A Class of Contracting Stream Operators

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Dedicated to John Tucker on the occasion of his 60th birthday.

Abstract

In [1] and [2], Tucker and Zucker present a model for the semantics of analog networks operating on streams from topological algebras. Central to their model is a parametrized stream operator representing the network along with a theory that concerns the existence, uniqueness, continuity, and computability of a fixed point of that stream operator. In this paper we narrow the scope from general topological algebras to algebras of streams that assume values only from a Banach space. This restriction facilitates the definition of a fairly broad class of stream operators to which the theory described in [2] applies.

As a demonstration in their original work, the authors provide two case studies: analog networks which model the behaviour of simple mass-spring-damper systems. The case studies showcase the theory well, but they seem to require the imposition of somewhat peculiar conditions on the parameters (the masses, the spring constants, and the damping coefficients). The extra conditions—while not catastrophic to the case studies—make them somewhat unsatisfying. We show here that while their original mass-spring-damper models do not fall within our new class, they can be trivially reconfigured into equivalent models that do. This modification obviates the extra conditions on the parameters.

Keywords: Analog computing, analog networks, continuous stream operations, continuous time streams, discrete time streams, fixed points, Hadamard's principle, contraction.

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1 Introduction

Analog computation concerns computation on continua rather than on discrete spaces. Where digital computation uses an abstract, symbolic encoding of data and explicitly written algorithms to operate upon them, analog computation uses—as its name would suggest—an analogy or transduction of measured data and a corresponding physical system which serves as a model of another system. The input data can be any sort of measurement (e.g. voltage, pressure, temperature, etc.) from the world outside the model, and it can be represented by any measurable quantity that is within the model. The model is set up to mimic the initial conditions of the system about which we wish to reason or make predictions, and then set in motion and observed. The "language" of analog computation comes directly from the laws of physics rather than from the minds of instruction set engineers and programming language designers.

Admittedly, digital computation often involves analogies as well. An array of bits in a digital computer might be used, for example, to directly represent the status of a series of locks in a canal. Metaphors for data structures, algorithms, and programming language constructs like binary "trees," "simulated annealing," and "inheritance" permeate the literature on digital computation. Hence, we might alternatively dichotomize computation into "algorithmic" and "non-algorithmic" paradigms, but the term, "analog computation" is already well-established and the notion of analogy is *inherent* in it (both in its representation of data and in its actual mechanisms of computation), while it appears only incidentally in digital computation, and often for only didactic purposes.

Putting aside such devices as the Antikythera Mechanism [3], slide rules, planimeters, and similar devices used to compute individual values (we might call them analog "calculators" rather than "computers"), likely the first recorded account of analog computation was written in 1836 by Gaspard-Gustave Coriolis [4], in which he described using gears and cylinders to integrate first-order differential equations. These ideas were further developed (or perhaps reinvented) in 1876 to tackle differential equations of arbitrary order by Lord Kelvin and his brother, James Thomson [5]. While Kelvin and Thomson's ideas were implemented to some extent in the "Argo" fire control system used by the Royal Navy [6], it was Vannevar Bush who designed and built what is likely the most advanced, mechanical analog computer and one of the most famous and practical computers of its day: the differential analyzer.

Claude Shannon, working as a research assistant in Bush's lab, defined a mathematical model of the differential analyzer and named it the "General Purpose Analog Computer" (or "GPAC") in [7]. The GPAC is an example of what could more generally be called an *analog network*, which may be visualized as a circuit: a directed graph in which the nodes are processing elements known as "modules" and the edges (known as "channels") act as wires or tubes to convey data streams (which are functions of time).

The network is merely a conceptual model, however, and is not intended to describe the actual appearance of the system. An electronic or hydraulic implementation of an analog network might physically resemble the directed graph itself, while a mechanical implementation often wouldn't. A module to perform scalar multiplication, for example, could be implemented as a step-up transformer or a transistor amplifier in an electronic circuit (both of which commonly appear in schematics), whereas the same module could be implemented mechanically as the physical interface between the teeth of two cogs of differing diameters (which does not so neatly suggest a node in a schematic). Hence, a physical system that bears no apparent resemblance to a network at all, may still qualify in our vernacular as an "analog network."

One of the main purposes of defining such a model is to determine the set of functions it is capable of generating, for if some physical device can reliably generate a particular function, it follows that this function is "computable" in the plainest and most intuitive sense of the word. Shannon proved[2] that the GPAC is capable of generating all and only the differentially algebraic functions. This is a very large class of functions, including polynomials of one real variable along with sinusoids, exponential functions, and solutions of ordinary differential equations consisting of these functions. It is not, however, without some disappointing limitations—Shannon's poster child being the well-known gamma function, which is not differentially algebraic.

Partially inspired by these limitations and partially by the assumption that the brain is a type of analog computer which is known to perform spatial as well as temporal integration, Lee Rubel defined the "Extended Analog Computer" (or EAC) in [10]. Rubel's EAC is theoretically capable of solving boundary value problems for partial differential equations, whereas the GPAC is limited (according to Shannon's definition) to initial value problems of ordinary differential equations. Jonathan Mills ran with Rubel's model, creating fully-functional analog computers inspired by the EAC from foam sheets typically used as packaging material and even blocks of salted gelatin [11]. There have been other implementations of analog computation that represent an even more profound departure from the GPAC model. Slime mold [12] and bees [13] have been used to solve small instances of the Travelling Salesperson Problem and generate near-optimal solutions to larger instances.

While models of analog computation offer one approach for investigating the computability of functions involving continua, there has been a parallel research effort focused on extending classical computability theory (as defined by Turing, Church, Kleene, etc.) into this realm: *computable analysis.* Pioneered primarily

[2]There were some problems with his proof which Marian Pour-El addressed and attempted to rectify in [8] using an alternative GPAC model. Ironically, there were problems of a similar nature (glossing over important steps which may actually be insurmountable) in her own approach, which were spotted and corrected by Daniel S. Graça and José Félix Costa in [9] using a third GPAC model.

by Andrzej Grzegorczyk [14, 15] and Daniel Lacombe [16], computable analysis puts real (and complex) analysis, functional analysis, and numerical analysis under the microscope of classical computability theory and asks the question central to most research on analog computation: which functions are computable? We already have a clear answer to that question in the domain of classical computability theory (i.e. for functions of the form $f : \mathbb{N} \to \mathbb{N}$), as all of the models of digital computation we've discovered so far are in agreement. This is, of course, the foundation for the famous Church-Turing Thesis.

Computability theory on continua has not yet reached the same degree of consensus, but much progress is being made. Olivier Bournez et al. showed in [17] that the GPAC is equivalent to Ker-I Ko's model of computability [18] as long as the GPAC is permitted to *approximate* functions (to arbitrary precision) rather than produce them in real time. Viggo Stoltenberg-Hansen and John Tucker used domain representability in [19] to prove that five different models of computation on topological algebras are equivalent (under some modest conditions). Further equivalence results (and exceptions) can be found in [20]. The matter is still not entirely settled, so the question of computability pertaining to functions with uncountable domains or codomains remains open for now.

In [1, 2], John Tucker and Jeff Zucker turn this question around and ask instead, given a particular analog network, under what conditions does it produce meaningful output, and under what conditions does this output vary continuously with the network's parameters? The significance of the latter question is grounded in the imperative of experimental physics known as "Hadamard's Principle," first articulated by Hadamard [21] and later refined by Courant and Hilbert [22]. Its fundamental tenet is that for the solution to a problem in physics to be practically applicable, it must vary continuously with the parameters of the system so that small discrepancies or inaccuracies in the input produce only small variations in the output. The stability of measurements in the presence of noise is an essential feature for a physical system to qualify as an analog computer.

Like the GPAC, the data streams carried by the analog networks in [1, 2] are functions of time. There are, of course, various ways of modelling time. The debate over whether spacetime is continuous, discrete, or even both simultaneously (see [23]) is ongoing, but regardless of the outcome of that debate, the majority of our physical laws and theories treat measurable quantities (including time) as real numbers. This may suggest using the whole real line as a model of time, but regardless of the duration a computer is allowed to run while solving a problem it must at some point actually be built, initialized, and started. For this reason, the authors chose to represent time using the only the nonnegative real numbers (as we do here, as well).

Unlike the GPAC, which uses real-valued streams, the streams in [1, 2] assume values from an arbitrary complete separable metric space $\mathcal{A}$. It is assumed that the space $\mathcal{A}$ is an algebra of some sort (or of multiple sorts), but no specific algebraic requirements are imposed on it. In this paper, we sacrifice some of that generality to investigate the results that can be obtained within the theory by taking $\mathcal{A}$ to be a Banach space.

There are three main sections to be covered: the infrastructure of the theory from [2], the new results attainable by using a Banach space, and finally some applications from elementary physics. In the first section we review the space of objects with which the theory is concerned: stream spaces and the operators defined on them. Specifically, we look at a handful of properties and a collection of algebraic operations that can be assembled to form a variety of stream operators. We present two fixed point theorems from prior work ([2, 24]) of which we make extensive use. Our main goal here is to produce operators which satisfy the antecedents of those theorems.

There are two operator properties of particular importance: causality and contraction. Causality is a basic requirement of the theory, without which we couldn't get off the ground at all, while contraction does most of the heavy lifting. This is not contraction in the usual sense, but rather it is a domain-restricted, conditional version of contraction introduced in [2]. We generalize this property here as a type of Lipschitz condition.

In the second section we conduct a thorough inventory of the pointwise stream operations induced by the algebraic operations of the Banach space, and examine the way each of them affects the aforementioned properties. While the pointwise operations yield a wide assortment of operators that satisfy the Lipschitz condition, the real engine behind our results is integration. An operator which satisfies the Lipschitz condition is all well and good, but in order to work with the two fixed point theorems, the operator must be *contracting,* and that is what integration provides. The integral (with respect to time) of a Lipschitz operator is contracting (in our modified sense of the term). All of these results are consolidated into a pair of lemmas (the Building Block and Continuity lemmas) and a single main theorem (the General Form Theorem).

In the third section, we move on to discuss two applications from mechanical physics. The first is the mass-spring-damper system described in [1, 2], which our general form is more than powerful enough to handle on its own. The second—which is only a simple pendulum—neatly highlights the limitations of that form, as it is apparently not general enough to apply to that system. If we introduce a predefined operator (the sin function, in this case), however, we can still apply the two main lemmas separately to do the work the theorem cannot.

This paper is relatively self-contained and requires no special background in analog computation or computable analysis, although it does assume some familiarity with elementary real analysis, functional analysis, and topology. It is also very closely associated with [2] (and to a lesser extent [1]), so the reader is strongly encouraged to read that first. Although no results concerning (digital) computability are discussed here, a great deal of the impetus for this work is to lay the foundation for a follow-up paper which will examine the operators we consider from the perspective of computable analysis. In this respect, we intend to mirror the two companion papers [2] and [25]—the latter of which delineates a set of extra conditions under which the operators in the former are computable (according to two different models of computability from computable analysis).

1.1 The Space of Streams

Let $\mathcal{A}$ be a complete metric space with the metric $d_{\mathcal{A}}$. We use the symbol $\mathbb{T}$ to represent time, taking[3] $\mathbb{T} = \mathbb{R}^+ \cup \{0\}$. We adopt $\mathcal{C}[\mathbb{T}, \mathcal{A}]^m$ (for some $m \in \mathbb{Z}^+$) as our fundamental *stream space:* the space of m-tuples of continuous functions from $\mathbb{T}$ into $\mathcal{A}$.

For $m = 1$ we define a family of pseudometrics[4] $\{d_{a,b} : a, b \in \mathbb{T} \text{ and } a \leq b\}$ where $\forall u, v \in \mathcal{C}[\mathbb{T}, \mathcal{A}]$,

$$d_{a,b}(u, v) = \max_{a \leq t \leq b} d_{\mathcal{A}}\left(u(t), v(t)\right)$$

Observe that if our stream space were instead $\mathcal{C}[[a, b], \mathcal{A}]$, then $d_{a,b}$ would be a metric. It is a pseudometric only because it "ignores" any differences between its arguments outside the interval $[a, b]$.

For $m \in \mathbb{Z}^+$ and $\mathbf{u} = (u_1, u_{2,}, \ldots, u_m)$, $\mathbf{v} = (v_1, v_2, \ldots, v_m) \in \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$ we define

$$d^m_{a,b}(\mathbf{u}, \mathbf{v}) = \max_{1 \leq k \leq m} d_{a,b}\left(u_k, v_k\right)$$

In practice, however, we will drop the superscript since no ambiguity is introduced by overloading the symbol $d_{a,b}$. Furthermore, it so often the case that we set $a = 0$ that typically we just write $d_b(\mathbf{u}, \mathbf{v})$ to mean $d^m_{0,b}(\mathbf{u}, \mathbf{v})$.

This family of pseudometrics induces the *local uniform topology* on $\mathcal{C}[\mathbb{T}, \mathcal{A}]^m$. A basis for this topology is given by open balls of the form

$$B_{T,\varepsilon}(\mathbf{u}) = \{\mathbf{v} \in \mathcal{C}[\mathbb{T}, \mathcal{A}]^m \;:\; d_T(\mathbf{u}, \mathbf{v}) < \varepsilon\}$$

for $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$, $T \in \mathbb{T}$, and $\varepsilon > 0$. See [2] for a discussion of its equivalence to the compact-open topology and the inverse limit topology in this context.

[3] Tucker and Zucker also develop their theory to address the case in which $\mathbb{T} = \mathbb{N}$, but here we'll be using only the continuum of nonnegative reals.

[4] A pseudometric is like a metric except that it is permitted to be zero even for distinct points. That is, if $d : X^2 \to Y$ is a pseudometric, then d is also a metric iff $\forall x, y \in X\ [d(x, y) = 0 \;\Rightarrow\; x = y]$.

In [2] it is shown that $\mathcal{C}[\mathbb{T},\mathcal{A}]^m$ is homeomorphic to $\mathcal{C}[\mathbb{T},\mathcal{A}^m]$, so the theory could just as easily be presented using only streams from $\mathcal{C}[\mathbb{T},\mathcal{A}]$, taking $\mathcal{A}=\mathcal{B}^m$ for some other space $\mathcal{B}$ whenever m-tuples are required. In many situations this would indeed be simpler, but analog networks typically carry many streams (one for each edge or channel in the network), so it makes sense to use m-tuples here.

There is, in fact, a metric that can be defined on $\mathcal{C}[\mathbb{T},\mathcal{A}]^m$ using the family of pseudometrics (see the "d_C" metric in [2]), but it is somewhat unwieldy in practice. So while the metric is important for showing that $\mathcal{C}[\mathbb{T},\mathcal{A}]^m$ (with the local uniform topology) is indeed metrizable, we continue to use the pseudometrics when actually reasoning about the space. In particular, we have the following lemma which is convenient for proving the continuity of stream functions:

Lemma 1.1. *A function* $f:\mathcal{C}[\mathbb{T},\mathcal{A}]^m\to\mathcal{C}[\mathbb{T},\mathcal{A}]^m$ *is continuous iff* $\forall\varepsilon>0\ \forall T\in\mathbb{T}\ \forall\mathbf{u}\in\mathcal{C}[\mathbb{T},\mathcal{A}]^m\ \exists\delta>0$ $\exists T'\in\mathbb{T}\ \forall\mathbf{v}\in\mathcal{C}[\mathbb{T},\mathcal{A}]^m$,

$$d_{T'}(\mathbf{u},\mathbf{v})<\delta\Rightarrow d_T\left(f(\mathbf{u}),f(\mathbf{v})\right)<\varepsilon$$

Finally, we will often form a product space of some metric space (X,d_X) and $\mathcal{C}[\mathbb{T},\mathcal{A}]^m$. An equivalent family of pseudometrics ("equivalent" in the sense that they collectively generate the same topology as the metric) on this product space can be defined as

$$d_T^{(X\times\mathcal{C}[\mathbb{T},\mathcal{A}]^m)}\left((x,\mathbf{u}),(y,\mathbf{v})\right)=\max\left\{d_X(x,y),d_T(\mathbf{u},\mathbf{v})\right\}$$

Again, without loss of specificity, we will drop the superscript and use simply d_T.

1.2 Properties of Stream Operators

Definition 1.2 (***Caus*** and ***WCaus***)**.** Let $F:\mathcal{C}[\mathbb{T},\mathcal{A}]^m\to\mathcal{C}[\mathbb{T},\mathcal{A}]^m$. If $\forall T\in\mathbb{T}\ \forall\mathbf{u},\mathbf{v}\in\mathcal{C}[\mathbb{T},\mathcal{A}]^m$,

$$(\forall t<T\ \mathbf{u}(t)=\mathbf{v}(t))\ \Rightarrow\ F(\mathbf{u})(T)=F(\mathbf{v})(T)$$

then we say that F satisfies ***Caus*** or $F\in$ ***Caus***. It is named as such since the property represents a form of causality. At each point in time, the value of $F(\mathbf{u})$ can be determined without any knowledge of future or present values of $\mathbf{u}$.

If instead,

$$(\forall t\le T\ \mathbf{u}(t)=\mathbf{v}(t))\ \Rightarrow\ F(\mathbf{u})(T)=F(\mathbf{v})(T)$$

then we say that F satisfies $\boldsymbol{WCaus}$ ("weak causality").

Remark 1.3. Causality conditions appear throughout control theory and signal processing (see [26] for example), and in several other contexts as well. Conditions almost identical to the two versions we define above (differing only in the domains and codomains of the operators involved), $\boldsymbol{WCaus}$ and $\boldsymbol{Caus}$, are identified in [27] and [28] as "retrospective" and "strongly retrospective," respectively.

Remark 1.4. As observed in [2], $F \in \boldsymbol{Caus}$ iff $F \in \boldsymbol{WCaus}$ and $\forall \mathbf{u}, \mathbf{v} \in \mathcal{C}[\mathbb{T}, \mathcal{A}]^m\ F(\mathbf{u})(0) = F(\mathbf{v})(0)$.

Definition 1.5 ($\boldsymbol{Lip}$). Let $F : \mathcal{C}[\mathbb{T}, \mathcal{A}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$. If $\exists \tau, \lambda \in \mathbb{R}^+ \cup \{0\}$ such that $\forall T \in \mathbb{T}\ \forall \mathbf{u}, \mathbf{v} \in \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$,

$$d_T(\mathbf{u}, \mathbf{v}) = 0 \ \Rightarrow\ d_{T+\tau}\left(F(\mathbf{u}), F(\mathbf{v})\right) \leq \lambda d_{T+\tau}(\mathbf{u}, \mathbf{v})$$

then we say that F satisfies $\boldsymbol{Lip}(\lambda, \tau)$ or $F \in \boldsymbol{Lip}(\lambda, \tau)$. The name is due to the similarity this property shares with the well-known Lipschitz continuity property from analysis (although traditionally α is in place of our λ).

Remark 1.6. It may seem as though $F \in \boldsymbol{Lip}(\lambda, \tau) \Rightarrow F \in \boldsymbol{WCaus}$. After all, if we take any $T \geq \tau$ and a pair of streams $\mathbf{u}, \mathbf{v} \in \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$ such that $d_T(\mathbf{u}, \mathbf{v}) = 0$, then certainly $d_{T-\tau}(\mathbf{u}, \mathbf{v}) = 0$. Hence, $d_{\mathcal{A}}\left(F(\mathbf{u})(T), F(\mathbf{v})(T)\right) \leq d_T\left(F(\mathbf{u}), F(\mathbf{v})\right) = d_{(T-\tau)+\tau}\left(F(\mathbf{u}), F(\mathbf{v})\right) \leq \lambda d_{(T-\tau)+\tau}(\mathbf{u}, \mathbf{v}) = \lambda d_T(\mathbf{u}, \mathbf{v}) = 0$. And therefore, $F(\mathbf{u})(T) = F(\mathbf{v})(T)$. Hence, any $F \in \boldsymbol{Lip}(\lambda, \tau)$ could be said to satisfy $\boldsymbol{WCaus}$ on $[\tau, \infty) \subseteq \mathbb{T}$.

There is, however, no way to establish the causality (weak or otherwise) of such an F on $[0, \tau)$, as the following example demonstrates.

Example 1.7 ($\boldsymbol{Lip} \not\Rightarrow \boldsymbol{WCaus}$). Take $\mathcal{A} = \mathbb{R}$ with the usual metric, let $\tau \in \mathbb{R}^+$, and choose $m = 1$. Define $F : \mathcal{C}[\mathbb{T}, \mathbb{R}] \to \mathcal{C}[\mathbb{T}, \mathbb{R}]$ as follows:

$$F(u)(t) = \begin{cases} \frac{1}{2}u(\tau) & \text{if } 0 \leq t \leq \tau \\ \frac{1}{2}u(t) & \text{if } t > \tau \end{cases}$$

Then $F \in \boldsymbol{Lip}(1/2, \tau)$ (and it's even continuous), but it does not satisfy $\boldsymbol{WCaus}$. To see this, consider $u(t) = t$ and $v(t) = -t$. Taking $T = 0$, we see that $\forall t \leq T\ u(T) = v(T) = 0$, but $F(u)(0) = \tau/2 \neq -\tau/2 = F(v)(0)$. Note that such an example would not be possible if we were to take $\mathbb{T} = \mathbb{R}$ ($\boldsymbol{Lip}(\lambda, \tau)$ would give us $\boldsymbol{WCaus}$ "for free" on such a stream space), but adapting the rest of the theory to work on $\mathcal{C}[\mathbb{R}, \mathcal{A}]$ would not be trivial and nor would it necessarily be an improvement overall (see Section 1 for an explanation).

Lemma 1.8. *If $F \in \boldsymbol{Lip}(\lambda, \tau)$ and $F \in \boldsymbol{WCaus}$ then $\forall \tau' \leq \tau\ \forall \lambda' \geq \lambda\ F \in \boldsymbol{Lip}(\lambda', \tau')$.*

Proof. Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$, $T \in \mathbb{T}$ and suppose $d_T(\mathbf{u}_1, \mathbf{u}_2) = 0$. For $\lambda' \geq \lambda$, it is obvious that $F \in \boldsymbol{Lip}(\lambda', \tau)$:

$$d_{T+\tau}(F(\mathbf{u}_1), F(\mathbf{u}_2)) \leq \lambda d_{T+\tau}(\mathbf{u}_1, \mathbf{u}_2) \leq \lambda' d_{T+\tau}(\mathbf{u}_1, \mathbf{u}_2)$$

The τ' assertion is less trivial. For $i = 1, 2$, define $\mathbf{u}_i^* \in \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$ as follows:

$$\mathbf{u}_i^*(t) = \begin{cases} \mathbf{u}_i(t) & \text{if } t < T + \tau' \\ \mathbf{u}_i(T + \tau') & \text{if } t \geq T + \tau' \end{cases}$$

Then for $0 < \tau' < \tau$,

$$\begin{aligned} d_{T+\tau'}\left(F(\mathbf{u}_1), F(\mathbf{u}_2)\right) &= d_{T+\tau'}\left(F\left(\mathbf{u}_1^*\right), F\left(\mathbf{u}_2^*\right)\right) \quad \text{since } F \in \boldsymbol{WCaus} \text{ and } d_{T+\tau'}(\mathbf{u}_i^*, \mathbf{u}_i) = 0 \\ &\leq d_{T+\tau}\left(F\left(\mathbf{u}_1^*\right), F\left(\mathbf{u}_2^*\right)\right) \quad \text{since } t < r \Rightarrow d_t(\mathbf{v}, \mathbf{w}) \leq d_r(\mathbf{v}, \mathbf{w}) \\ &\leq \lambda' d_{T+\tau}\left(\mathbf{u}_1^*, \mathbf{u}_2^*\right) \quad \text{since } F \in \boldsymbol{Lip}(\lambda', \tau) \\ &= \lambda' d_{T+\tau'}\left(\mathbf{u}_1^*, \mathbf{u}_2^*\right) \quad \text{since } \mathbf{u_i} \text{ are constant beyond } T + \tau' \\ &= \lambda' d_{T+\tau'}\left(\mathbf{u}_1, \mathbf{u}_2\right) \quad \text{since } d_{T+\tau'}(\mathbf{u}_i^*, \mathbf{u}_i) = 0 \end{aligned}$$

□

Remark 1.9. As in Remark 1.6, the only reason we must require F to satisfy $\boldsymbol{WCaus}$ in the proof of Lemma 1.8 is to establish the inequality for $T < \tau - \tau'$. For if $T \geq \tau - \tau'$ then

$$\begin{aligned} d_T(\mathbf{u}_1, \mathbf{u}_2) = 0 &\Rightarrow d_{T-\tau+\tau'}(\mathbf{u}_1, \mathbf{u}_2) = 0 \\ &\Rightarrow d_{(T-\tau+\tau')+\tau}\left(F\mathbf{u}_1, F\mathbf{u}_2\right) \leq \lambda' d_{(T-\tau+\tau')+\tau}(\mathbf{u}_1, \mathbf{u}_2) \\ &\qquad = \lambda' d_{T+\tau'}(\mathbf{u}_1, \mathbf{u}_2) \end{aligned}$$

This argument doesn't rely on $\boldsymbol{WCaus}$ at all, but it does require $T - \tau + \tau' \geq 0$ (so it isn't quite sufficient to show $F \in \boldsymbol{Lip}(\lambda', \tau')$).

Remark 1.10. Note that for any $\lambda \geq 0$, $\boldsymbol{WCaus}$ is actually equivalent to $\boldsymbol{Lip}(\lambda, 0)$. Putting this observation together with Lemma 1.8 yields the following result:

$$F \in Lip(\lambda, \tau) \cap \boldsymbol{WCaus} \Longleftrightarrow (\forall \tau' \leq \tau) \;\; F \in \boldsymbol{Lip}(\lambda, \tau')$$

Definition 1.11 (***Contr***)**.** Let $F : \mathcal{C}[\mathbb{T}, \mathcal{A}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$. If $F \in \boldsymbol{Lip}(\lambda, \tau)$ and $0 < \lambda < 1$ then we say that F satisfies $\boldsymbol{Contr}$ or $F \in \boldsymbol{Contr}$, named for the similarity this property shares with the notion of *contraction* on a metric space.

Remark 1.12. In order to be more consistent with [2] and to get the most general results possible, it would seem preferable to define $\boldsymbol{Lip}(\lambda, \tau)$ using the apparently weaker condition,

$$d_T(\mathbf{u}, \mathbf{v}) = 0 \Rightarrow d_{T,T+\tau}(F(\mathbf{u}), F(\mathbf{v})) \leq \lambda d_{T,T+\tau}(\mathbf{u}, \mathbf{v})$$

Call this condition $\boldsymbol{Lip}'(\lambda, \tau)$. One could not be faulted for thinking this definition is strictly more inclusive than $\boldsymbol{Lip}(\lambda, \tau)$, and it matches the definition of $\boldsymbol{Contr}(\lambda, \tau)$ in [2] much more closely. In fact, it turns out that the two definitions are equivalent (so we stand by Definition 1.5).

Proposition 1.13 (Equivalence of $\boldsymbol{Lip}$ definitions)**.** *Let* $F : \mathcal{C}[\mathbb{T}, \mathcal{A}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$*,* $\lambda \in \mathbb{R}^+$*,* $\tau \in \mathbb{T}$*. Then* $F \in \boldsymbol{Lip}(\lambda, \tau)$ *if and only if* $F \in \boldsymbol{Lip}'(\lambda, \tau)$*.*

Proof. Let $T \in \mathbb{T}$ and $\mathbf{u}, \mathbf{v} \in \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$ such that $d_T(\mathbf{u}, \mathbf{v}) = 0$.

($\Rightarrow$) Suppose $F \in \boldsymbol{Lip}(\lambda, \tau)$. Then

$$\begin{aligned} d_{T,T+\tau}(F(\mathbf{u}), F(\mathbf{v})) &\leq d_{T+\tau}(F(\mathbf{u}), F(\mathbf{v})) \\ &\leq \lambda d_{T+\tau}(\mathbf{u}, \mathbf{v}) \end{aligned}$$

The first inequality holds because $d_{T+\tau}(F(\mathbf{u}), F(\mathbf{v})) = \max\{d_T(F(\mathbf{u}), F(\mathbf{v})), d_{T,T+\tau}(F(\mathbf{u}), F(\mathbf{v}))\}$, and the second because $F \in \boldsymbol{Lip}(\lambda, \tau)$.

Now,

$$\begin{aligned} \lambda d_{T+\tau}(\mathbf{u}, \mathbf{v}) &= \lambda \max\{d_T(\mathbf{u}, \mathbf{v}),\ d_{T,T+\tau}(\mathbf{u}, \mathbf{v})\} \\ &= \lambda \max\{0, d_{T,T+\tau}(\mathbf{u}, \mathbf{v})\} \\ &= \lambda d_{T,T+\tau}(\mathbf{u}, \mathbf{v}) \end{aligned}$$

Therefore, $d_{T,T+\tau}(F(\mathbf{u}), F(\mathbf{v})) \leq \lambda d_{T,T+\tau}(\mathbf{u}, \mathbf{v})$.

($\Leftarrow$) Suppose $F \in \boldsymbol{Lip}'(\lambda, \tau)$. We must show that $d_{T+\tau}(F(\mathbf{u}), F(\mathbf{v})) \leq \lambda d_{T+\tau}(\mathbf{u}, \mathbf{v})$. As before, note that $\lambda d_{T+\tau}(\mathbf{u}, \mathbf{v}) = \lambda d_{T,T+\tau}(\mathbf{u}, \mathbf{v})$. Similarly, $d_{T+\tau}(F(\mathbf{u}), F(\mathbf{v})) = \max\{d_T(F(\mathbf{u}), F(\mathbf{v})), d_{T,T+\tau}(F(\mathbf{u}), F(\mathbf{v}))\}$.

Hence, we need to establish the following inequalities:

$$\begin{aligned} d_T(F(\mathbf{u}), F(\mathbf{v})) &\leq \lambda d_{T,T+\tau}(\mathbf{u}, \mathbf{v}) \\ d_{T,T+\tau}(F(\mathbf{u}), F(\mathbf{v})) &\leq \lambda d_{T,T+\tau}(\mathbf{u}, \mathbf{v}) \end{aligned}$$

The latter follows directly from the hypothesis, but the former requires a bit of work. We'll use an inductive approach for this. For the base case, suppose $0 \leq T < \tau$. Since $d_T(\mathbf{u}, \mathbf{v}) = 0$, it follows that $d_0(\mathbf{u}, \mathbf{v}) = 0$. Since $F \in \boldsymbol{Lip}'(\lambda, \tau)$,

$$\begin{aligned} d_\tau(F(\mathbf{u}), F(\mathbf{v})) &= d_{0,0+\tau}(F(\mathbf{u}), F(\mathbf{v})) \\ &\leq \lambda d_{0,0+\tau}(\mathbf{u}, \mathbf{v}) \\ &= \lambda d_\tau(\mathbf{u}, \mathbf{v}) \end{aligned}$$

Since $T < \tau$, $d_T(F(\mathbf{u}), F(\mathbf{v})) \leq d_\tau(F(\mathbf{u}), F(\mathbf{v}))$.

Since $T + \tau \geq \tau$, $\lambda d_\tau(\mathbf{u}, \mathbf{v}) \leq \lambda d_{T+\tau}(\mathbf{u}, \mathbf{v})$.

Putting these last three results together we get,

$$\begin{aligned} d_T(F(\mathbf{u}), F(\mathbf{v})) &\leq d_\tau(F(\mathbf{u}), F(\mathbf{v})) \\ &\leq \lambda d_\tau(\mathbf{u}, \mathbf{v}) \\ &\leq \lambda d_{T+\tau}(\mathbf{u}, \mathbf{v}) \\ &= \lambda d_{T,T+\tau}(\mathbf{u}, \mathbf{v}) \end{aligned}$$

Now, for the inductive step, let $n \in \mathbb{Z}^+$ and assume that $\forall t < n\tau \ \forall u, v \in \mathcal{C}[\mathbb{T}, \mathcal{A}]$,

$$d_t(\mathbf{u}, \mathbf{v}) = 0 \ \Rightarrow \ d_{t+\tau}(F(\mathbf{u}), F(\mathbf{v})) \leq \lambda d_{t+\tau}(\mathbf{u}, \mathbf{v})$$

Suppose $n\tau \leq T < (n+1)\tau$. We must show that

$$d_T(\mathbf{u}, \mathbf{v}) = 0 \ \Rightarrow \ d_{T+\tau}(F(\mathbf{u}), F(\mathbf{v})) \leq \lambda d_{T+\tau}(\mathbf{u}, \mathbf{v})$$

Since $d_T(\mathbf{u}, \mathbf{v}) = 0$ and $0 \leq T - \tau \leq T$, it follows that

$$d_{T-\tau}(\mathbf{u}, \mathbf{v}) = 0$$

So, by the inductive hypothesis and the fact that $T - \tau < n\tau$,

$$\begin{aligned} d_{(T-\tau)+\tau}(F(\mathbf{u}), F(\mathbf{v})) &\leq \lambda d_{(T-\tau)+\tau}(\mathbf{u}, \mathbf{v}) \\ &= \lambda d_T(\mathbf{u}, \mathbf{v}) \\ &= 0 \\ &\leq \lambda d_{T,T+\tau}(\mathbf{u}, \mathbf{v}) \end{aligned}$$

□

1.3 The Fixed Point Theorems

The intent of this paper is to identify a particular class of operators that satisfy the preconditions of two major theorems from [2, 24]. Here are those theorems.

Theorem TZ1. *If $F : \mathcal{C}[\mathbb{T}, \mathcal{A}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$ satisfies **Caus** and **Contr**, then F has a unique fixed point.*

Proof. See Theorem 1 in [2]. □

Remark 1.14. If P is some parameter space adjoined to the domain of F (i.e. F has the type $F : P \times \mathcal{C}[\mathbb{T}, \mathcal{A}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$), and $F(p, \cdot)$ has a unique fixed point stream for every p in some subset $V \subseteq P$, then we define the fixed-point function $\Phi : V \to \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$ such that $\forall p \in V$, $\Phi(p) = F(p, \Phi(p))$.

Theorem TZJ2. *Let P be a metric space and $F : P \times \mathcal{C}[\mathbb{T}, \mathcal{A}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$. Let $p \in P$ and let $V \subseteq P$ be a neighbourhood of p. Let $\tau, \lambda \in \mathbb{R}^+$ with $\lambda < 1$. Using the notation $F_r(\mathbf{u}) = F(r, \mathbf{u})$, suppose that for all $r \in V$ F_r satisfies **Caus** and **Lip**(λ, τ), and that for all $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$ F is continuous at $(p, \mathbf{u})$.*

Then $\Phi : V \to \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$ (as described in Remark 1.14, whose existence is assured by Theorem TZ1) is continuous at p.

Proof. See [24] □

Remark 1.15. The name "TZJ2" was chosen because Nick James needed a theorem very similar to John Tucker and Jeff Zucker's Theorem 2 from [2] (see page 3398), so he adapted it as shown above. In the original theorem, the continuity of $F(p, \mathbf{u})$ (for $p \in P, \mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{A}]^m$) is assumed with respect to only p, and not to $\mathbf{u}$. In its place is an assumption of *shift invariance* for F. We have essentially traded the assumption of shift invariance for a more demanding continuity assumption (i.e. continuity with respect to both p and $\mathbf{u}$).

It turns out that the two conditions are nearly interchangeable in this context. The particular brand of shift invariance used in [2] applies quite naturally to the type of integration introduced in Lemma 1.17 and used throughout the rest of the paper. The remainder of the operations in the Building Block Lemma (below) are all pointwise, and thus, in the parlance of signal processing, they are "memoryless" (i.e. they do not depend on past input values at all). A memoryless operation trivially satisfies (or preserves, depending on the operator in question) shift invariance. Hence, all of the main results below work out just as well if we exchange the additional continuity assumption with shift invariance (and use Theorem 2 from [2] in place of Theorem TZJ2 to derive our results). Our reasons for choosing continuity are that, unlike shift invariance, it places no restrictions on the parameter space P, it introduces no overhead in the exposition, and it is generally considered to be a more fundamental property.

1.4 Algebra of Streams over a Banach Space

The operators with which we are concerned in this paper operate on streams from $\mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, where $\mathcal{B}$ is a Banach space over a field of scalars $\mathcal{S}$. The norm on $\mathcal{B}$ will be denoted using double bars, $\|\cdot\|$, and it induces a metric $d_{\mathcal{B}}(x, y) = \|x - y\|$. The same m-tuple convention used for the stream metric will be used for the norms on both $\mathcal{B}$ and $\mathcal{S}$: $\|(u_1, \ldots, u_m)\| = \max_{1 \le k \le m} \|u_k\|$ and $|(a_1, \ldots, a_k)| = \max_{1 \le k \le m} |a_k|$. Furthermore, corresponding to each pseudometric d_T (for $T \in \mathbb{T}$) is a seminorm (or a "pseudonorm," using the vernacular in [29]) $\|\mathbf{u}\|_T = d_T(\mathbf{u}, \mathbf{0})$.

$\mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ inherits several properties directly from $\mathcal{B}$—almost enough to make it a Banach space itself. The addition operation on $\mathcal{B}$ naturally induces (pointwise) addition on $\mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ (the continuity of the sum of two streams is assured by the subadditivity of the norm on $\mathcal{B}$). Scalars from $\mathcal{S}$ operate on $\mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ as they do on $\mathcal{B}$ (e.g. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$, $(ab)\mathbf{u} = a(b\mathbf{u})$, etc.). It is shown in [2] that if $\mathcal{B}$ is separable and complete (which it is, being a Banach space), then so too is $\mathcal{C}[\mathbb{T}, \mathcal{B}]^m$. Similarly (although not addressed in [2]), the local convexity of $\mathcal{B}$ assures the local convexity of $\mathcal{C}[\mathbb{T}, \mathcal{B}]^m$.

This collection of properties ensures that $\mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ is at least a Fréchet space[5] over $\mathcal{S}$, but since the origin of

[5] A Fréchet space is like a Banach space, except it lacks a norm. In its place, however, a Fréchet space has a countable

$\mathcal{C}[\mathbb{T},\mathcal{B}]^m$ does not necessarily contain an open bounded neighbourhood[6], it follows from Theorem 1.39 in [30] that $\mathcal{C}[\mathbb{T},\mathcal{B}]^m$ is not normable. Hence it is not, itself, a Banach space.

For our purposes, however, a more useful observation is that $\mathcal{C}[\mathbb{T},\mathcal{S}]^m$ could *almost* serve as the set of scalars for the Fréchet space $\mathcal{C}[\mathbb{T},\mathcal{B}]^m$. Addition and multiplication on $\mathcal{S}$ induce corresponding pointwise operations under which $\mathcal{C}[\mathbb{T},\mathcal{S}]^m$ is closed, and which commute, associate, and distribute according to the field axioms. Pointwise multiplication of a stream from $\mathcal{C}[\mathbb{T},\mathcal{S}]^m$ with a stream from $\mathcal{C}[\mathbb{T},\mathcal{B}]^m$ produces a stream from $\mathcal{C}[\mathbb{T},\mathcal{B}]^m$. Being rife with zero divisors, however, $\mathcal{C}[\mathbb{T},\mathcal{S}]^m$ is not a field (it is only a commutative ring), and thus it cannot serve as a proper field of scalars in a topological vector space.

Despite this shortcoming, pairing $\mathcal{C}[\mathbb{T},\mathcal{S}]^m$ with $\mathcal{C}[\mathbb{T},\mathcal{B}]^m$ produces a useful algebra of pointwise operations—one which lays the foundation for matrix multiplication of streams in $\mathcal{C}[\mathbb{T},\mathcal{B}]^m$ by matrices in $\mathcal{C}[\mathbb{T},\mathcal{S}]^{m\times m}$. In fact, membership in a commutative ring is all that is required of the entries of a matrix in order to define a determinant (see [31]). That fact, in and of itself, is not immediately relevant to our research here, but it does suggest promising avenues of exploration in future research.

Most of the observations noted above follow readily, but the proof that pointwise multiplication between $\mathcal{C}[\mathbb{T},\mathcal{B}]^m$ and $\mathcal{C}[\mathbb{T},\mathcal{S}]^m$ works as we have claimed is not completely trivial. For brevity, we omit it here, but it is almost identical to the proof of Part (3) of the Continuity Lemma. See [24] for details.

Corollary 1.16. *If $A \in \mathcal{C}[\mathbb{T},\mathcal{S}]^{m\times m}$ and $\mathbf{u} \in \mathcal{C}[\mathbb{T},\mathcal{B}]^m$, then $A\mathbf{u} \in \mathcal{C}[\mathbb{T},\mathcal{B}]^m$*

While the algebraic operations on $\mathcal{C}[\mathbb{T},\mathcal{B}]^m$ facilitate the construction of many interesting stream operators, they would be of rather limited utility to the theory without integration (or something like it).

Lemma 1.17. *The Riemann integral[7] is well-defined on $\mathcal{C}[\mathbb{T},\mathcal{B}]^m$ and $\forall \mathbf{u} \in \mathcal{C}[\mathbb{T},\mathcal{B}]^m$ $\forall a,b \in \mathbb{T}$,*

$$\left\| \int_a^b \mathbf{u}(s)\, ds \right\| \leq \int_a^b \|\mathbf{u}(s)\|\ ds$$

Proof. See Theorems 2.1 and 5.1 in [32] for the definition and the inequality, respectively. □

Remark 1.18. Iterated integrals are of particular importance to our theory, but standard integral notation becomes a little cumbersome for representing them. So we'll be using the following notational conventions.

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collection of seminorms that induce its topology. See [30] for details.

[6] In this context a subset $X \subseteq \mathcal{C}[\mathbb{T},\mathcal{B}]^m$ is *bounded* if for every neighbourhood B of $\mathbf{0} \in \mathcal{C}[\mathbb{T},\mathcal{B}]^m$ there is an $R > 0$ such that for all $r \in \mathcal{S}$ with $|r| > R$, $X \subseteq rB$. This means that unless $\mathcal{B}$ is the trivial space $\mathcal{B} = \{\mathbf{0}\}$ (or perhaps a rather esoteric and pathological space) we have that for every $T \in \mathbb{T}$ and $\varepsilon > 0$, $B_{T,\varepsilon}(\mathbf{0}) = \{\mathbf{u} \in \mathcal{C}[\mathbb{T},\mathcal{B}]^m : d_T(\mathbf{u},\mathbf{0}) < \varepsilon\}$ is unbounded. This is because for any $r \in \mathcal{S}$ there is (for all the common Banach spaces, at least) a stream $\mathbf{u} \in \mathcal{C}[\mathbb{T},\mathcal{B}]^m$ such that $\|\mathbf{u}(T+1)\| > |r|$, and hence $B_{T,\varepsilon}(\mathbf{0}) \nsubseteq r\, B_{T+1,\varepsilon}(\mathbf{0})$.

[7] More accurately, the *generalized* Riemann integral, as defined in [32], based on the generalized integrals developed by McShane, Henstock, and Kurzweil [33].

Given $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, $a, t \in \mathbb{T}$ (with $a \leq t$), and $n \in \mathbb{N}$,

$$\begin{aligned} \int_a^{(0)} \mathbf{u}(t) &= \mathbf{u}(t) \\ \int_a^{(n+1)} \mathbf{u}(t) &= \int_a^t \left(\int_a^{(n)} \mathbf{u}(s) \right) ds \end{aligned}$$

Equivalently,

$$\int_a^{(n)} \mathbf{u}(t) = \int_a^t \int_a^{s_1} \int_a^{s_2} \cdots \int_a^{s_{n-1}} \mathbf{u}(s_n)\, ds_n\, ds_{n-1} \ldots ds_1$$

Lemma 1.19. *Let $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, $n \in \mathbb{Z}^+$, and $a, b \in \mathbb{T}$ with $a < b$. Then,*

$$\left\| \int_a^{(n)} \mathbf{u}(b) \right\| \leq \frac{(b-a)^n}{n!} \max_{a \leq t \leq b} \|\mathbf{u}(t)\|$$

Proof. The base case, for $n = 1$, follows from Lemma 1.17, along with the fact that for a real function, $f : \mathbb{R} \to \mathbb{R}$ (like the $\|\mathbf{u}(s)\|$ from the right-hand side of the inequality in Lemma 1.17), $\int_a^b f(s)\, ds \leq (b - a) \max_{a \leq s \leq b} |f(s)|$.

Now if we suppose that the inequality holds for all $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, $a, b \in \mathbb{T}$ $(a < b)$, and for some $n > 0$ then

$$\begin{aligned} \left\| \int_a^{(n+1)} \mathbf{u}(b) \right\| &= \left\| \int_a^b \int_a^{(n)} \mathbf{u}(s)\, ds \right\| \\ &\leq \int_a^b \left\| \int_a^{(n)} \mathbf{u}(s) \right\| ds \quad \text{(by Lemma 1.17)} \\ &\leq \int_a^b \frac{(s-a)^n}{n!} \max_{a \leq t \leq s} \|\mathbf{u}(t)\|\, ds \quad \text{(by the inductive hypothesis)} \\ &\leq \max_{a \leq t \leq b} \|\mathbf{u}(t)\| \int_a^b \frac{(s-a)^n}{n!}\, ds \\ &= \frac{(b-a)^{n+1}}{(n+1)!} \max_{a \leq t \leq b} \|\mathbf{u}(t)\| \end{aligned}$$

□

2 Operators Which Satisfy the Fixed Point Theorems

Having established in Section 1.4 some of the basic operations we can use to create stream operators, we can now proceed to examine the way the properties discussed in Section 1.2 are affected by these operations. In the Building Block and Continuity Lemmas (Lemmas 2.1 and 2.2 below), we will simply audit the effects of the algebraic operations so that when building operators from them or deconstructing operators in terms of them, we can directly calculate their properties. In the General Form Theorem (2.15), all these results are consolidated into the most general class of operators definable using these algebraic operations exclusively. The Building Block Lemma and the Continuity Lemma can also be used à la carte, however, with predefined operators that cannot be expressed using only the algebraic operations from Section 1.4 (see Section 3.2 for an example).

Lemma 2.1 (The Building Block Lemma)**.** *Given stream operators $F, G : \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, scalar stream $\boldsymbol{a} = (a_1, a_2, \ldots, a_m) \in \mathcal{C}[\mathbb{T}, \mathcal{S}]^m$, and matrix stream $A \in \mathcal{C}[\mathbb{T}, \mathcal{S}]^{m \times m}$, the properties of $\boldsymbol{Caus}$, $\boldsymbol{WCaus}$, and $\boldsymbol{Lip}(\lambda, \tau)$ are preserved by the basic stream operations as follows:*

1. **Primitive Operators**
   (a) Given $\mathbf{w} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, the constant operator $F_{\mathbf{w}}(\mathbf{v}) = \mathbf{w}$ satisfies $\boldsymbol{Caus}$ and $\boldsymbol{Lip}(0, \tau)$ for any $\tau \geq 0$.
   (b) The identity on $\mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ satisfies $\boldsymbol{WCaus}$ and $\boldsymbol{Lip}(1, \tau)$ for all $\tau \geq 0$.
2. **Addition of Operators**
   (a) $F, G \in \boldsymbol{WCaus} \Rightarrow (F + G) \in \boldsymbol{WCaus}$
   (b) $F, G \in \boldsymbol{Caus} \Rightarrow (F + G) \in \boldsymbol{Caus}$
   (c) $F \in \boldsymbol{Lip}(\lambda_F, \tau_F)$, $G \in \boldsymbol{Lip}(\lambda_G, \tau_G) \Rightarrow (F + G) \in \boldsymbol{Lip}\left(\lambda_F + \lambda_G, \min\{\tau_F, \tau_G\}\right)$
3. **Composition of Operators**
   (a) $F, G \in \boldsymbol{WCaus} \Rightarrow (F \circ G) \in \boldsymbol{WCaus}$
   (b) $F \in \boldsymbol{Caus}$ and $G \in \boldsymbol{WCaus} \Rightarrow (F \circ G), (G \circ F) \in \boldsymbol{Caus}$
   (c) If $F \in \boldsymbol{Lip}(\lambda_F, \tau_F)$, $G \in \boldsymbol{Lip}(\lambda_G, \tau_G)$, and $F, G \in \boldsymbol{WCaus}$ then $(F \circ G) \in \boldsymbol{Lip}\left(\lambda_F \lambda_G, \min\{\tau_F, \tau_G\}\right)$
4. **Pointwise Multiplication by a Scalar Stream**
   (a) $F \in \boldsymbol{WCaus} \Rightarrow \boldsymbol{a}F \in \boldsymbol{WCaus}$
   (b) $F \in \boldsymbol{Caus} \Rightarrow \boldsymbol{a}F \in \boldsymbol{Caus}$
   (c) Let $\alpha \geq 0$. If $F \in \boldsymbol{Lip}(\lambda, \tau)$ and $\forall t \in \mathbb{T} \max_{1 \leq i \leq m} |a_i(t)| \leq \alpha$ then $\boldsymbol{a}F \in \boldsymbol{Lip}(\alpha\lambda, \tau)$
5. **Pointwise Multiplication by a Scalar Matrix**
   (a) $F \in \boldsymbol{WCaus} \Rightarrow AF \in \boldsymbol{WCaus}$

(b) $F \in \boldsymbol{Caus} \Rightarrow AF \in \boldsymbol{Caus}$

(c) $F \in \boldsymbol{WCaus}$ and $A(0) = \mathbf{0} \Rightarrow AF \in \boldsymbol{Caus}$

(d) Let $\alpha \geq 0$. If $F \in \boldsymbol{Lip}(\lambda, \tau)$ and $\forall t \in \mathbb{T}\ \|A(t)\| \leq \alpha$ then $AF \in \boldsymbol{Lip}(\alpha\lambda, \tau)$

6. **Integration**

Define $F_{\int} : \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ *as follows:*

$$F_{\int}(\mathbf{u})(t) = \int_0^t \mathbf{u}(s)\, ds = \left(\int_0^t u_1(s)\, ds, \int_0^t u_2(s)\, ds, \ldots, \int_0^t u_m(s)\, ds\right)$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_m) \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$. *Then,*

(a) $F_{\int} \in \boldsymbol{Caus}$

(b) $F_{\int} \in \boldsymbol{Lip}(\lambda, \lambda)\ \forall \lambda \in \mathbb{R}^+$

Proof. □

(1a) $\forall \mathbf{u}, \mathbf{v} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \left[F_{\mathbf{w}}(\mathbf{u}) = F_{\mathbf{w}}(\mathbf{v})\right]$, so both results follow trivially.

(1b) $\forall T \in \mathbb{T}\ \forall \mathbf{u}, \mathbf{v} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \left[\left(\mathbf{u}\restriction_{[0,T]} = \mathbf{v}\restriction_{[0,T]} \Rightarrow\ \mathbf{id}(\mathbf{u})(T) = \mathbf{id}(\mathbf{v})(T)\right)\right.$ and $\left. d_{T+\tau}(\mathbf{u}, \mathbf{v}) \leq 1 \cdot d_{T+\tau}(\mathbf{id}(\mathbf{u}), \mathbf{id}(\mathbf{v}))\right]$.

(2a) $d_T(\mathbf{u}, \mathbf{v}) = 0 \Rightarrow (F + G)(\mathbf{u})(T) = F(\mathbf{u})(T) + G(\mathbf{u})(T) = F(\mathbf{v})(T) + G(\mathbf{v})(T) = (F + G)(\mathbf{v})$

(2b) By Remark 1.4, all that remains to be shown (given Part (2a)) is that $\forall \mathbf{u}, \mathbf{v} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \left[(F + G)(\mathbf{u})(0) = (F + G)(\mathbf{v})(0)\right]$. This follows directly from the fact that $\forall \mathbf{u}, \mathbf{v} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \left[F(\mathbf{u})(0) = F(\mathbf{v})(0)\right.$ and $\left.G(\mathbf{u})(0) = G(\mathbf{v})(0)\right]$.

(2c) Let $\tau = \min\{\tau_F, \tau_G\}$. By Lemma 1.8, $F \in \boldsymbol{Lip}(\lambda_F, \tau)$ and $G \in \boldsymbol{Lip}(\lambda_G, \tau)$. The result follows readily by taking $\mathbf{u}, \mathbf{v} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ and expanding

$$d_{T+\tau}((F + G)(\mathbf{u}), (F + G)(\mathbf{v}))$$

into

$$\max_{0 \leq t \leq T+\tau} \|F(\mathbf{u})(t) + G(\mathbf{u})(t) - F(\mathbf{v})(t) - G(\mathbf{v})(t)\|$$

Then finally rearranging the terms and using the subadditivity of $\|\cdot\|$ to obtain the result.

(3a) $F, G \in \boldsymbol{WCaus} \Rightarrow \forall T \in \mathbb{T}\ \forall \mathbf{u}, \mathbf{v} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \left[d_T(\mathbf{u}, \mathbf{v}) = 0 \Rightarrow d_T(G(\mathbf{u}), G(\mathbf{v})) = 0 \Rightarrow F(G(\mathbf{u}))(T) = F(G(\mathbf{v}))(T)\right]$.

(3b) Given any $\forall \mathbf{u}, \mathbf{v} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, it may be the case that $G(\mathbf{u})(0) \neq G(\mathbf{v})(0)$, but under F the image of all streams (including those two) at time $t = 0$ is the same. Thus $F \circ G \in \boldsymbol{Caus}$. As for $G \circ F$, we do know that $F(\mathbf{u})(0) = F(\mathbf{v})(0)$, and since $G \in \boldsymbol{WCaus}$, that equality "up to 0" is preserved: $G(F(\mathbf{u}))(0) = G(F(\mathbf{v}))(0)$.

(3c) By Lemma 1.8, $F \in \boldsymbol{Lip}(\lambda_F, \tau)$ and $G \in \boldsymbol{Lip}(\lambda_G, \tau)$, where $\tau = \min\{\tau_F, \tau_G\}$. So, given $T \in \mathbb{T}$ and $\mathbf{u}, \mathbf{v} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ such that $d_T(\mathbf{u}, \mathbf{v}) = 0$, it follows from (3a) that $d_T(F(G(\mathbf{u})), F(G(\mathbf{v}))) = 0$ also. Hence, $d_{T+\tau}(F(G(\mathbf{u})), F(G(\mathbf{v}))) \leq \lambda_1 d_{T+\tau}(G(\mathbf{u}), G(\mathbf{v})) \leq \lambda_F \lambda_G d_{T+\tau}(\mathbf{u}, \mathbf{v})$.

(4a) $d_T(\mathbf{u}, \mathbf{v}) = 0 \Rightarrow F(\mathbf{u})(T) = F(\mathbf{v})(T) \Rightarrow \boldsymbol{a}F(\mathbf{u})(T) = \boldsymbol{a}F(\mathbf{v})(T)$

(4b) $F(\mathbf{u})(0) = F(\mathbf{v})(0) \Rightarrow \boldsymbol{a}F(\mathbf{u})(0) = \boldsymbol{a}F(\mathbf{v})(0)$

(4c) Consider F as an m-tuple of functions: for $\mathbf{w} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ $F(\mathbf{w}) = (F_1(\mathbf{w}), \ldots, F_m(\mathbf{w}))$. Then,

$$\begin{aligned}
d_{T+\tau}(\boldsymbol{a}F(\mathbf{u}), \boldsymbol{a}F(\mathbf{v})) &= \max_{\substack{0 \leq t \leq T+\tau \\ 1 \leq k \leq m}} \|a_k(t)F_k(\mathbf{u})(t) - a_k(t)F_k(\mathbf{v})(t)\| \\
&= \max_{\substack{0 \leq t \leq T+\tau \\ 1 \leq k \leq m}} |a_k(t)| \, \|F_k(\mathbf{u})(t) - F_k(\mathbf{v})(t)\| \\
&\leq \max_{\substack{0 \leq t \leq T+\tau \\ 1 \leq k \leq m}} |a_k(t)| \max_{\substack{0 \leq t \leq T+\tau \\ 1 \leq k \leq m}} \|F_k(\mathbf{u})(t) - F_k(\mathbf{v})(t)\| \\
&\leq \alpha d_{T+\tau}(F(\mathbf{u}), F(\mathbf{v})) \\
&\leq \alpha \lambda d_{T+\tau}(\mathbf{u}, \mathbf{v})
\end{aligned}$$

(5a, 5b) Same as (4a, 4b).

(5c) $A(0) = \mathbf{0} \in \mathcal{C}[\mathbb{T}, \mathcal{S}]^{m \times m} \Rightarrow (AF)(\mathbf{u})(0) = (AF)(\mathbf{v})(0) = \mathbf{0} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$. The rest is given by (5a).

(5d) Similar to (4c), but using the matrix norm $\|A(t)\|$ in place of $|a_k(t)|$.

(6a) Let $\mathbf{u}, \mathbf{v} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ such that $\forall t < T$ $\mathbf{u}(t) = \mathbf{v}(t)$. Then, using the norm on $\mathcal{B}^m$,

$$\begin{aligned}
\left\| F_\int(\mathbf{u})(T) - F_\int(\mathbf{v})(T) \right\| &= \left\| \int_0^T \mathbf{u}(s)\, ds - \int_0^T \mathbf{v}(s)\, ds \right\| \\
&= \left\| \int_0^T (\mathbf{u}(s) - \mathbf{v}(s))\, ds \right\| \\
&\leq \int_0^T \|(\mathbf{u}(s) - \mathbf{v}(s))\|\, ds \\
&= 0
\end{aligned}$$

The linearity of the integral in the second line comes from Theorem 3.1 in [32], and the inequality arises from 1.17. Since $\|\cdot\|$ is a norm (rather than a mere seminorm), it follows that $F_\int(\mathbf{u})(T) = F_\int(\mathbf{v})(T)$.

(6b) Let $\mathbf{u}, \mathbf{v} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ such that $d_T(\mathbf{u}, \mathbf{v}) = 0$. Let $\lambda \geq 0$. Then,

$$
\begin{aligned}
d_{T+\lambda}\left(F_\int(\mathbf{u}), F_\int(\mathbf{v})\right) &= \max_{0 \leq t \leq T+\lambda} \left\| \int_0^t \mathbf{u}(s)\, ds - \int_0^t \mathbf{v}(s)\, ds \right\| \\
&= \max_{0 \leq t \leq T+\lambda} \left\| \int_0^t (\mathbf{u}(s) - \mathbf{v}(s))\ ds \right\| && (1) \\
&\leq \max_{0 \leq t \leq T+\lambda} \int_0^t \|\mathbf{u}(s) - \mathbf{v}(s)\|\ ds && (2) \\
&= \int_0^{T+\lambda} \|\mathbf{u}(s) - \mathbf{v}(s)\|\ ds && (3) \\
&= \int_T^{T+\lambda} \|\mathbf{u}(s) - \mathbf{v}(s)\|\ ds && (4) \\
&\leq \lambda \max_{T \leq s \leq T+\lambda} \|\mathbf{u}(s) - \mathbf{v}(s)\| && (5) \\
&= \lambda \max_{0 \leq s \leq T+\lambda} \|\mathbf{u}(s) - \mathbf{v}(s)\| && (6) \\
&= \lambda d_{T+\lambda}(\mathbf{u}, \mathbf{v})
\end{aligned}
$$

Step Justifications (unnumbered steps require no further elucidation):

(1) By Theorem 3.1 in [32].

(2) By Lemma 1.17. Note that this converts the Banach-valued integral to an ordinary integral over $\mathbb{R}$.

(3) Since the integrand is nonnegative, the maximum will be at $t = T + \lambda$.

(4) $d_T(\mathbf{u}, \mathbf{v}) = 0 \Rightarrow \int_0^T \|\mathbf{u}(s) - \mathbf{v}(s)\|\ ds = 0$.

(5) By Lemma 1.19.

(6) Since,

$$
\begin{aligned}
\max_{0 \leq s \leq T+\lambda} \|\mathbf{u}(s) - \mathbf{v}(s)\| &= \max\left\{ \max_{0 \leq s \leq T} \|\mathbf{u}(s) - \mathbf{v}(s)\|, \max_{T \leq s \leq T+\lambda} \|\mathbf{u}(s) - \mathbf{v}(s)\| \right\} \\
&= \max\left\{ 0, \max_{T \leq s \leq T+\lambda} \|\mathbf{u}(s) - \mathbf{v}(s)\| \right\} \\
&= \max_{T \leq s \leq T+\lambda} \|\mathbf{u}(s) - \mathbf{v}(s)\|
\end{aligned}
$$

Lemma 2.2 (The Continuity Lemma)**.** *Let (P, d_P) be a metric space (which will serve as a parameter space) and let $p \in P$. Let $F, G : P \times \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ and suppose that for all $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ F and*

G are continuous at $(p, \mathbf{u})$. Let $A : P \to \mathcal{C}[\mathbb{T}, \mathcal{S}]^{m\times m}$ be continuous at p. Then the following functions $H : P \times \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ defined below for $(r, \mathbf{u}) \in P \times \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, $t \in \mathbb{T}$, are all continuous at every point in $\{p\} \times \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \subseteq P \times \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$:

1. Addition: $H(r, \mathbf{u})(t) = (F + G)(r, \mathbf{u})(t) = F(r, \mathbf{u})(t) + G(r, \mathbf{u})(t)$
2. Composition: $H(r, \mathbf{u})(t) = F(r, G(r, \mathbf{u}))(\mathrm{t})$
3. Matrix Multiplication: $H(r, \mathbf{u})(t) = (AF)(r, \mathbf{u})(t) = A(r)(t)\, F(r, \mathbf{u})(t)$
4. Integration: $H(r, \mathbf{u})(t) = \int_0^t F(r, \mathbf{u})(s)\, ds$

Proof. The proof is straightforward, so we provide only (3) here. See [24] for the other parts.

(3) Let $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$. Let $\varepsilon > 0$, $T \in \mathbb{T}$. For the sake of tidiness, we'll overload the symbol $\|\cdot\|_T$ using it as a seminorm on both $\mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ and $\mathcal{C}[\mathbb{T}, \mathcal{S}]^{m\times m}$. In the latter case, $\|A(p)\|_T = \max_{0\le t\le T} \|A(p)(t)\|$, where $\|\cdot\|$ is the matrix norm on $\mathcal{B}^{m\times m}$. Let

$$\varepsilon' = \frac{1}{2}\left(\sqrt{\left(\|A(p)\|_T + \|F(p, \mathbf{u})\|_T\right)^2 + 4\varepsilon} - \|A(p)\|_T - \|F(p, \mathbf{u})\|_T\right)$$

Then $\exists \delta_F, \delta_A > 0$ $\exists T_F, T_A \in \mathbb{T}$ such that $\forall (r, \mathbf{v}) \in P \times \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$,

$$\begin{aligned} d_{T_F}\left((p, \mathbf{u}), (r.\mathbf{v})\right) < \delta_F \quad &\Rightarrow \quad d_T\left(F(p, \mathbf{u}), F(r.\mathbf{v})\right) < \varepsilon' \\ d_{T_A}\left((p, \mathbf{u}), (r.\mathbf{v})\right) < \delta_A \quad &\Rightarrow \quad d_T\left(A(p, \mathbf{u}), A(r.\mathbf{v})\right) < \varepsilon' \end{aligned}$$

Let $T' = \max\{T_A, T_F\}$ and take $(r, \mathbf{v}) \in P \times \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ such that $d_{T'}\left((p, \mathbf{u}), (r.\mathbf{v})\right) < \min\{\delta_F, \delta_A\}$. Then,

$$\begin{aligned} d_T\left(A(p)F(p, \mathbf{u}), A(r)F(r, \mathbf{v})\right) &= \|A(p)F(p, \mathbf{u}) - A(r)F(r, \mathbf{v})\|_T \\ &\le \|A(p) - A(r)\|_T \, \|F(p, \mathbf{u}) - F(r.\mathbf{v})\|_T \\ &\quad + \|A(p) - A(r)\|_T \, \|F(p, \mathbf{u})\|_T \\ &\quad + \|A(p)\|_T \, \|F(p, \mathbf{u}) - F(r.\mathbf{v})\|_T \\ &< \varepsilon \end{aligned}$$

$\square$

Remark 2.3. The Building Block Lemma and the Continuity Lemma naturally complement Theorems TZ1 and TZJ2, respectively. The former suggests ways to construct operators that satisfy Theorem TZ1 and the latter merely assures us that there will be no unpleasant surprises when we hope for them to satisfy Theorem TZJ2. The key observation here is that most operators we might build from these theorems—starting with the identity operator as our foundation—will satisfy only $\boldsymbol{WCaus}$ and $\boldsymbol{Lip}(\lambda, \tau)$ for some $\lambda \geq 1$. There are only two operations in the list that can be applied to modify such an operator into one which will satisfy $\boldsymbol{Caus}$ and $\boldsymbol{Lip}(\lambda, \tau)$ for a $\lambda < 1$:

- integration, and
- multiplication by a matrix stream $A(t)$ that begins at $\mathbf{0}$ (at $t = 0$) and whose norm remains bounded by some $\lambda < 1$.

Remark 2.4. This suggests the following class of operators, at least as a starting point.

Corollary 2.5. *Let (P, d_P) be a metric space (of parameter values), let $p \in P$, and let $V \subseteq P$ be a neighbourhood of p. Let $\boldsymbol{y} : P \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ be continuous at p. Let $A, B : P \to \mathcal{C}[\mathbb{T}, \mathcal{S}]^{m \times m}$ be functions such that*

- *A and B are continuous at p*
- *$\forall r \in V \ B(r)(0) = \mathbf{0} \in \mathcal{B}^{m \times m}$*
- *$\exists M_A, M_B \in \mathbb{R}^+ \ \forall t \in \mathbb{T} \ \forall r \in V \ \|A(r)(t)\| \leq M_A$ and $\|B(r)(t)\| \leq M_B < 1$*

Define $F : P \times \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, as follows for $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, $r \in P$:

$$F(r, \mathbf{u})(t) = \boldsymbol{y}(r)(t) + B(r)(t)\mathbf{u}(t) + \int_0^t A(r)(s)\mathbf{u}(s)\, ds$$

Then for every $r \in V$, the function $F(r, \cdot) : \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ has a fixed point $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, and its fixed point function $\Phi : V \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ (as described in Remark 1.14) is continuous at p.

Remark 2.6. Since $\mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ and $\mathcal{C}[\mathbb{T}, \mathcal{S}]^{m \times m}$ are closed under their various algebraic operations and since integration is linear, a single $\boldsymbol{y}$, B, and A are clearly sufficient here (e.g. the sum of two constant streams $\boldsymbol{y}_1(r)(t) + \boldsymbol{y}_2(r)(t)$ could obviously be expressed using a single constant stream $\boldsymbol{y}(r)(t)$ and likewise for the other terms). Nested integrals, however, cannot be simplified into a single integral. So Corollary 2.5 can be generalized further in the following way.

Corollary 2.7. *Let (P, d_P), p, V, $\boldsymbol{y}$, and B be as defined in Corollary 2.5. Let $n \in \mathbb{Z}^+$ and let $A_1, A_2, \ldots, A_n$ all be as A is defined (all continuous at p and all having bounded norms throughout V and $\mathbb{T}$). Then the same results can be obtained by defining F as follows (using the notation introduced in Remark 1.18):*

$$F(r, \mathbf{u})(t) \quad = \quad \boldsymbol{y}(r)(t) + B(r)(t)\mathbf{u}(t) + \sum_{k=1}^{n} \int_0^{(k)} \left(A_k(r)\mathbf{u}\right)(t)$$

Remark 2.8. Corollary 2.7 is the most general result that can be obtained directly (and exclusively) from the Building Block Lemma and the Continuity Lemma, but with a bit of extra work, we can go further to tackle infinite series of nested integrals instead of merely finite sums of them.

Lemma 2.9. *Let*[8] *$M \in \mathbb{R}^+$ and let $A_1, A_2, A_3, \ldots \in \mathcal{C}[\mathbb{T}, \mathcal{S}]^{m\times m}$ be a sequence of matrix streams such that $\forall t \in \mathbb{T}\ \forall k \in \mathbb{Z}^+\ \|A_k(t)\| \le M$. Then the following operator is well-defined on $\mathcal{C}[\mathbb{T}, \mathcal{B}]^m$:*

$$F(\mathbf{u})(t) \quad = \quad \sum_{k=1}^{\infty} \int_0^{(k)} \left(A_k\mathbf{u}\right)(t) \tag{7}$$

Proof. For $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, $n \in \mathbb{Z}^+$, and $t \in \mathbb{T}$, define the partial sum F_n as follows:

$$F_n(\mathbf{u})(t) \quad = \quad \sum_{k=1}^{n} \int_0^{(k)} \left(A_k\mathbf{u}\right)(t)$$

Then by Lemma 1.19, for any $T \in \mathbb{T}$, and any $N > 0$ and $n > N$,

$$\begin{aligned}
d_T\left(F_n(\mathbf{u}), F_N(\mathbf{u})\right) &= \left\|F_n(\mathbf{u}) - F_N(\mathbf{u})\right\|_T \\
&= \left\| \sum_{k=N+1}^{n} \int_0^{(k)} \left(A_k\mathbf{u}\right)(t) \right\|_T \\
&\le \sum_{k=N+1}^{n} \left\| \int_0^{(k)} \left(A_k\mathbf{u}\right)(t) \right\|_T \\
&= \sum_{k=N+1}^{n} \max_{0\le t\le T} \left\| \int_0^{(k)} \left(A_k\mathbf{u}\right)(t) \right\| \\
&\le M \left\|\mathbf{u}\right\|_T \sum_{k=N+1}^{n} \max_{0\le t\le T} \frac{t^k}{k!} \\
&= M \left\|\mathbf{u}\right\|_T \sum_{k=N+1}^{n} \frac{T^k}{k!}
\end{aligned}$$

Given $\mathbf{u}$ and T, this distance can be made arbitrarily small by making N sufficiently large (and keeping $n > N$). Thus, for every $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, $\{F_n(\mathbf{u})\}_{n=1}^{\infty}$ is a locally uniform Cauchy sequence and since $\mathcal{C}[\mathbb{T}, \mathcal{B}]^m$

[8] In fact, the lemma holds if M is any function of the form $M : \mathbb{T} \to \mathbb{R}^+$, but we can't make use of this generality here and it becomes merely inconvenient for our purposes.

is complete, $\lim_{n\to\infty} F_n(\mathbf{u})$ exists (and hence defines $F(\mathbf{u})$). ☐

Lemma 2.10. *For all $\lambda > 0$ the operator $F : \mathcal{C}[\mathbb{T},\mathcal{B}]^m \to \mathcal{C}[\mathbb{T},\mathcal{B}]^m$ defined in Equation* (7) *(Lemma 2.9) satisfies* $\boldsymbol{Caus}$ *and* $\boldsymbol{Lip}(\lambda,\tau)$ *with $\tau = \frac{\lambda}{M+\lambda}$ (where M is the upper bound for $\|A_k(t)\|$ indicated in Lemma 2.9).*

Proof. By the Building Block Lemma, Parts (1b), (2b), (3b), and (6a), $F_n \in \boldsymbol{Caus}$ for every $n \in \mathbb{Z}^+$. Locally uniform convergence implies pointwise convergence, so if for some $T \in \mathbb{T}$, $\mathbf{u},\mathbf{v} \in \mathcal{C}[\mathbb{T},\mathcal{B}]^m$ $\forall n \in \mathbb{Z}^+$ $F_n(\mathbf{u})(T) = F_n(\mathbf{v})(T)$, then the same is true of the limits of each side of the equation as well. This is true whether $T = 0$ or $T > 0$. Thus, it follows that the limit F also satisfies $\boldsymbol{Caus}$.

Let $T \in \mathbb{T}$ and $\mathbf{u},\mathbf{v} \in \mathcal{C}[\mathbb{T},\mathcal{B}]^m$ such that $d_T(\mathbf{u},\mathbf{v}) = 0$. Let $\lambda > 0$ and $\tau = \frac{\lambda}{M+\lambda}$. Then by the Building Block Lemma (1b), (3c), (5d), and (6b), $\forall n \in \mathbb{Z}^+$ the operator $\mathbf{u} \mapsto \int_0^{(n)} (A_n\mathbf{u})$ satisfies $\boldsymbol{Lip}(\tau^n M,\tau)$. Thus, using (2c),

$$F_n \in \boldsymbol{Lip}\left(M\sum_{k=1}^{n}\tau^k,\tau\right)$$

Hence, $\forall \mathbf{u},\mathbf{v} \in \mathcal{C}[\mathbb{T},\mathcal{B}]^m$ $\forall T \in \mathbb{T}$, if $d_T(\mathbf{u},\mathbf{v}) = 0$ then

$$d_{T+\tau}\left(F_n(\mathbf{u}),F_n(\mathbf{v})\right) \le M\sum_{k=1}^{n}\tau^k d_{T+\tau}(\mathbf{u},\mathbf{v})$$

Now, $d_{T+\tau}\left(F(\mathbf{u}),F(\mathbf{v})\right) = d_{T+\tau}\left(\lim_{n\to\infty} F_n(\mathbf{u}),\lim_{n\to\infty} F_n(\mathbf{v})\right)$, and since $d_{T+\tau}$ is continuous,

$$\begin{aligned}
d_{T+\tau}\left(\lim_{n\to\infty} F_n(\mathbf{u}),\lim_{n\to\infty} F_n(\mathbf{v})\right) &= \lim_{n\to\infty} d_{T+\tau}\left(F_n(\mathbf{u}),F_n(\mathbf{v})\right)\\
&\le M\sum_{k=1}^{\infty}\tau^k d_{T+\tau}(\mathbf{u},\mathbf{v})\\
&= \frac{M\tau}{1-\tau}d_{T+\tau}(\mathbf{u},\mathbf{v})\\
&= \lambda d_{T+\tau}(\mathbf{u},\mathbf{v})
\end{aligned}$$

☐

Remark 2.11. Lemmas 2.9 and 2.10 offer conditions sufficient to guarantee a fixed point for integral series operator (7) using Theorem TZ1. We now wish to augment the domain of this operator with a parameter space and determine a (ideally modest) set of conditions to be imposed on the matrix streams $\{A_n\}_1^\infty$ to ensure such an operator is continuous at a given point in its parameter space (the main requirement demanded by Theorem TZJ2).

Lemma 2.12 (The Equicontinuity Lemma)**.** *Let $(X, \mathfrak{T}_X)$ be a topological space (where $\mathfrak{T}_X$ is the topology on X) and (Y, d_Y) be a metric space. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : X \to Y$ that converges pointwise to a function $f : X \to Y$. If $\{f_n\}_{n=1}^{\infty}$ is equicontinuous at a point $x \in X$, then f is continuous at x.*

Proof. If $\{f_n\}_{n=1}^{\infty}$ is equicontinuous at x, then $\exists \delta_x : \mathbb{R}^+ \to \mathfrak{T}_X$ such that $\forall \varepsilon > 0$ $x \in \delta_x(\varepsilon)$ and $\forall n \in \mathbb{Z}^+$ $\forall y \in X$ $y \in \delta_x(\varepsilon) \Rightarrow d_Y(f_n(x), f_n(y)) < \varepsilon$. Since f_n converges pointwise to f, $\exists N : X \times \mathbb{R}^+ \to \mathbb{N}$ such that $\forall y \in X$ $\forall \varepsilon > 0$ $\forall k > N(y, \varepsilon)$ $d(f_k(y), f(y)) < \varepsilon$. Let $\varepsilon > 0$. Let $y \in \delta(\varepsilon/3)$. Choose any $n > \max\{N(x, \varepsilon/3), N(y, \varepsilon/3)\}$. Then,

$$\begin{aligned} d_Y(f(x), f(y)) &\leq d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_n(y)) + d_Y(f_n(y), f(y)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

□

Lemma 2.13. *Let (P, d_P) be a metric space and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of functions $A_n : P \to \mathcal{C}[\mathbb{T}, \mathcal{S}]^{m \times m}$. For each $n \in \mathbb{Z}^+$ define $H_n : P \times \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ as $H_n(r, \mathbf{v})(t) = (A_n(r)(t))(\mathbf{v}(t))$ (pointwise matrix multiplication). If $\{A_n\}_{n=1}^{\infty}$ are equicontinuous at a point $p \in P$ and $\exists M : \mathbb{T} \to \mathbb{R}^+$ such that $\forall T \in \mathbb{T}$ $\forall n \in \mathbb{Z}^+$ $\|A_n\|_T \leq M(T)$, then $\{H_n\}_{n=1}^{\infty}$ are equicontinuous at $(p, \mathbf{u})$ for every $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$.*

Proof. Let $\delta_A : \mathbb{R}^+ \times \mathbb{T} \to \mathbb{R}^+$ be the modulus of continuity for $\{A_n\}_{n=1}^{\infty}$ at p. That is, $\forall \varepsilon > 0$ $\forall T \in \mathbb{T}$ $\forall n \in \mathbb{Z}^+ \forall r \in P$,

$$d_P(r, p) < \delta_A(\varepsilon, T) \Rightarrow \|A_n(p) - A_n(r)\|_T < \varepsilon$$

We can then derive a modulus of continuity for $\{H_n\}_{n=1}^{\infty}$ using only δ_A, $M(T)$, and a stream $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ (and, in particular, *not* using n) by following the proof of the Continuity Lemma part (3), taking F to be the projection function $F : P \times \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, defined for $(r, \mathbf{v}) \in P \times \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ as $F(r, \mathbf{v}) = \mathbf{v}$. Specifically, given $\varepsilon > 0$, $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, and $T \in \mathbb{T}$, we take

$$\varepsilon' = \frac{1}{2}\left(\sqrt{(M(T) + \|\mathbf{u}\|_T)^2 + 4\varepsilon} - M(T) - \|\mathbf{u}\|_T\right)$$

(cf. proof of The Continuity Lemma (2.2), part (3)). Then define $\delta(\varepsilon, T, \mathbf{u}) = \min\{\varepsilon', \delta_A(\varepsilon', T)\}$. □

Lemma 2.14. *Let (P, d_P) be a metric space and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : P \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ which is equicontinuous at every point in some set $Q \subseteq P$. Define $F_n : P \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ as follows for $r \in P$,*

$n \in \mathbb{Z}^+$, and $t \in \mathbb{T}$:

$$F_n(r) \quad = \quad \sum_{k=1}^{n} \int_0^{(k)} (f_k(r))\,(t)$$

Then $\{F_n\}_{n=1}^{\infty}$ is equicontinuous on Q.

Proof. Since $\{f_n\}_{n=1}^{\infty}$ is equicontinuous on Q, there is a modulus of continuity function $\delta_f : \mathbb{R}^+ \times \mathbb{T} \times Q \to \mathbb{R}^+$ such that $\forall \varepsilon > 0\ \forall n \in \mathbb{Z}^+\ \forall T \in \mathbb{T}\ \forall q \in Q\ \forall p \in P\ d_P(p,q) < \delta_f(\varepsilon, T, q) \Rightarrow d_T(f_n(p), f_n(q)) < \varepsilon$. Define $\delta(\varepsilon, T, p) = \delta_f(e^{-T}\varepsilon, T, p)$. Then for any $T \in \mathbb{T}$, $n \in \mathbb{Z}^+$, and $q \in Q$ such that $d_P(p,q) < \delta(\varepsilon, T, p)$,

$$\begin{aligned}
d_T(F_n(p), F_n(q)) &= \left\| \sum_{k=1}^{n} \int_0^{(k)} (f_k(p)) - \sum_{k=1}^{n} \int_0^{(k)} (f_k(q)) \right\|_T \\
&= \left\| \sum_{k=1}^{n} \int_0^{(k)} (f_k(p) - f_k(q)) \right\|_T \\
&\leq \sum_{k=1}^{n} \left\| \int_0^{(k)} (f_k(p) - f_k(q)) \right\|_T \\
&= \sum_{k=1}^{n} \max_{0 \leq t \leq T} \left\| \int_0^{(k)} (f_k(p) - f_k(q))\,(t) \right\| \\
&\leq \sum_{k=1}^{n} \max_{0 \leq t \leq T} \frac{t^k}{k!} \max_{0 \leq s \leq t} \|(f_k(p) - f_k(q))\,(s)\| \qquad (8) \\
&= \sum_{k=1}^{n} \max_{0 \leq t \leq T} \frac{t^k}{k!} d_t(f_k(p), f_k(q)) \\
&\leq \sum_{k=1}^{n} \frac{T^k}{k!} d_T(f_k(p), f_k(q)) \\
&< e^{-T}\varepsilon \sum_{k=1}^{n} \frac{T^k}{k!} \\
&< e^{-T}\varepsilon \sum_{k=1}^{\infty} \frac{T^k}{k!} \\
&= e^{-T}\varepsilon e^{T} = \varepsilon
\end{aligned}$$

The inequality in (8) is from 1.19. ☐

Theorem 2.15 (The General Form Theorem)**.** *Let (P, d_p) be a metric space (of parameters). Let $V \subseteq P$ be a neighbourhood of a point $p \in P$. Let $\boldsymbol{y} : P \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ be continuous at p. Let $B, A_1, A_2, \ldots : P \to \mathcal{C}[\mathbb{T}, \mathcal{S}]^{m \times m}$ be functions such that*

- *B is continuous at p, and $\{A_n\}_{n=1}^{\infty}$ are equicontinuous at p,*

- $\forall r \in V\ B(r)(0) = \mathbf{0} \in \mathcal{C}[\mathbb{T}, \mathcal{S}]^{m\times m}$, *and*

- $\exists M_A, M_B \in \mathbb{R}^+\ \forall r \in V\ \forall t \in \mathbb{T}\ \forall n \in \mathbb{Z}^+\ \|A_n(r)(t)\| \leq M_A$ *and* $\|B(r)\| \leq M_B < 1$

Define $F : P \times \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ *as follows for* $r \in P$, $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, *and* $t \in \mathbb{T}$*:*

$$F(r, \mathbf{u})(t) \quad = \quad \boldsymbol{y}(r)(t) + B(r)(t)\mathbf{u}(t) + \sum_{k=1}^{\infty} \int_0^{(k)} \left(A_k(r)\mathbf{u}\right)(t) \tag{9}$$

For each $r \in P$ *define* $F_r : \mathcal{C}[\mathbb{T}, \mathcal{B}]^m \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ *as* $F_r(\mathbf{u}) = F(r, \mathbf{u})$.

Then for each $r \in P$, F_r *has a unique fixed point* $\Phi(r)$*, and the fixed point function* $\Phi : V \to \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ *for* F *is continuous at* p.

Proof. First we'll show that $\forall r \in P\ F_r$ satisfies ***Caus*** and ***Contr***. Theorem TZ1 informs us that these conditions are sufficient to guarantee that F_r has a unique fixed point for all $r \in P$. Finally we show that for every $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$, F is continuous at $(p, \mathbf{u})$, and thus, Theorem TZJ2 provides the conclusion.

[$F_r \in$ ***Caus***] Lemma 2.9 establishes the fact that $F_r(\mathbf{u})$ converges to a stream for all $(r, \mathbf{u}) \in P \times \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$. Using the Building Block Lemma, Parts (1a), (1b), (3b), (5a), (5c), and (2.10), we find that each of the main three terms satisfies ***Caus*** for any fixed $r \in P$. Part (2b) assembles them to show that F_r, itself, satisfies ***Caus***.

[$F_r \in$ ***Contr***] Let $\lambda_\Sigma = \frac{1-M_B}{2}$ and $\tau = \frac{\lambda_\Sigma}{M_A+\lambda_\Sigma}$. From (1a) the first term of F_r satisfies ***Lip***$(0, \tau)$. From (1b) and (5c) the second term satisfies ***Lip***(M_B, τ). From Lemma 2.10, the third term (the summation) satisfies ***Lip***(λ_Σ, τ). Putting the three results together, we conclude from (2c) that for all $r \in P$, F_r satisfies ***Lip***(λ, τ) with $\lambda = 0 + M_B + \lambda_\Sigma = \frac{1+M_B}{2} < 1$.

[F continuous at $(p, \mathbf{u})$] By Lemma 2.13, the set of integrands is equicontinuous at every point in the set $Q = \{p\} \times \mathcal{C}[\mathbb{T}, \mathcal{B}]^m$. Thus, by Lemma 2.14, the set of partial sums $\left\{\sum_{k=1}^{n} \int_0^{(k)} \left(A_k(r)\mathbf{u}\right)(t)\right\}_{n=1}^{\infty}$ is equicontinuous at every point in Q. Since the series converges pointwise, the Equicontinuity Lemma then asserts that its limit is continuous at every point of Q. It is then trivial to use the Continuity Lemma to show that F is continuous at every point in Q, and hence by Theorem TZJ2, Φ is continuous at p. ∎

Remark 2.16. The attentive reader may notice that Part (2) of the Continuity Lemma has so far been neglected. While it is unnecessary for the General Form Theorem, its value is in facilitating the analysis of operators that are constructed by more exotic means than the algebraic operations on $\mathcal{C}[\mathbb{T}, \mathcal{B}]^m$ and $\mathcal{C}[\mathbb{T}, \mathcal{S}]^{m\times m}$. See Section 3.2 for an example.

3 Applications

3.1 The Mass-Spring-Damper System Revisited

3.1.1 Case Study 1

The simple mass-spring-damper system (see Figure 1) was introduced in [1] as an analog network case study. The system is typically expressed as a second-order, homogeneous ODE with constant coefficients:

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = f(t)$$

where M is the mass, D is the damping coefficient, K is the spring constant, f is the forcing function, and x is the displacement. The initial conditions are given as

$$\begin{aligned} x(0) &= x_0 \in \mathbb{R} \text{ (initial displacement)} \\ \dot{x}(0) &= v_0 \in \mathbb{R} \text{ (initial velocity)} \end{aligned}$$

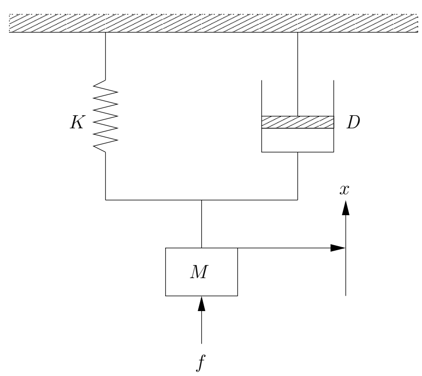

Figure 1: Mass-Spring-Damper System

It is typical to reduce the second-order equation to a first-order system using the substitutions $v(t) = \dot{x}(t)$ and $a(t) = \dot{v}(t)$. Integrating this system with respect to t and solving for the initial conditions gives us a system of integral equations equivalent to the original initial value problem:

$$
\begin{aligned}
a(t) &= \frac{f(t) - Dv(t) - Kx(t)}{M} \\
v(t) &= \int_0^t a(s)\,ds + v_0 \\
x(t) &= \int_0^t v(s)\,ds + x_0
\end{aligned}
$$

This system is the mass-spring-damper system as it is represented in [1] and [2] as their first case study. For each parameter choice $p = (M, K, D, v_0, x_0, f)$, it induces the operator $F_p : \mathcal{C}[\mathbb{T}, \mathbb{R}]^3 \to \mathcal{C}[\mathbb{T}, \mathbb{R}]^3$ defined for $\mathbf{u}(t) = (a, v, x)^\intercal(t)$ as

$$
F_p \begin{bmatrix} a \\ v \\ x \end{bmatrix} (t) = \begin{bmatrix} \frac{1}{M}\left(f(t) - Dv(t) - Kx(t)\right) \\ \int_0^t a(s)\,ds + v_0 \\ \int_0^t v(s)\,ds + x_0 \end{bmatrix} \tag{10}
$$

A fixed point of this operator represents both a solution the original initial value problem (with the given parameters), and the semantics for the analog network shown in Figure 2 (which is a slightly less formal version of the one used by Tucker and Zucker):

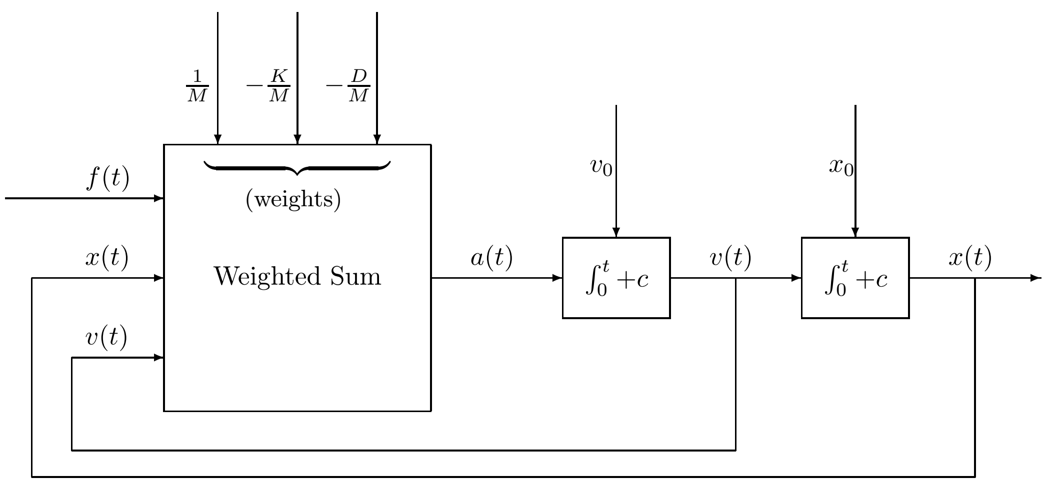

Figure 2: Analog Network for Simple Mass-Spring-Damper System

Tucker and Zucker prove that this operator F_p satisfies the ***Contr*** condition if $M > \max\{K, 2D\}$, and hence their theory guarantees the existence of a fixed point under that condition. It is unclear whether this is a necessary condition, however, but while it may be weakened to some degree, it cannot be disposed altogether, as the following example demonstrates.

Example 3.1. Let $T \geq 0$. Take the constants $M = D = K = 1$, $v_0 = x_0 = 0$, and let $\mathbf{u}_1 = (a_1, v_1, x_1)^\intercal, \mathbf{u}_2 = (a_2, v_2, x_2)^\intercal$ be stream tuples such that $x_1 = x_2 = a_1 = a_2$ (for all time), and $v_1 \restriction_{[0,T]} = v_2 \restriction_{[0,T]}$ but $\exists t \in (T, T+\tau]$ such that $v_1(t) \neq v_2(t)$. For convenience, write $(a_i', v_i', x_i')^\intercal = F_p(\mathbf{u}_i)$ for $i = 1, 2$. Then for any $\tau > 0$, $\lambda < 1$, and any input stream f,

$$\begin{aligned}
d_{T+\tau}(F_p\mathbf{u}_1, F_p\mathbf{u}_2) &= \max\{d_{T+\tau}(a_1', a_2'),\, d_{T+\tau}(v_1', v_2'),\, d_{T+\tau}(x_1', x_2')\} \\
&\geq d_{T+\tau}(a_1', a_2') \\
&= d_{T+\tau}\left(\frac{f - Dv_1 - Kx_1}{M}, \frac{f - Dv_2 - Kx_2}{M}\right) \\
&= d_{T+\tau}\left((f - v_1 - x_1), (f - v_2 - x_2)\right) \\
&= \max_{0 \leq t \leq T+\tau} |(f(t) - v_1(t) - x_1(t)) - (f(t) - v_2(t) - x_2(t))| \\
&= \max_{T \leq t \leq T+\tau} |(f(t) - v_1(t) - x_1(t)) - (f(t) - v_2(t) - x_2(t))| \\
&= \max_{T \leq t \leq T+\tau} |v_2 - v_1| \\
&= d_{T+\tau}(v_1, v_2) \\
&= \max\{d_{T+\tau}(v_1, v_2),\, 0,\, 0\} \\
&= \max\{d_{T+\tau}(v_1, v_2),\, d_{T+\tau}(x_1, x_2),\, d_{T+\tau}(a_1, a_2)\} \\
&= d_{T+\tau}(\mathbf{u}_1, \mathbf{u}_2) \\
&> \lambda d_{T+\tau}(\mathbf{u}_1, \mathbf{u}_2)
\end{aligned}$$

Thus, F_p with $p = (1, 1, 1, 0, 0, f)$ does not satisfy the ***Contr*** condition.

3.1.2 A More Robust Formulation

Example 3.1 shows that there *are* parameter values which cause Tucker and Zucker's model of the mass-spring-damper system to fail to satisfy ***Contr***, and hence also to fail to satisfy their special condition, $M > \max\{K, 2D\}$. In other words, the special condition is not simply an artifact of calculation (or an "idle threat," as it were); it does identify systems which do not satisfy ***Contr***. While somewhat disappointing, it is not completely unexpected that such systems would exist. In particular, it is conceivable to think we might see the ***Contr*** condition fail in regions of the parameter space in which the system behaves erratically or in which the system is most sensitive to parameter variation. Oddly enough, that does not appear to be the case.

Recall that there are three types of behaviour a mass-spring-damper system can exhibit (see [34], for example, or almost any elementary text on ordinary differential equations): overdamped, critically damped, and underdamped. An overdamped system behaves as if submerged in molasses—if the mass is displaced (and no other forcing function acts on it), it gradually and monotonically returns to the equilibrium position. A critically damped system monotonically returns to its equilibrium position as well, but as quickly as possible (like an optimized overdamped system). An underdamped system will oscillate with exponentially decreasing amplitude.

The value of $\zeta = D/\sqrt{4MK}$ determines which behaviour a system will exhibit. If $\zeta > 1$ the system is overdamped, if $\zeta < 1$ the system is underdamped, and if $\zeta = 1$ it is critically damped. Since the motion of an underdamped system is the least constrained, we might expect that if Tucker and Zucker's condition ($M > \max\{K, D\}$) is to fail, an underdamped system is where it would happen; and likewise, if it ever holds, surely it would hold for an overdamped system. In fact, for each type of behaviour there is a system which satisfies the special condition and a system which doesn't.

Example 3.2. Let $D \in \mathbb{R}^+$ and set $M = 3D$. Then

$$\zeta = \frac{D}{\sqrt{12DK}} = \sqrt{\frac{D}{12K}}$$

The system is overdamped if $K < {}^{D}\!/_{12}$, critically damped if $K = {}^{D}\!/_{12}$, and underdamped if $K > {}^{D}\!/_{12}$. As long as $K < 3D$ (which leaves plenty of wiggle room), we have $M > \max\{K, 2D\}$. So there are systems of every type which satisfy the condition.

Now let $K \in \mathbb{R}^+$, and let $M = K$. Then

$$\zeta = \frac{D}{2\sqrt{MK}} = \frac{D}{2K}$$

The system is overdamped if $D > 2K$, critically damped if $D = 2K$, and underdamped if $D < 2K$. Regardless of the value of D, $M \leq \max\{K, 2D\}$. So there are also systems of every type which do not satisfy the condition.

Fortunately, by simply making the acceleration stream implicit, we can rearrange the system into an equivalent one that satisfies the ***Contr*** condition for any choice of $M, K, D > 0$ (so while the special condition was not merely an artifact of calculation, it was only an idiosyncracy of that particular model of the system).

Define the operator $G : P \times \mathcal{C}[\mathbb{T}, \mathbb{R}]^2 \to \mathcal{C}[\mathbb{T}, \mathbb{R}]^2$ as follows for $p = (M, K, D, v_0, x_0, f) \in (\mathbb{R}^+)^3 \times \mathbb{R}^2 \times \mathcal{C}[\mathbb{T}, \mathbb{R}] = P$ and $(v, x)^\intercal \in \mathcal{C}[\mathbb{T}, \mathbb{R}]^2$ (cf. (10)):

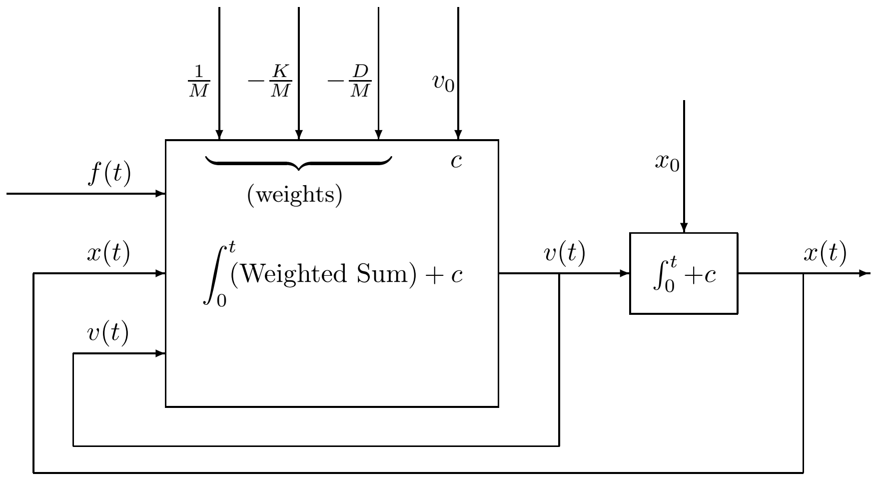

Figure 3: Revised Mass-Spring-Damper Network

$$G\left(p, \begin{bmatrix} v \\ x \end{bmatrix}\right)(t) = \begin{bmatrix} \frac{1}{M}\int_0^t \left(f(s) - Dv(s) - Kx(s)\right)\, ds + v_0 \\ \int_0^t v(s)\, ds + x_0 \end{bmatrix} \tag{11}$$

For convenience, we'll use the notation $G_p(\mathbf{u}) = G(p, \mathbf{u})$ for $\mathbf{u} \in \mathcal{C}[\mathbb{T}, \mathbb{R}]^2$ and $p \in P$.

The corresponding network is shown in Figure 3. We will now show that G satisfies the conditions demanded of F from the General Form Theorem. Define

$$A_1(p)(t) = \begin{bmatrix} -\frac{D}{M} & -\frac{K}{M} \\ 1 & 0 \end{bmatrix} \text{ and } \boldsymbol{y}(p)(t) = \begin{bmatrix} \frac{1}{M}\int_0^t f(s)\, ds + v_0 \\ x_0 \end{bmatrix}$$

Let all the other matrices from the General Form Theorem (B and A_k for $k = 2, 3, \ldots$) be zero. Rewrite Equation (11) as follows to put it in the form of (9):

$$G\left(p, \begin{bmatrix} v \\ x \end{bmatrix}\right)(t) \quad = \quad \boldsymbol{y}(p)(t) + \int_0^{(1)} \left(A_1(p) \begin{bmatrix} v \\ x \end{bmatrix}\right)(t)$$

It is relatively straightforward to show that $\boldsymbol{y}$ and A_1 are continuous on P—and hence, on any neighbourhood $V \subseteq P$ of p. So take V to be the open ball of radius $\frac{M}{2}$, centred at $p = (M, K, D, v_0, x_0, f)$. More precisely,

$$
\begin{aligned}
V &= P \cap \left(\frac{M}{2}, \frac{3M}{2}\right) \times \left(K - \frac{M}{2}, K + \frac{M}{2}\right) \times \left(D - \frac{M}{2}, D + \frac{M}{2}\right) \\
&\quad \times \left(v_0 - \frac{M}{2}, v_0 + \frac{M}{2}\right) \times \left(x_0 - \frac{M}{2}, x_0 + \frac{M}{2}\right) \\
&\quad \times \left\{g \in \mathcal{C}[\mathbb{T}, \mathbb{R}] \,:\, d_C(f, g) < \frac{M}{2}\right\}
\end{aligned}
$$

Let $M_A = 1 + \frac{2D+2K}{M}$. Then, as required by the General Form Theorem, $\forall p' = (M', K', D', v_0', x_0', f') \in V$ $\forall t \in \mathbb{T}$,

$$
\begin{aligned}
\|A_1(r)(t)\| &= \sup\left\{ \left\| \begin{bmatrix} -\frac{D'}{M'} & -\frac{K'}{M'} \\ 1 & 0 \end{bmatrix} \mathbf{u} \right\| \,:\, \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2 \text{ and } \|\mathbf{u}\| \le 1 \right\} \\
&= \max\left\{ \frac{D' + K'}{M'}, 1 \right\} \\
&\le \max\left\{ \frac{D + \frac{M}{2} + K + \frac{M}{2}}{M - \frac{M}{2}}, 1 \right\} = M_A
\end{aligned}
$$

There is no matrix stream B, so using the proof of the General Form Theorem, a straightforward calculation reveals that $G \in \boldsymbol{Lip}(\lambda, \tau)$ for

$$
\lambda = \frac{1}{2} \quad \text{and} \quad \tau = \frac{M}{4D + 4K + 3M}
$$

The remaining antecedents of the General Form Theorem follow trivially for G. Hence, for every $p = (M, K, D, v_0, x_0, f)$, G_p has a unique fixed point $\Phi(p)$, and the corresponding fixed-point function

$$
\Phi : \left(\mathbb{R}^+\right)^3 \times \mathbb{R}^2 \times \mathcal{C}[\mathbb{T}, \mathbb{R}] \to \mathcal{C}[\mathbb{T}, \mathbb{R}]^2
$$

for G is also continuous.

The characterization of G as a "formulation" of F is justified by the fact that any fixed point of G_p uniquely specifies a fixed point for F_p and vice versa. In particular,

$$
\begin{bmatrix} v \\ x \end{bmatrix} \text{ is a fixed point for } G_p \quad \Leftrightarrow \quad \begin{bmatrix} \frac{1}{M}(f - Dv - Kx) \\ v \\ x \end{bmatrix} \text{ is a fixed point for } F_p
$$

Hence, as intuition would suggest, Tucker and Zucker's theory can indeed be applied to mass-spring-damper systems with any positive values for K, D, and M. Admittedly G is not strictly equivalent to F (being two-dimensional), but if an explicit acceleration stream is desired, it can introduced to the system like this:

$$
\begin{aligned}
A'(p)(t) &= \begin{bmatrix} 0 & \frac{D^2}{M^2} - \frac{K}{M} & \frac{DK}{M^2} \\ 0 & -\frac{D}{M} & -\frac{K}{M} \\ 0 & 1 & 0 \end{bmatrix} \\
\boldsymbol{y}'(p)(t) &= \begin{bmatrix} \frac{1}{M}\left(f(t) - \frac{D}{M}\int_0^t f(s)\,ds - v_0 - x_0\right) \\ \frac{1}{M}\int_0^t f(s)\,ds + v_0 \\ x_0 \end{bmatrix} \\
F'_p \begin{bmatrix} a \\ v \\ x \end{bmatrix}(t) &= \int_0^t A'(p)(s) \begin{bmatrix} a \\ v \\ x \end{bmatrix}(s)\,ds + \boldsymbol{y}'(p)(t)
\end{aligned}
$$

Alternatively, we could skip the order-reduction step and simply integrate the original ODE twice with respect to t, solving for the constants of integration using the initial conditions to yield

$$
\begin{aligned}
x(t) &= \frac{1}{M}\int_0^t \left(\int_0^s (f(r) - Kx(r))\,dr + Dx(s) + v_0\right) ds + x_0 \\
&= \frac{1}{M}\int_0^t \left(-K\int_0^s x(r)\,dr + Dx(s)\right) ds + \frac{1}{M}\left(\int_0^t \int_0^s f(r)\,dr\,ds + tv_0\right) + x_0
\end{aligned}
$$

In this case we use 1×1 "matrices," setting

$$
\begin{aligned}
A_1(p) &= \frac{D}{M} \\
A_2(p) &= -\frac{K}{M} \\
B(p) &= A_3 = A_4 = \cdots = 0 \\
\boldsymbol{y}(p)(t) &= \frac{1}{M}\left(\int_0^{(2)} f(t) + tv_0\right) + x_0
\end{aligned}
$$

Finally, returning to the issue of molasses-submerged systems and similarly whimsical contrivances (along with more practical ones), observe that the matrices employed in this application have made no use of the dimension of time, which is built into the model. Thus, K, D, and M can be made to vary smoothly over time if, for example, one wishes to model such systems as a mass-spring-damper in a medium of varying viscosity and/or temperature.

3.1.3 Case Study 2

The second case study in [1] involves a coupled mass-spring-damper system: two MSD systems with one connected to the mass of the other. The authors derive a similar system of integral equations (with two of everything involved in Case Study 1) and determine that the system satisfies ***Contr*** as long as $M_1 > max(2K_1, 2D_1)$ and $M_2 > max(2K_1 + 2K_2, 2D_2)$. Fortunately, this can be modified in the same way as Case Study 1 to yield an equivalent system that satisfies ***Contr*** for any parameter values. Just as in the simpler version, the corresponding parametrized operator is continuous, and hence, Theorem TZJ2 can be applied to it to obtain a continuous fixed-point function $\Phi : (\mathbb{R}^+)^6 \times \mathbb{R}^4 \to \mathcal{C}[\mathbb{T}, \mathbb{R}]^k$ (where k can be chosen to be 2, 4, or 6, depending on whether acceleration and velocity are to be explicitly represented by streams).

3.2 Simple Pendulum

The simple, frictionless pendulum with a single, rigid arm, constrained to move within a vertical plane is another staple of elementary mechanics. It is represented using the following second-order ODE (see [35]):

$$\ddot{\theta}(t) \quad = \quad -\frac{g}{\ell}\sin(\theta(t)) \tag{12}$$

where $\theta(t)$ is the angle formed by the bob and its equilibrium position at time t, g is the gravitational constant, and ℓ is the length of the arm. Using the order-reduction trick from the last example, let $\phi = \dot{\theta}$. Then (12) can be represented by the following equivalent system:

$$\begin{aligned} \phi(t) \quad &= \quad -\int_0^t \frac{g}{\ell}\sin(\theta(s))\ ds + \phi_0 \\ \theta(t) \quad &= \quad \int_0^t \phi(s)\, ds + \theta_0 \end{aligned}$$

Our parameter space is $P = \mathbb{R}^+ \times \mathbb{R}^2$ (condense g/ℓ into a single, positive parameter, leaving ϕ_0 and θ_0 as real numbers). In this case, the General Form Theorem is of no help at all since the sin function is nonlinear. We can, however, still use the Building Block Lemma directly and treat the sin function as a sort of magically-bestowed, primitive operator like the identity and the constant functions from (1a) and (1b) of the Building

Block Lemma. Define[9] $G : P \times \mathcal{C}[\mathbb{T}, \mathbb{R}]^2 \to \mathcal{C}[\mathbb{T}, \mathbb{R}]^2$ and $\boldsymbol{y} : P \to \mathcal{C}[\mathbb{T}, \mathbb{R}]^2$ as follows:

$$G\left(p, \begin{bmatrix} \phi \\ \theta \end{bmatrix}\right)(t) = \begin{bmatrix} -\frac{g}{\ell} \sin(\theta(t)) \\ \phi(t) \end{bmatrix} \text{ and } \boldsymbol{y}(p)(t) = \begin{bmatrix} \phi_0 \\ \theta_0 \end{bmatrix}$$

We can then define $F : P \times \mathcal{C}[\mathbb{T}, \mathbb{R}]^2 \to \mathcal{C}[\mathbb{T}, \mathbb{R}]^2$ like this:

$$F\left(p, \begin{bmatrix} \phi \\ \theta \end{bmatrix}\right)(t) = \int_0^{(1)} G\left(p, \begin{bmatrix} \phi \\ \theta \end{bmatrix}\right)(t) + \boldsymbol{y}(p)(t)$$

While G is defined above as a stream operator, it actually uses only the current value of the input stream (see Footnote 9). Hence, it clearly satisfies $\boldsymbol{WCaus}$. It takes a bit of work to develop the formal details, but differentiating G_p with respect to ϕ and θ reveals that G_p satisfies $\boldsymbol{Lip}(\lambda_G, \tau)$ for any $\tau \in \mathbb{R}^+$ and $\lambda_G = \max\{1, g/\ell\}$. This is because the magnitude of the slope of the first component of G (with respect to θ rather than t) never exceeds g/ℓ, and the slope of the second (with respect to ϕ) is always 1.

Thus, we can apply the Building Block Lemma to deduce that F_p satisfies $\boldsymbol{Caus}$ and $\boldsymbol{Lip}(1/2, \tau)$ for $\tau = \frac{1}{2}\min\{1, \ell/g\}$. It is clear by inspection that G is continuous, and hence, by the Continuity Lemma, so is F. Hence, by Theorem TZJ2, so is the fixed point function for F.

The continuity of the fixed point may be somewhat surprising in this case since, for any g/ℓ, there is a certain critical angular velocity (or position/velocity pair) which will be precisely the right amount to turn the bob upright and leave it there forever in its unstable equilibrium position. Even the slightest amount less and the bob falls back down on the side from which it approached the vertical. The slightest amount more, and it goes over the top, swinging back down on the other side. This would seem to represent a discontinuity at that point of critical velocity, but in fact, it doesn't. Theorem TZJ2 assures of us this, but it offers little in the way of insight.

What drives our perception of a discontinuity is the abrupt change in the *asymptotic* behaviour of the system in response to arbitrarily small changes in the initial conditions. Qualitatively speaking, there is a profound difference between a pendulum that falls back down and one that remains upright. What this observation fails to consider is the length of time the bob spends in a near-upright position. As the initial velocity approaches that critical value which leaves the pendulum upright forever, the bob spends more and more time in that very slow-moving limbo state in which it would appear to have an uncertain future.

[9] Note that G could instead be defined using the simpler form $G : P \times \mathbb{R}^2 \to \mathbb{R}^2$, but such a definition—while certainly more elegant here—introduces awkwardness in the next step.

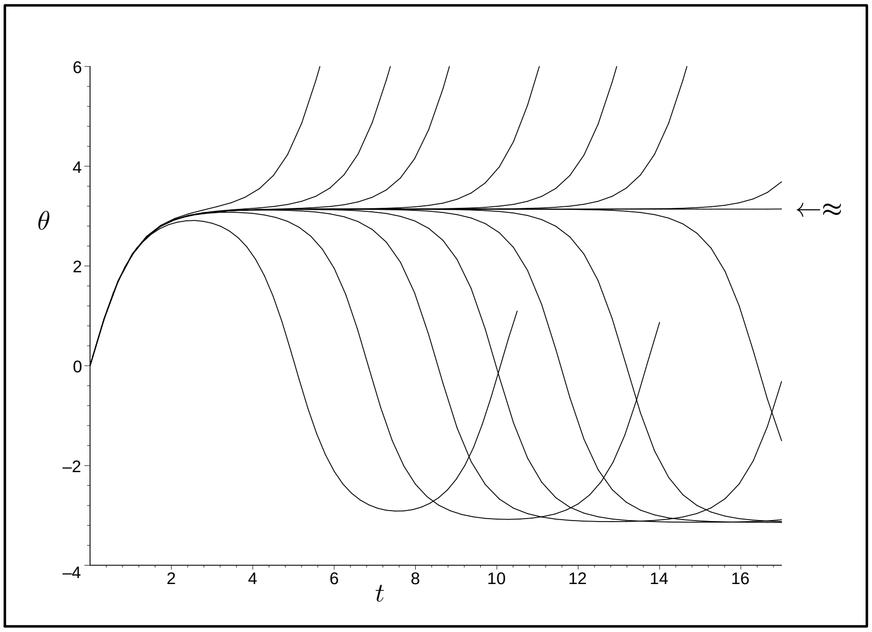

Figure 4: Pendulum trajectories approaching perfect equilibrium

Now consider this fact in light of the topology on $\mathcal{C}[\mathbb{T}, \mathbb{R}]$. Increasingly large values of T must be used to encounter any significant difference (with respect to the pseudometrics d_T) between the trajectory of the perpetually upright bob, and those with sufficiently similar initial velocity. This phenomenon is plotted in Figure 4. Each curve corresponds to the trajectory of the bob, starting at $\theta(0) = 0$ (hanging straight down initially), with a certain initial velocity. The trajectory marked "$\approx$" is the one corresponding to the perfect amount of initial velocity to push the bob upright and leave it there forever. The trajectories that slope downward are produced by less initial velocity (some are truncated in the plot for the sake of clarity), and those which slope upward are produced by an excessive initial velocity, which pushes the bob over the top. The important thing to note is that—regardless of whether too little or too much initial velocity is involved—the time at which the non-upright trajectories distinguish themselves becomes later and later, the closer their initial velocity is to the critical value. From this, we may conclude that while instability likely always results in a (locally) smaller modulus of continuity, it does not necessarily imply actual discontinuity—i.e. the modulus of continuity will get very small around an unstable equilibrium point, but it may still remain strictly positive.

4 Future Work

4.1 Develop Building Blocks to Handle Cases Like the Pendulum

It is, of course, disappointing that for all the power of the General Form Theorem, it is still insufficient to handle an application as basic as the most simple pendulum from elementary mechanics. This is one price we pay for keeping our Banach space $\mathcal{B}$ distinct from its set of scalars $\mathcal{S}$. By letting $\mathcal{B} = \mathcal{S}$ (which holds for many common vector spaces anyway), we can introduce exponentiation, which in turn, allows for power series and hence trigonometric and exponential operators. This is not, by any means, a straightforward addition to the theory, however. Consider, for example, the prospect of including the rather modest, pointwise squaring operator to $\mathcal{C}[\mathbb{T}, \mathbb{R}]$:

Example 4.1. Let $\mathbf{id}^2 : \mathcal{C}[\mathbb{T}, \mathbb{R}] \to \mathcal{C}[\mathbb{T}, \mathbb{R}]$ be defined as follows for $u \in \mathcal{C}[\mathbb{T}, \mathbb{R}]$:

$$\mathbf{id}^2(u)(t) = (u(t))^2$$

$\mathbf{id}^2$ certainly satisfies $\boldsymbol{WCaus}$, but what about $\boldsymbol{Lip}(\lambda, \tau)$? Let $\lambda, \tau \in \mathbb{R}^+$ and consider the following two streams:

$$\begin{aligned} u(t) &= \frac{\lambda + 1}{\tau} t \\ v(t) &= 0 \end{aligned}$$

Then $d_0(u, v) = 0$, but $d_{0+\tau}(\mathbf{id}^2(u), \mathbf{id}^2(v)) = (\lambda + 1)^2 > \lambda(\lambda + 1) = \lambda d_{0+\tau}(u, v)$. Thus, $\forall \lambda, \tau \in \mathbb{R}^+$ $\mathbf{id}^2 \notin \boldsymbol{Lip}(\lambda, \tau)$.

The problem here is ultimately due to the fact that the derivative of $f(x) = x^2$ is unbounded on $\mathbb{R}$. No matter how leniently we choose λ (and τ), we can always find a steep enough stream to deny $\mathbf{id}^2$ its coveted membership in the class of $\boldsymbol{Lip}(\lambda, \tau)$ operators. One way we might be able to circumvent this problem is by developing a nested exhaustion $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \cdots \subseteq \mathcal{B}$ of the codomain of the streams. After all, given any

$R, \tau \in \mathbb{R}^+$ and any $u, v \in \mathcal{C}[\mathbb{T}, [-R, R]] \subseteq \mathcal{C}[\mathbb{T}, \mathbb{R}]$, we see that

$$\begin{aligned} d_{T+\tau}(\mathbf{id}^2(u), \mathbf{id}^2(v)) &= \max_{0 \leq t \leq T+\tau} \left| u^2(t) - v^2(t) \right| \\ &= \max_{0 \leq t \leq T+\tau} |u(t) + v(t)| \, |u(t) - v(t)| \\ &\leq 2R \max_{0 \leq t \leq T+\tau} |u(t) - v(t)| \\ &= 2R d_{T+\tau}(u, v) \end{aligned}$$

Therefore, $\mathbf{id}^2$ can be said to satisfy the $\boldsymbol{Lip}(2R, \tau)$ condition on $\mathcal{C}[\mathbb{T}, [-R, R]]$.

Returning to the example of the pendulum, recall the Maclaurin series for $\sin(t)$:

$$\sin(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}$$

Let s_n be the derivative of the n^{th} partial sum:

$$s_n(t) = \sum_{k=0}^{n} \frac{(-1)^k t^{2k}}{(2k)!}$$

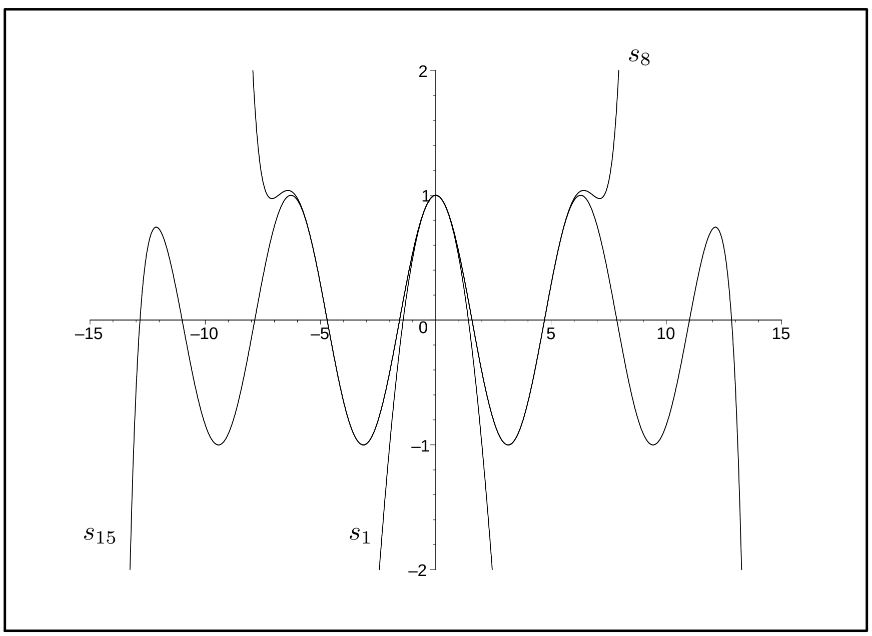

Figure 5: Partial sums in the Maclaurin expansion for $\cos(t)$

The preimage of any bounded interval centred at 0 (say, $[-1, 1]$) continues to expand as we examine successively larger partial sums. Turning our attention to Figure 5, we see that the preimage of $[-1, 1]$ under

s_1 is $[-2, 2]$. Under s_8, it's roughly $[-6.1, 6.1]$, and by s_{15}, the preimage has expanded to approximately $[-13.1, 13.1]$. As $n \to \infty$, this preimage of $[-1, 1]$ under s_n approaches $\mathbb{R}$.

So while none of these partial sums satisfy the ***Lip*** condition on $\mathcal{C}[\mathbb{T}, \mathbb{R}]$, the operator to which they converge does. This observation offers some hope that with a bit of care, power series might be incorporated into the theory. The main value of doing so is in their tremendous versatility. We could, of course, simply throw specific analytic functions like sin into the theory individually, but it would be far more powerful (and elegant) to catch them all in a single net. Furthermore, it allows for greater generality. The sin function traditionally assumes only real or complex values, but its power series expansion could be used to define versions of it (along with several other functions) on more exotic spaces.

4.2 Other Lines of Inquiry

- A network model (or rather a mathematical model that can be interpreted as a network model) along with a system of fixed-point semantics is presented in [2]. The authors describe a set of conditions sufficient to guarantee that the model operates properly. The abstractness of the model necessarily imposes a corresponding level of abstractness on these conditions. This paper is meant to be a companion work in which some of that abstractness is sacrificed in an attempt to get closer to a more concrete, GPAC-like result—a result in which a tangible class of functions is identified that satisfies Tucker and Zucker's conditions. [2] is, however, only the first of a two-part series, the second of which examines their model from the framework of computable analysis (specifically, the computable analysis covered in [36]). Hence, a natural second step will be to follow this paper with a corresponding entry that applies computable analysis to the Building Block and Continuity Lemmas, and to the General Form Theorem.

- Even with the pendulum included, this theory, as it stands currently, cannot be applied to most of the dynamical systems from elementary physics (instances of the wave equation, heat diffusion, and even merely the double pendulum). Its reliance on explicit formulas is perhaps the biggest limiting factor. Most of the common systems of partial differential equations and differential-algebraic equations cannot be represented explicitly the way the pendulum and the mass-spring-damper system can be written: with isolated (stream) variables exclusively on the left-hand side and potentially more complicated expressions on the right. While it is certainly more powerful than a direct application of Banach's Contraction Mapping Principle, the dependence upon this form is quite frustrating. It would enhance the theory tremendously if it could be adapted somehow to be applicable to some of the implicit forms. Note that, unlike the GPAC, there is no obvious reason the model presented here could not be applied to functions of more than one variable. While we insist on having at least one nonnegative real variable, others could easily be included in the Banach space and the parameter space (e.g. our "streams" could be continuous functions of the form $u : \mathbb{T} \to \mathcal{L}^2(\mathbb{R})$).

- As mentioned in the introductory remarks of Section 1.4, it is a somewhat intriguing coincidence that our theory involves the use of square matrices whose elements are taken from what turns out be a commutative ring with identity ($\mathcal{C}[\mathbb{T}, \mathcal{S}]$), and that this just happens to be the minimal algebraic structure necessary to define determinants [31]. Whether any of the myriad uses for determinants is applicable to the theory is unknown to us, but it would seem to warrant at least a cursory investigation.

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