_k \;=_{df}\; F\restriction_{k\tau} : \mathcal{C}_k \;\to\; \mathcal{C}_k.$$

Note that by Lemma 2.1.2, $\mathcal{C}_k$ is complete. Also, by causality of F and Lemma 3.2.6, for all $\boldsymbol{u} \in \mathcal{C}[\mathbb{T}, A]^m$,

$$F_k(\boldsymbol{u}\restriction_{k\tau}) \;=\; F(\boldsymbol{u})\restriction_{k\tau}\,. \tag{3.17}$$

We will construct a solution, namely a fixed point $\boldsymbol{v}$ of F, in stages. At stage k we will have a $k\tau$-approximate fixed point, i.e., a stream $\boldsymbol{v}_k$ such that

$$F(\boldsymbol{v}_k)\restriction_{k\tau} \;=\; \boldsymbol{v}_k\restriction_{k\tau} \tag{3.18a}$$

and for all l

$$l < k \;\;\Longrightarrow\;\; \boldsymbol{v}_k\restriction_{l\tau} \;=\; \boldsymbol{v}_l\restriction_{l\tau}\,. \tag{3.18b}$$

Stage $k = 1$: Define the sequence

$$\boldsymbol{v}_1^{(0)},\; \boldsymbol{v}_1^{(1)},\; \boldsymbol{v}_1^{(2)},\; \ldots\;,\; \boldsymbol{v}_1^{(n)},\; \ldots$$

by: $\boldsymbol{v}_1^{(0)}$ is any stream in $\boldsymbol{ran}(F)$, and for all n

$$\boldsymbol{v}_1^{(n+1)} \;=\; F(\boldsymbol{v}_1^{(n)}).$$

Note that for all n

$$\boldsymbol{v}_1^{(n)}(0) \;=\; \boldsymbol{v}_1^{(0)}(0).$$

by Lemma 3.4.4 and induction on n. Hence, putting

$$D_1 \;=_{df}\; \mathsf{d}_\tau(\boldsymbol{v}_1^{(0)}\,,\, \boldsymbol{v}_1^{(1)})$$

it follows from the contraction property of F and induction on n that

$$\mathsf{d}_\tau(\boldsymbol{v}_1^{(n)}\,,\, \boldsymbol{v}_1^{(n+1)}) \;\le\; \lambda^n \cdot D_1. \tag{3.19}$$

Next consider the sequence

$$\boldsymbol{w}_1^{(0)}\,,\; \boldsymbol{w}_1^{(1)}\,,\; \boldsymbol{w}_1^{(2)}\,,\; \ldots \tag{3.20}$$

of streams in $\mathcal{C}_1$, defined by

$$\boldsymbol{w}_1^{(n)} \;=_{df}\; \boldsymbol{v}_1^{(n)}\restriction_\tau\,.$$

By (3.17) with $k = 1$, it follows that for $n = 0, 1, 2, \ldots$

$$F_1(\boldsymbol{w}_1^{(n)}) \;=\; \boldsymbol{w}_1^{(n+1)}.$$

The sequence (3.20) can be seen to be Cauchy, by choosing (for any $\epsilon > 0$) N such that

$$\lambda^N \;<\; \frac{(1-\lambda)\cdot\epsilon}{D_1}, \tag{3.21}$$

assuming $D_1 > 0$ (otherwise $\boldsymbol{v}_1^{(0)}$ is a τ-approximate fixed point, and we can simply put $\boldsymbol{v}_1 = \boldsymbol{v}_1^{(0)}$). For then, for $m > n \geq N$,

$$\begin{aligned}
\mathsf{d}_\tau(\boldsymbol{w}_1^{(n)}, \boldsymbol{w}_1^{(m)}) &\leq \mathsf{d}_\tau(\boldsymbol{w}_1^{(n)}, \boldsymbol{w}_1^{(n+1)}) + \ldots + \mathsf{d}_\tau(\boldsymbol{w}_1^{(m-1)}, \boldsymbol{w}_1^{(m)}) \\
&\leq (\lambda^n + \lambda^{n+1} + \ldots + \lambda^m) \cdot D_1 && \text{by (3.19)} \\
&< \frac{\lambda^n}{(1-\lambda)} D_1 \\
&\leq \frac{\lambda^N}{(1-\lambda)} D_1 \\
&< \epsilon && \text{by (3.21)}
\end{aligned}$$

Hence, by the completeness of $\mathcal{C}_1$, the sequence (3.20) converges to a limit $\boldsymbol{w}_1 \in \mathcal{C}_1$. Hence, also, the sequence

$$F_1(\boldsymbol{w}_1^{(0)}),\ F_1(\boldsymbol{w}_1^{(1)}),\ F_1(\boldsymbol{w}_1^{(2)}),\ \ldots \tag{3.22}$$

converges to $F_1(\boldsymbol{w}_1)$, since by the contraction property of F,

$$\mathsf{d}_\tau(F_1(\boldsymbol{w}_1^{(n)}),\ F_1(\boldsymbol{w}_1)) \ \leq\ \lambda \cdot \mathsf{d}_\tau(\boldsymbol{w}_1^{(n)},\ \boldsymbol{w}_1).$$

Since (3.22) is actually the sequence (3.20) shifted by 1, it follows that it also converges to $\boldsymbol{w}_1$, and so

$$F_1(\boldsymbol{w}_1) \ =\ \boldsymbol{w}_1.$$

Hence if we define

$$\boldsymbol{v}_1 \ =_{df}\ \boldsymbol{ext}_\tau(\boldsymbol{w}_1)$$

(see Notation 3.2.3(c)), it follows that $\boldsymbol{v}_1$ is a τ-approximate fixed point of F.

Stage $k+1$: Now suppose we have a $k\tau$-approximate fixed point $\boldsymbol{v}_k$. Define the sequence

$$\boldsymbol{v}_{k+1}^{(0)},\ \boldsymbol{v}_{k+1}^{(1)},\ \boldsymbol{v}_{k+1}^{(2)},\ \ldots\ ,\ \boldsymbol{v}_{k+1}^{(n)},\ \ldots \tag{3.23}$$

by
$$\boldsymbol{v}_{k+1}^{(0)} \ =\ \boldsymbol{v}_k$$
and for all n
$$\boldsymbol{v}_{k+1}^{(n+1)} \ =\ F(\boldsymbol{v}_{k+1}^{(n)}).$$

Note that for all n, $\boldsymbol{v}_{k+1}^{(n)}$ is $k\tau$-equivalent to $\boldsymbol{v}_k$ and is a $k\tau$-approximate fixed point of F, by Lemma 3.4.3 and induction on n. (Here causality of F is used again.) Putting

$$D_{k+1} \ =_{df}\ \mathsf{d}_{(k+1)\tau}(\boldsymbol{v}_{k+1}^{(0)},\ \boldsymbol{v}_{k+1}^{(1)}),$$

we can prove from the contraction property of F, by induction on n:

$$\mathsf{d}_{(k+1)\tau}(\boldsymbol{v}_{k+1}^{(n)},\ \boldsymbol{v}_{k+1}^{(n+1)}) \ \leq\ \lambda^n \cdot D_{k+1}.$$

Next consider the sequence

$$\boldsymbol{w}^{(0)}_{k+1}\,,\ \boldsymbol{w}^{(1)}_{k+1}\,,\ \boldsymbol{w}^{(2)}_{k+1}\,,\ \ldots \tag{3.24}$$

of streams in $\mathcal{C}_{k+1}$, defined by

$$\boldsymbol{w}^{(n)}_{k+1} \ =_{df}\ \boldsymbol{v}^{(n)}_{k+1}\!\restriction_{(k+1)\tau}\,.$$

By (3.17) again, it follows that for $n = 0, 1, 2, \ldots$

$$F_{k+1}(\boldsymbol{w}^{(n)}_{k+1}) \ = \ \boldsymbol{w}^{(n+1)}_{k+1}.$$

The sequence (3.24) can be seen to be Cauchy, by choosing (for any $\epsilon > 0$) N such that

$$\lambda^N \ < \ \frac{(1-\lambda)\cdot\epsilon}{D_{k+1}}, \tag{3.25}$$

assuming $D_{k+1} > 0$ (otherwise $\boldsymbol{v}^{(0)}_{k+1}$ is a $(k+1)\tau$-approximate fixed point, and we can simply put $\boldsymbol{v}_{k+1} = \boldsymbol{v}^{(0)}_{k+1}$). For then, for $m > n \geq N$, we can show, as with Stage 1, that

$$\begin{aligned} \mathsf{d}_{(k+1)\tau}(\boldsymbol{w}^{(n)}_{k+1},\, \boldsymbol{w}^{(m)}_{k+1}) \ &\leq \ \frac{\lambda^N}{(1-\lambda)}\ D_{k+1} \\ &< \ \epsilon \qquad\qquad \text{by (3.25)} \end{aligned}$$

Hence, by the completeness of $\mathcal{C}_{k+1}$, the sequence (3.24) converges to a limit $\boldsymbol{w}_{k+1} \in \mathcal{C}_{k+1}$. Hence, also, the sequence

$$F_{k+1}(\boldsymbol{w}^{(0)}_{k+1}),\ F_{k+1}(\boldsymbol{w}^{(1)}_{k+1}),\ F_{k+1}(\boldsymbol{w}^{(2)}_{k+1}),\ \ldots \tag{3.26}$$

converges to $F_{k+1}(\boldsymbol{w}_{k+1})$, since by the contraction property of F,

$$\mathsf{d}_{(k+1)\tau}(F_{k+1}(\boldsymbol{w}^{(n)}_{k+1}),\, F_{k+1}(\boldsymbol{w}_{k+1})) \ \leq \ \lambda\cdot\mathsf{d}_{(k+1)\tau}(\boldsymbol{w}^{(n)}_{k+1},\, \boldsymbol{w}_{k+1}).$$

Since (3.26) is actually the sequence (3.24) shifted by 1, it follows that it also converges to $\boldsymbol{w}_{k+1}$, and so

$$F_{k+1}(\boldsymbol{w}_{k+1}) \ = \ \boldsymbol{w}_{k+1}.$$

Hence if we define

$$\boldsymbol{v}_{k+1} \ =_{df}\ \boldsymbol{ext}_{(k+1)\tau}(\boldsymbol{w}_{k+1}),$$

it follows that $\boldsymbol{v}_{k+1}$ is a $(k+1)\tau$-approximate fixed point of F. Further, since for all n,
$\boldsymbol{v}^{(n)}_{k+1}\!\restriction_{k\tau} = \boldsymbol{v}_k\!\restriction_{k\tau}$,
it follows (using the fact that a uniform limit is also a pointwise limit) that

$$\boldsymbol{v}_{k+1}\!\restriction_{k\tau} \ = \ \boldsymbol{v}_k\!\restriction_{k\tau}\,.$$

Hence we have a sequence of streams

$$\boldsymbol{v}_0,\ \boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \ldots$$

satisfying (3.18). Finally, we can specify the required fixed point of F as the unique stream $\boldsymbol{v}$ such that for all k

$$\boldsymbol{v}{\restriction}_{k\tau} = \boldsymbol{v}_k{\restriction}_{k\tau}\ .$$

To conclude the proof, we must consider one more point: is the fixed point $\boldsymbol{v}$ constructed above actually a stream? In other words, is it *continuous* as a function from $\mathbb{T}$ to A^m? But this follows from the above construction of $\boldsymbol{v}$, as an iterated limit of sequences of sequences of approximations, all of which converge locally uniformly, and hence preserve continuity (in fact local uniform continuity), as does the end result $\boldsymbol{v}$. □

The network stream transformers in our two running examples are contracting, as we have seen (Examples 3.3.5). What about causality? To investigate this, we first define a related concept. Let, again, F be as in (3.1).

Definition 3.4.5 (Causality and weak causality). Let $T \geq 0$.

(a) F is *causal w.r.t.* T, or satisfies $\boldsymbol{Caus}(T)$, if for all $(\boldsymbol{a}, \boldsymbol{x}) \in A^q \times \mathcal{C}[\mathbb{T}, A]^p$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}[\mathbb{T}, A]^m$

$$\boldsymbol{u}{\restriction}_{<T} = \boldsymbol{v}{\restriction}_{<T} \implies F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{u})(T) = F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{v})(T).$$

(b) F is *weakly causal w.r.t.* T, or satisfies $\boldsymbol{WCaus}(T)$, if for all $(\boldsymbol{a}, \boldsymbol{x}) \in A^q \times \mathcal{C}[\mathbb{T}, A]^p$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}[\mathbb{T}, A]^m$

$$\boldsymbol{u}{\restriction}_{T} = \boldsymbol{v}{\restriction}_{T} \implies F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{u})(T) = F_{\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{v})(T).$$

(Compare Definition 3.2.4.)

Remarks 3.4.6. (a) For $\mathbb{T} = \mathbb{N}$, $\boldsymbol{WCaus}(T)$ is clearly weaker than $\boldsymbol{Caus}(T)$.

(b) For $\mathbb{T} = \mathbb{R}$, $\boldsymbol{WCaus}(T)$ is equivalent to $\boldsymbol{Caus}(T)$ if $T > 0$, by continuity.

(c) However $\boldsymbol{WCaus}(0)$ is strictly weaker than $\boldsymbol{Caus}(0)$ (see Remark 3.2.5(c)).

Now let N be an analog network, as in our first running example, and F^N the corresponding network state function (Example 3.1.4(1)).

Lemma 3.4.7. *For an analog network N:*

(a) *For $T \geq 0$, if all the module functions of N are weakly causal w.r.t. T, then so is F^N.*

(b) *Same, with "weakly causal" replaced by "causal".*

Proof: For (a): Note that the composition of two weakly causal functions is weakly causal. The result follows by induction on the number of modules in N. Similarly for (b). □

Remark 3.4.8. In the case of an SCA network, we cannot use the above argument, since the module functions are not stream transformers, but functions from data tuples to data (see Remark 3.1.5), and so the concept of causality does not apply to them. But in any case the network state function is clearly causal, by (3.15) (Example 3.3.5(2)).

Examples 3.4.9 (Fixed point; Applying Theorem 1).

• ***Example 1: Analog networks***. The module functions of the standard modules used in the two case studies (see Examples 3.3.5(1)): *pointwise addition*, *scalar multiplication*, and *integration* [TZ07, §4], are all weakly causal with respect to any $T \geq 0$. Hence (considering first the network N_1) the network state function F^{N_1} (cf. equations (3.9))) is weakly causal. By Remark 3.4.6(b), F^{N_1} is then causal with respect to any $T > 0$.

Caus(0) follows by inspection of equations (3.9). Note that in this case ***Caus***(0) fails if equation (3.9c$'$) is used instead of (3.9c) in the specification of F^{N_1}.

Hence Theorem 1 can be applied to show the existence of a fixed point of F^{N_1} for all values of the system parameters M, K, D for which $M > \max(K, 2D)$, and all initial values x_0, v_0.

Theorem 1 can be similarly applied to the network N_2, to show the existence of a fixed point of F^{N_2} for *all* positive values of M, K, D, and all initial values x_0, v_0 [JZ11].

The second case study of [TZ07] (the iterated mass/spring/damper system) can be handled by similar considerations.

• ***Example 2: SCAs***. As stated above (Example 3.3.5(2)), causality, as well as contraction, apply automatically to the SCA network state function F^N, and so the existence of the network stream transformer Φ^N can be justified by the fixed point construction of Theorem 1. However this use of Theorem 1 is not necessary, since Φ^N can be constructed directly from the module functions simply by a simultaneous primitive recursion (see the discussion in Example 3.3.5(2)).

4 Continuity of fixed point of contracting operators

In preparation for the investigation of the continuity of the fixed point in this section, we consider another property of operators: invariance under time shift.

4.1 Shift invariance of operators

The common analog modules, such as those treated in [TZ07], satisfy a modified version of (the usual notion of) invariance, i.e., *invariance relative to initial values*. First, we must divide our parameters into two classes: the "system parameters" and "initial constants". The latter can be thought of as *initial values* of (some of) the stream variables, and appear typically as constants of integration. Unlike the system parameters, they appear in "updated form" in the formulation of the invariance property (Definition 4.1.3 below).

For example, in case study 1 in [TZ07], with the network N_1 (Example 3.3.5(1)) there are 1 input stream variable, 3 non-input stream variables a, v, x (acceleration, velocity and displacement respectively), 3 system parameters M, K, D, and 2 initial constants x_0, v_0 associated with x, v respectively. Note that there is no initial constant associated with the acceleration a. The network N_2 for the same system (Example 3.3.5(1) again) is similar, except that it has only 2 non-input stream variables x, v (each with its associated initial value). Hence, in general:

> *supposing there are m stream variables $u_1, \dots, u_m$, we assume there are also s initial parameters $a_1, \dots, a_s$ for some s, $0 \le s \le m$, where a_i is associated with u_i for $i = 1, \dots, s$.*

We will denote system parameters by $\boldsymbol{c}, \dots$ and initial constants by $\boldsymbol{a}, \dots$.

So assume, from now on, that our operators have the form

$$F\colon A^r \times A^s \times \mathcal{C}[\mathbb{T}, A]^p \ \to (\mathcal{C}[\mathbb{T}, A]^m \ \to \mathcal{C}[\mathbb{T}, A]^m) \tag{4.1}$$

$(0 \le r,\ 0 \le s \le m,\ p > 0,\ m > 0)$ or, in uncurried form,

$$F\colon A^r \times A^s \times \mathcal{C}[\mathbb{T}, A]^p \times \mathcal{C}[\mathbb{T}, A]^m \ \to\ \mathcal{C}[\mathbb{T}, A]^m \tag{4.2}$$

where A^r contains the r system parameters[6] $\boldsymbol{c} = (c_1, \dots, c_r)$, A^s contains the s initial constants $\boldsymbol{a} = (a_1, \dots, a_s)$, and $\mathcal{C}[\mathbb{T}, A]^p$ contains the p input streams $\boldsymbol{x}$. We think of $\boldsymbol{a}$ as the *initial values* of the first s of the m non-input stream variables $\boldsymbol{u}$. Then for $\boldsymbol{c} \in A^r$, $\boldsymbol{a} \in A^s$ and $\boldsymbol{x} \in \mathcal{C}[\mathbb{T}, A]^p$, $F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}$ is the operator

$$F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}} = F(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}, \ \cdot\ ) :\ \mathcal{C}[\mathbb{T}, A]^m \ \to\ \mathcal{C}[\mathbb{T}, A]^m. \tag{4.3}$$

These operators are essentially of the same form as those shown in (3.1)–(3.3); they differ only in that the space of parameters A^q shown there is divided here into the spaces A^r and A^s of *system parameters* and *initial constants* respectively $(q = r + s)$.

---

[6] More generally, the system parameters may range over *subspaces* of A; see, e.g., Example 4.2.13(1) where $A = \mathbb{R}$, and the parameters range over $\mathbb{R}^{>0}$. We are ignoring that innocuous complication here for the sake of simplicity.

Remark 4.1.1 (Continuity of F). The assumption of continuity of the operator F in Theorem 2 and elsewhere is made with respect to the typing of F in (4.2). In fact, as a study of the proofs of Theorems 2 and 4 will show, we need only assume continuity (or uniform continuity) of F for the first 3 arguments, i.e., for $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in A^r \times A^s \times \mathcal{C}[\mathbb{T}, A]^p$.

Notation 4.1.2. For $\boldsymbol{u} \in \mathcal{C}[\mathbb{T}, A]^m$ and $s \leq m$ we write $\boldsymbol{u}^s$ for the s-tuple of streams $(u_1, \dots, u_s)$, and then for $T \geq 0$ we write $\boldsymbol{u}^s(T)$ for the s-tuple $(u_1(T), \dots, u_s(T))$.

Definition 4.1.2 (Compatibility of non-input streams with initial parameters). A tuple $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{u}) \in A^r \times A^s \times \mathcal{C}[\mathbb{T}, A]^p \times \mathcal{C}[\mathbb{T}, A]^m$ is said to be *compatible*, or $\boldsymbol{u}$ is said to be *compatible* with $\boldsymbol{a}$, if $\boldsymbol{u}^s(0) = \boldsymbol{a}$.

Now an operator F (as in (4.2)) is said to be *shift invariant* if its behaviour is invariant under time shifts, with *suitable changes made with the initial parameters*. More precisely:

Definition 4.1.4 (Shifted stream tuple). For any $\boldsymbol{u} \in \mathcal{C}[\mathbb{T}, A]^m$ and $T \geq 0$, we define the *shifted stream tuple* $\boldsymbol{shift}_T(\boldsymbol{u})$ by

$$\boldsymbol{shift}_T(\boldsymbol{u})(t) \;=_{df}\; \boldsymbol{u}(T+t).$$

Remark 4.1.5. For use in the proof of Theorem 2, we note that

$$\mathsf{d}_\tau(\boldsymbol{shift}_T(\boldsymbol{u}),\, \boldsymbol{shift}_T(\boldsymbol{v})) \;=\; \mathsf{d}_{T,\,T+\tau}(\boldsymbol{u}, \boldsymbol{v}).$$

Definition 4.1.6 (Shift invariance with updated initial values). An operator F as in (4.2) is *shift invariant*, or satisfies ***Invar***, if for all $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{u}) \in A^r \times A^s \times \mathcal{C}[\mathbb{T}, A]^p \times \mathcal{C}[\mathbb{T}, A]^m$ and $T \geq 0$, if

$$F(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{u})\restriction_T \;=\; \boldsymbol{u}\restriction_T, \tag{4.4}$$

then

(i) $\boldsymbol{u}^s(0) \;=\; \boldsymbol{a}$, and

(ii) $F(\boldsymbol{c},\, \boldsymbol{u}^s(T),\, \boldsymbol{shift}_T(\boldsymbol{x}),\, \boldsymbol{shift}_T(\boldsymbol{u})) \;=\; \boldsymbol{shift}_T(F(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{u}))$.

Remarks 4.1.7. (a) Equation (4.4) says that $\boldsymbol{u}$ is a T-approximate fixed point of $F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}$. (Note that the usual definition of shift invariance of operators makes no reference to approximate fixed points.)

(b) Clause (i) says that the inputs are compatible (Definition 4.1.3).

(c) Clause (ii) is the "invariance property" of F, subject to the "updating" of the initial values from $\boldsymbol{u}^s(0)$ $(= \boldsymbol{a})$ to $\boldsymbol{u}^s(T)$.

(d) In the special case $T = 0$, the shift invariance condition says that $F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{u})(0) = \boldsymbol{u}(0)$ only if $\boldsymbol{u}^s(0) \;=\; \boldsymbol{a}$, i.e., $\boldsymbol{u}$ can be a 0-approximate fixed point of $F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}$ only if $\boldsymbol{u}$ is compatible with $\boldsymbol{a}$.

(e) The network state functions of case studies 1 and 2 are shift invariant.

(f) Note that in case study 1, with network N_1, the input stream is f, the remaining streams are (x, v, a), and the initial parameters are (x_0, v_0). Compatibility of inputs here means that $x(0) = x_0$ and $v(0) = v_0$, but $a(0)$ can be arbitrary. Applying F_{N_1} to (x, v, a) gives (x', v', a') where, again, $x'(0) = x_0$ and $v'(0) = v_0$, but now $a'(0) = (f(0) - Kx_0 - Dv_0)/M$. Applying F to (x', v', a') leads to the same 0-values, i.e.,

$$(x_0,\ v_0,\ (f(0) - Kx_0 - Dv_0)/M))$$

is a 0-fixed point of F^{N_0}. With network N_2, the situation is simpler: there are only two non-input streams (x, v), with initial values (x_0, v_0), which form a 0-fixed point of F^{N_2}. Compatibility of inputs here simply means that $x(0) = x_0$ and $v(0) = v_0$.

4.2 Continuity of fixed point

Consider a (total) operator F as in (4.1) satisfying causality. Let $U \subseteq A^r \times A^s \times \mathcal{C}[\mathbb{T}, A]^p$ be such that $F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}} \colon \mathcal{C}[\mathbb{T}, A]^m \ \to\ \mathcal{C}[\mathbb{T}, A]^m$ is contracting, with modulus $\lambda_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}$ and increment $\tau_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}$, for all $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U$.

Remark 4.2.1 (Openness of U). We will assume, for convenience, that U is open. Although the theory could be (re-)formulated without this assumption (by referring to the interior of U, when necessary), it is a reasonable assumption which smoothes the exposition.

Then by Theorem 1, for all $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U$, $F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}$ has a unique fixed point $\mathsf{FP}(F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}})$. Define

$$\Phi \colon A^r \times A^s \times \mathcal{C}[\mathbb{T}, A]^p \ \rightharpoonup\ \mathcal{C}[\mathbb{T}, A]^m \tag{4.5a}$$

by: $\boldsymbol{dom}(\Phi) = U$, and for $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U$,

$$\Phi(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \ =\ \mathsf{FP}(F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}). \tag{4.5b}$$

In our analog network example, F represents a state function for a network with r constants $\boldsymbol{c} \in A^r$, s initial values $\boldsymbol{a} \in A^s$, p input channels with input streams $\boldsymbol{x} \in \mathcal{C}[\mathbb{T}, A]^p$, and m modules. The output channels of the network will form a subset of the m module output channels. For simplicity, we can assume that *all* the module output channels are also network output channels, with output streams $\boldsymbol{y} \in \mathcal{C}[\mathbb{T}, A]^m$. The input/output function for the network, or *network function*, will then be the stream transformer Φ.

Remark 4.2.2 (Families of contraction moduli and increments). We assume that the contraction modulus and increment vary with the constants, initial values and input streams $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U$. Thus we have *families* of contraction moduli $\langle \lambda_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}} \mid (\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U \rangle$ and increments $\langle \tau_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}} \mid (\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U \rangle$ such that $F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}$ is contracting with respect to $(\lambda_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}},\ \tau_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}})$, for all $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U$. For later use, in formulating our theorems, we will write (boldface) '$\boldsymbol{\lambda}$' and '$\boldsymbol{\tau}$' for the *functions* corresponding to these two families, i.e.,

$$\begin{aligned} \boldsymbol{\lambda} &\colon A^r \times A^s \times \mathcal{C}[\mathbb{T}, A]^p \ \rightharpoonup\ \mathbb{R} \\ \text{and} \qquad \boldsymbol{\tau} &\colon A^r \times A^s \times \mathcal{C}[\mathbb{T}, A]^p \ \rightharpoonup\ \mathbb{T} \end{aligned} \tag{4.6a}$$

which are defined (at least) on U, such that for all $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U$:

$$\begin{aligned} \boldsymbol{\lambda}(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) &= \lambda_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}, \\ \boldsymbol{\tau}(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) &= \tau_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}. \end{aligned} \tag{4.6b}$$

From Theorem 1 we know of the existence of the fixed point function Φ as in (4.5). In this section we investigate the continuity of Φ. We will need further assumptions, namely continuity of F, local boundedness of $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$, causality of F (as in Theorem 1), and also its shift invariance.

Two lemmas (4.2.3 and 4.2.6) follow, showing how the properties of causality and shift invariance are inherited from F to the fixed point function Φ.

Lemma 4.2.3 (Causality of fixed point). *If F is causal, then so is its fixed point function Φ.*

Proof (outline): Show, by induction on k, that the $k\tau$-initial segment of the $k\tau$-approximate fixed point, as constructed in the proof of Theorem 1, depends only on the $k\tau$-initial segment of the input, using causality of F. Note also that this property (i.e., that the $k\tau$-initial segment depends only on the $k\tau$-initial segment of the input) is preserved by $k\tau$-approximate uniform limits. $\square$

Definition 4.2.4 (Closure of domain under shifts). Given Φ as in (4.5a), with $\boldsymbol{dom}(\Phi) = U$, we say that U is *closed under shifts w.r.t.* Φ if for all $T > 0$ and all $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})$:

$$(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U \quad \Longrightarrow \quad (\boldsymbol{c},\ \Phi(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})^s(T),\ \boldsymbol{shift}_T(\boldsymbol{x})) \in U.$$

Remark 4.2.5. This closure condition is satisfied trivially if U is of the form

$$U = V \times A^s \times \mathcal{C}[\mathbb{T}, A]^p$$

for some $V \subseteq A^r$. This is, in fact, the case with the two case studies.

Lemma 4.2.6. *Suppose F is shift invariant, and $F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}$ has a fixed point $\Phi(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})$ for all $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U$, and U is closed under shifts w.r.t. Φ. Then Φ is shift invariant on U, in the sense that for all $T > 0$, and all $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U$, if $\Phi(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) = \boldsymbol{v}$, then*

$$\Phi(\boldsymbol{c},\ \boldsymbol{v}^s(T),\ \boldsymbol{shift}_T(\boldsymbol{x})) = \boldsymbol{shift}_T(\boldsymbol{v}). \tag{4.7}$$

Proof: For $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U$, and $T > 0$, since $F(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{v}$, we have, by invariance of F,

$$F(\boldsymbol{c},\ \boldsymbol{v}^s(T),\ \boldsymbol{shift}_T(\boldsymbol{x}),\ \boldsymbol{shift}_T(\boldsymbol{v})) = \boldsymbol{shift}_T(\boldsymbol{v}). \tag{4.8}$$

By shift closure of U w.r.t. Φ,

$$(\boldsymbol{c},\ \boldsymbol{v}^s(T),\ \boldsymbol{shift}_T(\boldsymbol{x})) \in U.$$

Hence, and by (4.8), (4.7) follows. $\square$

Notation 4.2.7 (Basic open sets of $\mathcal{C}[\mathbb{T}, A]^m$).
Given $T > 0$, $\epsilon > 0$ and $\boldsymbol{u} \in \mathcal{C}[\mathbb{T}, A]^m$, we write

$$\mathbf{N}_T(\boldsymbol{u}, \epsilon) \ =_{df} \ \{ \boldsymbol{v} \in \mathcal{C}[\mathbb{T}, A]^m \mid \mathsf{d}_T(\boldsymbol{v}, \boldsymbol{u}) < \epsilon \}.$$

The collection of such neighbourhoods forms an open base for the topology on $\mathcal{C}[\mathbb{T}, A]^m$. In fact, we could restrict ϵ to be (for example) of the form 2^{-n} for $n = 1, 2, \ldots$, and restrict T to vary over positive integers, or positive integral multiples of some real $\tau > 0$.

More generally, we can define an open base for, e.g., the space $A^r \times \mathcal{C}[\mathbb{T}, A]^m$ by

$$\mathbf{N}_T((\boldsymbol{a}, \boldsymbol{u}), \epsilon) \ =_{df} \ \{ (\boldsymbol{b}, \boldsymbol{v}) \in \mathcal{C}[\mathbb{T}, A]^m \mid \mathsf{d}_A(\boldsymbol{a}, \boldsymbol{b}) < \epsilon \wedge \mathsf{d}_T(\boldsymbol{u}, \boldsymbol{v}) < \epsilon \}$$

for all $(\boldsymbol{a}, \boldsymbol{u}) \in A^r \times \mathcal{C}[\mathbb{T}, A]^m$, $T > 0$ and $\epsilon > 0$.

As further preparation for Theorem 2, we also define two conditions on the families of contraction moduli and increments, weaker than continuity.

Definition 4.2.8 (Local boundedness of contraction moduli and increments).

(a) The family $\langle \lambda_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}} \mid (\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U \rangle$ is *locally bounded* at $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U$ if there exists $\lambda_0 < 1$ such that for all $(\boldsymbol{c}', \boldsymbol{a}', \boldsymbol{x}')$ sufficiently near $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})$, $\lambda_{\boldsymbol{c}',\boldsymbol{a}',\boldsymbol{x}'} < \lambda_0$.

(b) The family $\langle \tau_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}} \mid (\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U \rangle$ is *locally bounded* at $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U$ if there exists $\tau_0 > 0$ such that for all $(\boldsymbol{c}', \boldsymbol{a}', \boldsymbol{x}')$ sufficiently near $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})$, $\tau_{\boldsymbol{c}',\boldsymbol{a}',\boldsymbol{x}'} > \tau_0$.

Remark 4.2.9. Clearly, continuity of $\boldsymbol{\lambda}$ or $\boldsymbol{\tau}$ implies local boundedness.

Theorem 2 (Continuity of FP). *Given stream operators F and $F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}$ as in* (4.1) *and* (4.3)*, an open set $U \subseteq A^r \times A^s \times \mathcal{C}[\mathbb{T}, A]^p$, and families of contraction moduli $\boldsymbol{\lambda} = \langle \lambda_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}} \mid (\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U \rangle$ and increments $\boldsymbol{\tau} = \langle \tau_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}} \mid (\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U \rangle$, suppose*

(i) *$F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}} \in \boldsymbol{Contr}(\lambda_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}, \tau_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}})$ for all $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U$,*

(ii) *F is causal,*

(iii) *F is shift invariant,*

(iv) *F is continuous on U,*

(v) *$\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$ are locally bounded on U, and*

(vi) *U is closed under shifts w.r.t. Φ,*

where Φ is the fixed point function for F as in (4.5)*, given by Theorem 1.*
Then Φ is continuous on U.

Proof: Choose any $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \in U$. Note first that we can assume without loss of generality that $\lambda_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}$ can be (re-)defined so as to be *constant* near $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})$, since, by the local boundedness assumption (v), $\lambda_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}$ is less than some $\lambda < 1$ near $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})$. We can then take this λ to be the modulus of contraction at and near $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})$.

Similarly, we can assume that the contraction increment $\tau_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}$ is *constant* near $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})$, since, by local boundedness again, its value is greater than some $\tau > 0$ near $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})$. By Lemma 3.3.3, we can take this τ to be the contraction increment at and near $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})$.

We continue (from the proof of Theorem 1) with the notation $\mathcal{C}_k =_{df} \mathcal{C}[[0, k\tau], A]^m$, $F_k =_{df} F\restriction_{k\tau}$ and $\Phi_k =_{df} \Phi\restriction_{k\tau}$.

Recall that

$$\boldsymbol{v} \ =_{df} \ \Phi(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}) \ = \ \mathsf{FP}(F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}) \tag{4.9}$$

is obtained as the limit of a sequence

$$\boldsymbol{v}_1, \ \boldsymbol{v}_2, \ \ldots, \ \boldsymbol{v}_k, \ \ldots \tag{4.10}$$

where for each k, $\boldsymbol{v}_k$ is a $k\tau$-approximate fixed point of $F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}$, where, in turn,
(i) $\boldsymbol{v}_1$ is a τ-approximate limit of a sequence of 0-approximate fixed points

$$\boldsymbol{v}_1^{(0)}, \ \boldsymbol{v}_1^{(1)}, \ \boldsymbol{v}_1^{(2)}, \ \ldots, \ \boldsymbol{v}_1^{(n)}, \ \ldots \tag{4.11}$$

with $\boldsymbol{v}_1^{(0)}$ any stream in $\boldsymbol{ran}(F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}})$, say

$$\boldsymbol{v}_1^{(0)} \ = \ F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{u}_0) \tag{4.12}$$

for any $\boldsymbol{u}_0$ (cf. Lemma 3.4.4), and $\boldsymbol{v}_1^{(n+1)} = F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{v}_1^{(n)})$;
(ii) $\boldsymbol{v}_{k+1}$ is a $(k+1)\tau$-approximate limit of a sequence of $k\tau$-approximate fixed points (repeating (3.23)):

$$\boldsymbol{v}_{k+1}^{(0)}, \ \boldsymbol{v}_{k+1}^{(1)}, \ \boldsymbol{v}_{k+1}^{(2)}, \ \ldots, \ \boldsymbol{v}_{k+1}^{(n)}, \ \ldots \tag{4.13}$$

with $\boldsymbol{v}_{k+1}^{(0)} = \boldsymbol{v}_k$ and $\boldsymbol{v}_{k+1}^{(n+1)} = F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}(\boldsymbol{v}_{k+1}^{(n)})$.

We will show that for all k, Φ_k is continuous. The result follows from Lemmas 3.2.8(b)(ii) (putting $F = \Phi$ and $T_k = k\tau$) and 4.2.3.

First, some notation. For any $(\boldsymbol{c}', \boldsymbol{a}', \boldsymbol{x}') \in U$, we write (parallelling the notation of (4.9)–(4.13)):

$$\boldsymbol{v}' \ =_{df} \ \Phi(\boldsymbol{c}', \boldsymbol{a}', \boldsymbol{x}') \ = \ \mathsf{FP}(F_{\boldsymbol{c}',\boldsymbol{a}',\boldsymbol{x}'}) \tag{4.9$'$}$$

which is the limit of the sequence

$$\boldsymbol{v}_1', \ \boldsymbol{v}_2', \ \ldots, \ \boldsymbol{v}_k', \ \ldots \tag{4.10$'$}$$

where $\boldsymbol{v}_1'$ is a τ-approximate limit of the sequence

$$\boldsymbol{v}_1'^{(0)}, \ \boldsymbol{v}_1'^{(1)}, \ \boldsymbol{v}_1'^{(2)}, \ \ldots, \ \boldsymbol{v}_1'^{(n)}, \ \ldots \tag{4.11$'$}$$

with

$$\boldsymbol{v}_1'^{(0)} = F_{\boldsymbol{c}',\boldsymbol{a}',\boldsymbol{x}'}(\boldsymbol{u}_0), \tag{4.12$'$}$$

(the same $\boldsymbol{u}_0$ as in (4.12)), and $\boldsymbol{v'}_1^{(n+1)} = F_{\boldsymbol{c'},\boldsymbol{a'},\boldsymbol{x'}}(\boldsymbol{v'}_1^{(n)})$; and $\boldsymbol{v'}_{k+1}$ is a $(k+1)\tau$-approximate limit of the sequence

$$\boldsymbol{v'}_{k+1}^{(0)},\ \boldsymbol{v'}_{k+1}^{(1)},\ \boldsymbol{v'}_{k+1}^{(2)},\ \ldots,\ \boldsymbol{v'}_{k+1}^{(n)},\ \ldots \tag{4.13$'$}$$

with $\boldsymbol{v'}_{k+1}^{(0)} = \boldsymbol{v'}_k$ and $\boldsymbol{v'}_{k+1}^{(n+1)} = F_{\boldsymbol{c'},\boldsymbol{a'},\boldsymbol{x'}}(\boldsymbol{v'}_{k+1}^{(n)})$.

We will show that for all $k = 1, 2, \ldots,$ Φ_k is continuous at $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})$, i.e.,

$$\textit{for}\ \ (\boldsymbol{c'}, \boldsymbol{a'}, \boldsymbol{x'}{\restriction}_{k\tau})\ \textit{sufficiently "close to"}\ (\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}{\restriction}_{k\tau}),\ \boldsymbol{v'}_k{\restriction}_{k\tau}\ \textit{is "close to"}\ \boldsymbol{v}_k{\restriction}_{k\tau}. \tag{4.14}$$

The proof is by induction on k.

Basis: $k = 1$. By assumption F is continuous, and so

$$\textit{for any fixed } n,\ \boldsymbol{v}_1^{(n)} \textit{ depends continuously on } (\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}). \tag{4.15}$$

Choose $\epsilon > 0$. Then, putting

$$D_1 \ =_{df}\ \mathsf{d}_\tau(\boldsymbol{v}_1^{(0)}, \boldsymbol{v}_1^{(1)}), \tag{4.16}$$

and assuming $D_1 > 0$ (otherwise $\boldsymbol{v}_1^{(0)}{\restriction}_\tau = \boldsymbol{v}_1^{(1)}{\restriction}_\tau = \boldsymbol{v}_1^{(2)}{\restriction}_\tau = \ldots = \boldsymbol{v}_1{\restriction}_\tau$, and the argument becomes much simpler), choose N such that

$$\lambda^N \ <\ \frac{(1-\lambda)\epsilon}{9D_1} \tag{4.17}$$

and choose $\delta > 0$ such that (1) $\mathbf{N}_\tau((\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}), \delta) \subseteq U$, (2) the contraction modulus and increment have constant values λ and τ in $\mathbf{N}_\tau((\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}), \delta)$ (already used in (4.17) and (4.16)!), and for all $(\boldsymbol{c'}, \boldsymbol{a'}, \boldsymbol{x'}) \in \mathbf{N}_\tau((\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}), \delta)$, we have: (3)

$$\mathsf{d}_\tau(\boldsymbol{v'}_1^{(0)}, \boldsymbol{v}_1^{(0)}) \ < D_1 \tag{4.18}$$

(by (4.15), with $n = 0$), and (4)

$$\mathsf{d}_\tau(\boldsymbol{v'}_1^{(1)}, \boldsymbol{v}_1^{(1)}) \ < D_1 \tag{4.19}$$

(again by (4.15), with $n = 1$), and finally (5)

$$\mathsf{d}_\tau(\boldsymbol{v'}_1^{(N)}, \boldsymbol{v}_1^{(N)}) \ <\ \epsilon/3 \tag{4.20}$$

(again by (4.15) with $n = N$). Then

$$\begin{aligned} \mathsf{d}_\tau(\boldsymbol{v'}_1^{(0)}, \boldsymbol{v'}_1^{(1)}) \ &\le\ \mathsf{d}_\tau(\boldsymbol{v'}_1^{(0)}, \boldsymbol{v}_1^{(0)}) \ + \mathsf{d}_\tau(\boldsymbol{v}_1^{(0)}, \boldsymbol{v}_1^{(1)}) \ +\ \mathsf{d}_\tau(\boldsymbol{v}_1^{(1)}, \boldsymbol{v'}_1^{(1)}) \\ &<\ 3 \cdot D_1 \end{aligned} \tag{4.21}$$

by (4.16), (4.18) and (4.19). Further, for all $n > N$,

$$\begin{array}{rcll} \mathsf{d}_\tau(\boldsymbol{v}_1^{(N)}, \boldsymbol{v}_1^{(n)}) & \le & \mathsf{d}_\tau(\boldsymbol{v}_1^{(N)}, \boldsymbol{v}_0^{(N+1)}) + \ldots + \mathsf{d}_\tau(\boldsymbol{v}_0^{(n-1)}, \boldsymbol{v}_0^{(n)}) & \\ & \le & (\lambda^N + \lambda^{N+1} + \ldots + \lambda^n) \cdot D_1 & \text{by (3.19)} \\ & < & \dfrac{\lambda^N}{(1-\lambda)} \cdot D_1 & \\ & < & \epsilon/9 & \text{by (4.17)} \end{array}$$

and so (letting $n \to \infty$)

$$\mathsf{d}_\tau(\boldsymbol{v}_1^{(N)}, \boldsymbol{v}_1) \le \epsilon/9 < \epsilon/3. \tag{4.22}$$

Similarly, for all $n > N$,

$$\begin{array}{rcll} \mathsf{d}_\tau(\boldsymbol{v'}_1^{(N)}, \boldsymbol{v'}_1^{(n)}) & < & \lambda^N \cdot (1-\lambda)^{-1} \cdot \mathsf{d}_\tau(\boldsymbol{v'}_1^{(0)}, \boldsymbol{v'}_1^{(1)}) & \\ & < & \lambda^N \cdot (1-\lambda)^{-1} \cdot 3 \cdot D_1 & \text{by (4.21)} \\ & < & \epsilon/3 & \text{by (4.17)} \end{array}$$

and so (letting $n \to \infty$)

$$\mathsf{d}_\tau(\boldsymbol{v'}_1^{(N)}, \boldsymbol{v'}_1) \le \epsilon/3. \tag{4.23}$$

Hence

$$\begin{array}{rcll} \mathsf{d}_\tau(\boldsymbol{v'}_1, \boldsymbol{v}_1) & \le & \mathsf{d}_\tau(\boldsymbol{v'}_1, \boldsymbol{v'}_1^{(N)}) + \mathsf{d}_\tau(\boldsymbol{v'}_1^{(N)}, \boldsymbol{v}_1^{(N)}) + \mathsf{d}_\tau(\boldsymbol{v}_1^{(N)}, \boldsymbol{v}_1) & \\ & < & \epsilon/3 + \epsilon/3 + \epsilon/3 & \text{by (4.23), (4.20), (4.22)} \\ & = & \epsilon, & \end{array}$$

proving the continuity of Φ_1 at $(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})$.

Induction step: Assume Φ_k is continuous. We must show that Φ_{k+1} is continuous, i.e., prove (4.14) for $k \leftarrow k+1$. Put

$$\begin{array}{rcl} \boldsymbol{a}_k & =_{df} & \boldsymbol{v}^s(k\tau) \\ \boldsymbol{a'}_k & =_{df} & \boldsymbol{v'}^s(k\tau). \end{array}$$

From *closure under shifts* of U follows

$$(\boldsymbol{c},\, \boldsymbol{a}_k,\, \boldsymbol{shift}_{k\tau}(\boldsymbol{x})) \in U \qquad \text{and} \qquad (\boldsymbol{c'},\, \boldsymbol{a'}_k,\, \boldsymbol{shift}_{k\tau}(\boldsymbol{x'})) \in U.$$

Further, from *causality* of F follows causality of Φ by Lemma 4.2.3, and from *shift invariance* of F follows shift invariance of Φ by Lemma 4.2.6; hence (by Lemma 3.2.6)

$$\begin{array}{rcl} \Phi_1(\boldsymbol{c},\, \boldsymbol{a}_k,\, \boldsymbol{shift}_{k\tau}(\boldsymbol{x}) \restriction_\tau) & = & \boldsymbol{shift}_{k\tau}(\boldsymbol{v}) \restriction_\tau \\ \Phi_1(\boldsymbol{c'},\, \boldsymbol{a'}_k,\, \boldsymbol{shift}_{k\tau}(\boldsymbol{x'}) \restriction_\tau) & = & \boldsymbol{shift}_{k\tau}(\boldsymbol{v'}) \restriction_\tau . \end{array} \tag{4.23}$$

Now by continuity of Φ_1 (proved above), and of $\boldsymbol{shift}_{k\tau}$ (easily shown), given $\epsilon > 0$ there exists $\delta_1 > 0$ such that for all $(\boldsymbol{c}', \boldsymbol{a}', \boldsymbol{x}') \in U$,

$$\mathsf{d}_\tau\big((\boldsymbol{c}',\ \boldsymbol{a}'_k,\ \boldsymbol{shift}_{k\tau}(\boldsymbol{x}')),\ (\boldsymbol{c},\ \boldsymbol{a}_k,\ \boldsymbol{shift}_{k\tau}(\boldsymbol{x})\big)\ <\ \delta_1 \qquad \Longrightarrow\ \mathsf{d}_\tau\big(\boldsymbol{shift}_{k\tau}(\boldsymbol{v}'),\ \boldsymbol{shift}_{k\tau}(\boldsymbol{v})\big)\ <\ \epsilon$$

and so, by Remark 4.1.5:

$$\mathsf{d}_{k\tau,\,(k+1)\tau}((\boldsymbol{c}', \boldsymbol{a}', \boldsymbol{x}'), (\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}))\ <\ \delta_1 \quad \Longrightarrow \quad \mathsf{d}_{k\tau,\,(k+1)\tau}(\boldsymbol{v}', \boldsymbol{v})\ <\ \epsilon. \tag{4.24}$$

Now by the induction hypothesis (i.e., (4.14) for k), there exists $\delta_2 > 0$ such that

$$\mathsf{d}_{k\tau}((\boldsymbol{c}', \boldsymbol{a}', \boldsymbol{x}'), (\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}))\ <\ \delta_2 \quad \Longrightarrow \quad \mathsf{d}_{k\tau}(\boldsymbol{v}', \boldsymbol{v})\ <\ \epsilon. \tag{4.25}$$

Taking $\delta = \min(\delta_1, \delta_2)$ and combining (4.25) and (4.24), we obtain

$$\mathsf{d}_{(k+1)\tau}((\boldsymbol{c}', \boldsymbol{a}', \boldsymbol{x}'), (\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}))\ <\ \delta \quad \Longrightarrow \quad \mathsf{d}_{(k+1)\tau}(\boldsymbol{v}', \boldsymbol{v})\ <\ \epsilon,$$

proving continuity of Φ_{k+1}, as desired. $\square$

Remark 4.2.10 (Strongly contracting operators). If assumption (i) is replaced by:

(i′) $F_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}} \in \boldsymbol{SContr}(\lambda_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}}\,,\, \tau_{\boldsymbol{c},\boldsymbol{a},\boldsymbol{x}})$

(cf. Remark 3.3.6) then the proof of Theorem 2 becomes *much* easier (see [TZ07], proof of Theorem 2(b)).

Remark 4.2.11 (Theorem 2 in terms of module functions). Suppose F is the network state function for an analog network N. Then Theorem 2 holds if assumption (iv) is replaced by:

(iv′) *the module functions of* N *are continuous.*

This follows from Remark 3.1.6.

Remark 4.2.12 (Assumption of shift invariance). Nicholas James (personal communication) has succeeded in proving Theorem 2 without the assumptions of shift invariance (iii) and closure of U under shifts (vi). Details will be given in a future publication.

Examples 4.2.13 (Continuity; Applying Theorem 2).

• ***Example 1: Analog networks.*** We return to case study 1, the mass/spring/damper system, discussed in Example 3.3.5(1) and 3.4.9(1). From now on we will only consider network N_2 for this system, with state function

$$F = F^{N_2} \colon\ U \times \mathcal{C}[\mathbb{T}, \mathbb{R}]^2 \ \to\ \mathcal{C}[\mathbb{T}, \mathbb{R}]^2$$

where

$$U \ =_{df} \ (\mathbb{R}^{>0})^3 \times \mathbb{R}^2 \times \mathcal{C}[\mathbb{T}, \mathbb{R}],$$

with

$$F_{(M,\, K,\, D),\, (x_0,\, v_0), f} \colon\ \mathcal{C}[\mathbb{T}, \mathbb{R}]^2 \ \to\ \mathcal{C}[\mathbb{T}, \mathbb{R}]^2$$

given by

$$F_{(M,\, K,\, D),\, (x_0,\, v_0), f}\,(x, v) \ = \ F\big((M,\, K,\, D),\, (x_0,\, v_0), f,\ (x, v)\big)$$

for any $((M,\, K,\, D),\, (x_0,\, v_0), f) \in U$. Then taking any (fixed) $\lambda_0 < 1$ and

$$\tau_{M,K,D} \ = \ \lambda_0 \cdot \ \min\left(\frac{M}{K+D},\ 1\right), \tag{4.26}$$

it follows from the considerations in Example 3.3.5(1) (cf. (3.13), (3.14)) that

$$F_{(M,\, K,\, D),\, (x_0,\, v_0), f} \ \in \ \boldsymbol{Contr}(\lambda_0,\, \tau_{M,K,D}).$$

and hence, by Theorem 1, F has a *fixed point* function $\Phi \colon U \ \to\ \mathcal{C}[\mathbb{T}, \mathbb{R}]^2$ with

$$\Phi((M,\, K,\, D),\, (x_0,\, v_0), f) \ = \ \mathsf{FP}(F_{(M,\, K,\, D),\, (x_0,\, v_0), f})$$

for all $((M,\, K,\, D),\, (x_0,\, v_0), f) \in U$. Further, since each of the module functions (pointwise addition, scalar multiplication and integration) is continuous, so is F, from Remark 3.1.6 (and cf. Remark 4.2.11). Moreover, U is clearly closed under shifts, by Remark 4.2.5, and λ_0 (being constant) and $\tau_{M,K,D}$ are clearly locally bounded on U. Further, from equations (3.11) F can be seen to be shift invariant.

Hence Theorem 2 can be applied to prove the continuity of Φ on U.

Theorem 2 can similarly be applied to the second case study in [TZ07] (an iterated mass/spring/damper system). We omit details.

• ***Example 2: SCAs.*** Here we have a very simple special case of Theorem 2:

Theorem 2′. *If the module functions of an SCA network are continuous, then so is the network function* Φ.

This is because Φ is defined from the module functions by primitive recursion, which preserves continuity [TTZ09, Lemma 7.1.1].

Note that in this case Φ is total, i.e., if

$$F \colon A^r \times A^m \times \mathcal{C}[\mathbb{T}, A]^p \times \mathcal{C}[\mathbb{T}, A]^m \ \to \ \ \mathcal{C}[\mathbb{T}, A]^m$$

(putting $s = m$ in (4.2)), then

$$U \;=\; \boldsymbol{dom}(\Phi) \;=\; A^r \times A^m \times \mathcal{C}[\mathbb{T}, A]^p.$$

Caution! It is actually the *cartesian form of* Φ, $\boldsymbol{cart}\,(\Phi)$, that is defined by primitive recursion, and hence continuous, where if

$$\Phi\colon U \;\to\; \mathcal{C}[\mathbb{T}, A]^m$$

then

$$\boldsymbol{cart}\,(\Phi)\colon U \;\times \mathbb{T} \;\to\; A^m$$

is defined by

$$\boldsymbol{cart}\,(\Phi)(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}, t) \;=\; \Phi(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})(t).$$

However one can check that for any stream-valued function f,

$$\boldsymbol{cart}\,(f) \text{ continuous} \quad\Longrightarrow\quad f \text{ continuous,} \tag{4.27}$$

at least in the case that $\mathbb{T} = \mathbb{N}$.

Discussion 4.2.14 (Hadamard's principle; the significance of continuity). As explained in the Introduction (§1.4), the reason for the importance of establishing continuity of the fixed point function under the conditions given in Theorem 2, is that it implies *stability* of the fixed point Φ, as the solution to the specification

$$F(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x}, \Phi(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})) \;=\; \Phi(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{x})$$

under the stated conditions. The significance of this is related to Hadamard's principle [Had52] which, as (re-)formulated by Courant and Hilbert ([CH53, pp. 227ff.],[Had64]) states that for a scientific problem to be well posed, the solution must (apart from existing and being unique) depend continuously on the data.

An important aspect of Hadamard's principle is that it can be viewed as making classical experimental physics possible. Suppose, for example, that one wants to verify any of the well-known relations of classical physics — Hooke's Law or Charles's Law, for example — by taking measurements and drawing a graph of the relationship between the "independent" and "dependent variables" — force vs displacement of a spring in the first example, and temperature vs volume of a gas (at constant pressure) in the second. (The first of these two examples was used implicitly in our first case study.) The experimental results, and consequent graph, only make sense on the assumption that the function that one is attempting to plot is continuous, so that small discrepancies or inaccuracies in the inputs produce only small variations in the outputs. Moreover, this is needed to guarantee *repeatability* of experiments. The stability of measurements in the presence of noise is an essential feature for a physical system to qualify as an analog computer.

Actually, in the formulation of Hadamard's principle, "continuously" should perhaps be replaced by "piecewise continuously", to accommodate discontinuities at phase changes, for example, the gas/liquid interface in connection with Charles's Law.

Further discussions on this topic, from different perspectives, have been given by Beeson [Bee85, p. 368] and Myrvold [Myr95].

5 Concluding remarks

Stream processing occurs everywhere, often without being recognised as such. There are many occasions where a theoretical analysis of computation has led to models of stream processing [Ste97].

In this paper we have used standard topological notions to model stream processing in continuous and discrete time, in a uniform way. An essential technique in this paper has been to lower the type of higher order stream operators by an "uncurrying" process (see Remark 4.1.1, and the comment at the end of Example 4.2.13(2)). This allows the use of standard and relatively elementary technical concepts from topology and (in future work) computability theory. We have used, as running examples, two simple, commonly found paradigms of stream processing, which we previously studied independently [TZ07, TTZ09].

The basic mathematical theory of stream processing raises some intriguing questions about the role of natural assumptions on stream operators such as continuity, causality and shift invariance. The theory presented is designed to be close to examples of systems which are rich in physical properties.

Clearly, the semantic modelling of analog systems benefits most from our approach, as analog computers are both complicated and neglected. Of course, there are many further examples of stream processing (such as dataflow networks and hybrid embedded systems) to be investigated.

5.1 Computability of operations on streams

At the heart of our theory are questions about the computability of stream processing. There are several different approaches to computability on topological spaces, which converge [SHT99]. In a companion paper [TZ11] we will address the question of the computability of Φ. We consider two models of computability on A, and hence on $\mathcal{C}[\mathbb{T}, A]$: *concrete*, based on representations constructed from $\mathbb{N}$, and *abstract*, independent of representations, and based on effective approximability by a high level imperative programming language ***WhileCC****** (that is, the ***While*** language with a "countable choice" operator and finite arrays). The equivalence between these was established in [TZ04]. With Theorems 1 and 2 in mind we prove:

Theorem (Concrete computability): *If, in addition to the assumptions in Theorem 2 (and under some further reasonable assumptions), F is concretely computable, then so is Φ.*

We then use the equivalence between abstract and concrete computability discussed above to prove:

Theorem (Abstract computability): *If, in addition to the assumptions in Theorem 2 (and under some further reasonable assumptions), F is* ***WhileCC**** *approximably computable, then so is Φ.*

5.2 Future research

The study of continuity of stream operators (and their computability [TZ11]) provides a rich source of topics for future research. We mention two such topics here.

(1) ***Partial and nondeterministic module functions.*** From considerations of *continuity*, we are led to consider module functions that are nondeterministic (or many-valued) and partial [TZ04, TZ05].

These features will complicate the theory considerably — for example, in the case of SCAs, it would require replacing a single global clock by a system of local clocks [TTZ09, §8.2(1)]. However, they constitute an important generalisation, because of the desirability of continuity by Hadamard's principle (see the Introduction and Discussion 4.2.14).

Continuity considerations are especially significant with *hybrid systems*, at analog-digital interfaces [NK93].

(2) ***Generalisation of stream concept.*** The considerations in (1) will lead to the investigation of streams which are also partial and nondeterministic.

The use of *piecewise continuous* streams (in the case $\mathbb{T} = \mathbb{R}^{\geq 0}$) forms another important generalisation of the stream concept.

References

[Bee85] M. Beeson. *Foundations of Constructive Mathematics.* Springer-Verlag, 1985.

[BSHT98] J. Blanck, V. Stoltenberg-Hansen, and J.V. Tucker. Domain representation of streams. In B. Möller and J.V. Tucker, editors, *Prospects for hardware foundations*, volume 1546 of *Lecture Notes in Computer Science.* Springer-Verlag, 1998.

[CH53] R. Courant and D. Hilbert. *Methods of Mathematical Physics, Vol. II.* Interscience, 1953. Translated and revised from the German edition [1937].

[Cla10] C. Clare. *Technology for modelling: Electrical analogies, engineering practice, and the development of analogue computing.* Springer-Verlag, 2010.

[CT00] M. Chen and J.V. Tucker. Constructive volume geometry. *Computer Graphics Forum*, 19:281–293, 2000.

[Eng89] R. Engelking. *General Topology.* Heldermann Verlag, 1989.

[Had52] J. Hadamard. *Lectures on Cauchy's Problem in Linear Partial Differential Equations.* Dover, 1952. Translated from the French edition [1922].

[Had64] J. Hadamard. *La Théorie des Équations aux Dérivées Partielles.* Éditions Scientifiques, 1964.

[HY61] J.G. Hocking and G.S. Young. *Topology.* Addison-Wesley, 1961.

[Joh06] K. Johnson. *Algebraic Specifications of Spatial Data Types with Applications to Constructive Volume Geometry.* PhD thesis, Department of Computer Science, Swansea University, Swansea, Wales, 2006.

[JZ11] N. James and J. Zucker. A class of contracting stream operators. In preparation, 2011.

[Kel55] J.L. Kelley. *General Topology*. Van Nostrand, 1955. Reprinted Springer-Verlag, 1975.

[Kle52] S.C. Kleene. *Introduction to Metamathematics*. North Holland, 1952.

[Myr95] W.C. Myrvold. Computability in quantum mechanics. In W. DePauli-Schimanovich, E. Köhler, and F. Stadler, editors, *The Foundational Debate: Complexity and Constructivity in Mathematics and Physics*, pages 33–46. Kluwer, 1995.

[NK93] A. Nerode and W. Kohn. Models for hybrid systems: Automata, topologies, controllability, observability. In R.L. Grossman, A. Nerode, A.P. Ravn, and H. Rischel, editors, *Hybrid Systems*, volume 736 of *Lecture Notes in Computer Science*, pages 317–356. Springer-Verlag, 1993.

[OW97] A.V. Oppenheim and A.S. Willsky. *Signals and Systems (2nd ed.)*. Prentice Hall, 1997.

[Roy63] H.L. Royden. *Real Analysis*. Macmillan, 1963.

[Rud76] W. Rudin. *Principles of Mathematical Analysis* (3rd ed.). McGraw-Hill, 1976.

[SHT99] V. Stoltenberg-Hansen and J.V. Tucker. Concrete models of computation for topological algebras. *Theoretical Computer Science*, 219:347–378, 1999.

[Sim63] G.F. Simmons. *Introduction to Topology and Modern Analysis*. McGraw-Hill, 1963.

[Sma01] James S. Small. *The analogue alternative: The electronic analogue computer in Britain and the USA, 1930–1975*. Routledge, 2001.

[Son90] E.D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer-Verlag, 1990.

[Ste97] R. Stephens. A survey of stream processing. *Acta Informatica*, 34:491–541, 1997.

[TT91] B.C. Thompson and J.V. Tucker. Algebraic specification of synchronous concurrent algorithms and architectures (Revised). Research Report 9-91, Department of Computer Science, Swansea University, Swansea, Wales, 1991.

[TTZ09] B.C. Thompson, J.V. Tucker, and J.I. Zucker. Unifying computers and dynamical systems using the theory of synchronous concurrent algorithms. *Applied Mathematics and Computation*, 215:1386–1403, 2009.

[TZ94] J.V. Tucker and J.I. Zucker. Computable functions on stream algebras. In H. Schwichtenberg, editor, *Proof and Computation: NATO Advanced Study Institute International Summer School at Marktoberdorf, 1993*, pages 341–382. Springer-Verlag, 1994.

[TZ00] J.V. Tucker and J.I. Zucker. Computable functions and semicomputable sets on many-sorted algebras. In S. Abramsky, D. Gabbay, and T. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 5, pages 317–523. Oxford University Press, 2000.

[TZ04] J.V. Tucker and J.I. Zucker. Abstract versus concrete computation on metric partial algebras. *ACM Transactions on Computational Logic*, 5:611–668, 2004.

[TZ05] J.V. Tucker and J.I. Zucker. Computable total functions, algebraic specifications and dynamical systems. *Journal of Logic and Algebraic Programming*, 62:71–108, 2005.

[TZ06] J.V. Tucker and J.I. Zucker. Abstract versus concrete computability: The case of countable algebras. In V. Stoltenberg-Hansen and J. Väänänen, editors, *Logic Colloquium '03, Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic, held in Helsinki, Finland, August 14–20, 2003*, volume 24 of *Lecture Notes in Logic*, pages 377–408. Association for Symbolic Logic, 2006.

[TZ07] J.V. Tucker and J.I. Zucker. Computability of analog networks. *Theoretical Computer Science*, 371:115–146, 2007.

[TZ11] J.V. Tucker and J.I. Zucker. Computability of operators on continuous and discrete time streams. In preparation, 2011.