Department of Computing and Software

Faculty of Engineering — McMaster University

Domain Information System (DIS): From Theory to Semantics

by

Alicia Marinache, Ridha Khedri, Wendy MacCaull

CAS Report Series **CAS-25-02-RK**

Department of Computing and Software **August 2025**

Information Technology Building
McMaster University
1280 Main Street West Hamilton, Ontario, Canada L8S 4K1

Copyright © 2025

Domain Information System (DIS): From Theory to Semantics

Alicia Marinache[1], Ridha Khedri[1], and Wendy MacCaull[2]

[1] Department of Computing and Software, Faculty of Engineering,
McMaster University, Hamilton, Ontario, Canada

[2] Department of Mathematics, Statistics and Computer Science, Faculty of Science,
St. Francis Xavier University, Antigonish, Nova Scotia, Canada

Technical Report CAS-25-02-RK
Department of Computing and Software
McMaster University

August 18, 2025

Abstract

his report develops the formal semantics of the Domain Information System (DIS) framework by constructing a mathematical interpretation of its core structures and proving that this interpretation constitutes a model of the DIS theory. DIS distinguishes between two foundational components: the Domain Data View (DDV), which captures structured data using cylindric algebra, and the Domain Ontology (DOnt), which models conceptual knowledge via Boolean lattices, commutative monoids, and rooted graphs. A mapping operator aligns data instances to ontology concepts, supporting formal reasoning across layers. We demonstrate that the proposed interpretation satisfies the axioms of the DIS theory, confirming its internal consistency and providing a sound foundation for reasoning about data-aware knowledge systems.

Keywords: knowledge representation, reasoning, data layer, conceptual layer

Contents

1 Introduction

Structured data is increasingly central to knowledge-driven applications, yet aligning it with formally defined ontological frameworks remains a persistent challenge [KVS20, DGLL$^+$18]. Traditional approaches to ontology engineering often assume a fixed conceptual model and rely on post hoc mappings to relate that model to underlying datasets. This separation leads to inconsistencies, costly integration processes, and limitations in reasoning, particularly as systems evolve or need to be reused across domains.

To address this, we adopt the DIS framework [Mar16], a formalism that explicitly decouples data from domain knowledge while maintaining a semantically coherent integration between them. DIS introduces two distinct but connected theories: a DDV, which represents structured data through algebraic constructs, and a DOnt, which defines the conceptual structure using Boolean lattices and relational graphs. These components are linked via a mapping operator that ensures data-level structures correspond directly to domain-level concepts. This formal separation allows the ontology to evolve independently of data, and vice versa, while enabling precise reasoning across both.

In this technical report, we provide a modular formalisation of the DIS. In Section 2, we present the theory and its syntax. In Section 3 we introduce an interpretation for DIS, and in Section 4 we verify that this interpretation is a model for the DIS theory.

2 Domain Information System: Theory

The DIS is a modular framework for knowledge representation that structurally separates data from domain knowledge. It is composed of three main components: the DOnt, which models conceptual knowledge through algebraic and graph-based structures, the DDV, which captures structured data using diagonal-free cylindric algebra [THM71], and a mapping operator τ that links data to domain concepts.

The DOnt provides a semantic foundation by combining a commutative idempotent monoid of atomic and composite concepts, a Boolean lattice that defines a hierarchy of concepts through composition and negation, and a family of rooted graphs, each representing semantic dependencies or conceptual hierarchies within the ontology. In this context, rooted graphs are directed acyclic graphs in which all paths terminate at a unique root, allowing for context-aware navigation and modularisation of concepts. Formally, for a rooted graph $G_{t_i} = \big(C_i, R_i, t_i\big)$, its root is given by $t_i \in C_i$ s.t. $\forall k \in C_i.\ k = t_i \vee (k, t_i) \in {R_i}^+$, where ${R_i}^+$ is the transitive closure of R_i.

Definition 2.1 (Domain Ontology)**.** Let $\mathcal{C} = \big(C, \oplus, \mathbf{e}_C\big)$ be a commutative idempotent monoid. Let $\sqsubseteq_C$ be the natural order on $\mathcal{C}$, defined as $\forall c, d \in C.\ c \sqsubseteq_C d \iff c \oplus d = d$. Let $\mathcal{L} = \big(L, \sqsubseteq_C\big)$ be a free Boolean lattice, generated from a set of atoms in C. Let I be a finite set of indices, and $\mathcal{G} = \{G_{t_i}\}_{t_i \in L, i \in I}$ a set of graphs rooted in L. A *domain ontology* is the structure $\mathcal{O} \stackrel{\text{def}}{=} \big(\mathcal{C}, \mathcal{L}, \mathcal{G}\big)$. $\square$

To model data, the DDV uses a cylindric algebra where each dimension corresponds to a dataset attribute. The cylindrification operators $\mathbf{c}_\kappa$ are indexed over the elements of a set $\mathcal{U}$ isomorphic to the atoms of L, where L is the carrier set of the Boolean lattice $\mathcal{L}$ of the domain ontology, as described in Definition 2.1.

Definition 2.2 (Domain Data View associated with an ontology)**.** Let $\mathcal{O} = \big(\mathcal{C}, \mathcal{L}, \mathcal{G}\big)$ be a domain ontology. A *domain data view* associated with $\mathcal{O}$ is a diagonal-free cylindric algebra $\mathcal{A} = \big(A, +, \star, -, 0_A, 1_A, \{\mathbf{c}_\kappa\}_{\kappa \in \mathcal{U}}\big)$, where $\mathcal{U}$ represents the finite set of attributes of the considered dataset. □

The two components are integrated through the mapping operator τ. This operator assigns to each data element of the DDV a concept in the DOnt, capturing its structural and semantic meaning.

Definition 2.3 (Domain Information System)**.** Let $\mathcal{O}$ be a domain ontology, $\mathcal{A}$ its associated domain data view, and $\tau : A \to L$ a mapping that relates the elements of A to the elements of the Boolean lattice in $\mathcal{O}$. A *Domain Information System (DIS)* is the structure $\mathcal{I} = (\mathcal{O}, \mathcal{A}, \tau)$ for which the following axioms hold, for any $a, b \in A$:

- $\tau(0_A) = \mathbf{e}_C$
- $\tau(1_A) = \top_\mathcal{L}$
- $\tau(a + b) = \tau(a) \oplus \tau(b)$ □

These definitions are governed by the axioms listed below, which define the algebraic, graph-based, and mapping properties of a well-formed DIS $\mathcal{I} = (\mathcal{O}, \mathcal{A}, \tau)$. Let $a, b \in A, k, k_1, k_2 \in C, \kappa, \lambda \in \mathcal{U}, i \in \mathbb{N}, G_{t_i} \in \mathcal{G}$, with $G_{t_i} = \big(C_i, R_i, t_i\big)$

(A1) $\big(C, \oplus, \mathbf{e}_C\big)$ is a commutative idempotent monoid

(A2) $\big(L, \oplus, \otimes, \ominus, \mathbf{e}_C, \top_\mathcal{L}\big)$ is a Boolean algebra

(A3) $L \subseteq C$

(A4) $C_i \subseteq C$

(A5) $R_i \subseteq C_i \times C_i$

(A6) $(k_1, k_2) \in R_i \implies k_1 \neq k_2$

(A7) $t_i \in L \wedge \forall k \in C_i.\ k = t_i \vee (k, t_i) \in {R_i}^+$

(A8) $(k_1, k_2) \in R_i \implies k_2 = t \vee (k_2, t_i) \in R_i^+$

(A9) $k \in C \implies k \in L \vee \exists G_{t_i} = \big(C_i, R_i, t_i\big) \in \mathcal{G}.\ k \in C_i$

(A10) $\big(A, +, \star, -, 0_A, 1_A\big)$ is a Boolean algebra

(A11) $\mathbf{c}_\kappa(0) = 0$

(A12) $a \leq \mathbf{c}_\kappa(a)$

(A13) $\mathbf{c}_\kappa(a \star \mathbf{c}_\kappa(b)) = \mathbf{c}_\kappa(a) \star \mathbf{c}_\kappa(b)$

(A14) $\mathbf{c}_\kappa(\mathbf{c}_\lambda(a)) = \mathbf{c}_\lambda(\mathbf{c}_\kappa(a))$

(A15) $\tau(0_A) = \mathbf{e}_C$

(A16) $\tau(1_A) = \top_{\mathcal{L}}$

(A17) $\tau(a+b) = \tau(a) \oplus \tau(b)$ $\square$

Axiom (A1) describes the monoid of concepts $\mathcal{C}$. Axioms (A2)$-$(A3) describe the Boolean lattice $\mathcal{L}$. Axioms (A4)$-$(A9) describe the family of rooted graphs $\mathcal{G}$, as well as the boundary set on C. Axioms (A10)$-$(A14) describe the cylindric algebra $\mathcal{A}$. Axioms (A15)$-$(A17) describe the operator τ. With every monoid there is a natural order. In the monoid $\mathcal{C}$ (of the DOnt component), this is expressed as the partial order $\sqsubseteq_{\mathcal{C}}$, defined as $k_1 \sqsubseteq_{\mathcal{C}} k_2 \overset{\text{def}}{\iff} k_1 \oplus k_2 = k_2$. In the cylindric algebra $\mathcal{A}$ (of the DDV component), the natural order denoted by $\leq$ is defined as $a \leq b \overset{\text{def}}{\iff} a + b = b$.

3 Domain Information System: Model

3.1 Domain Data View Model

To model the DDV component of a DIS instance $\mathcal{I}$, we begin with a finite set of sorts, called the universe and denoted by $\mathcal{U} = \{S_1, S_2, \dots, S_n\}$. These sorts are derived from the attributes of the underlying dataset and are interpreted as finite sets of values associated with those attributes. While two sorts may share the same value domain, for example sorts StartYear and EndYear might both contain the value 2025, they are treated as semantically distinct due to their differing roles and interpretation in the domain ontology (see Section 3.4). An *s-value* is an ordered pair (s, v), where $s \in \mathcal{U}$ is a sort and $v \in s$ is an element of that sort. Two s-values with identical values but different sorts (e.g., $(\mathit{StartYear}, \mathit{2025})$ and $(\mathit{EndYear}, \mathit{2025})$) are not equivalent, as they encode different semantic meanings. An *s-datum* represents a structured data tuple and is defined as a nonempty set of s-values such that each sort appears at most once.We refer to the set of sorts that appear in an s-data as its *structure*.

The DDV is modelled as the mathematical structure $\mathcal{M}_{\mathcal{A}} = \big(A, +, \star, -, \mathbf{0}, \mathbf{1}, \{\mathbf{c}_\kappa\}_{\kappa \in \mathcal{U}}\big)$. Each element of carrier set A is interpreted as a set of well-formed s-datums, i.e., an s-data. The Boolean operators $+, \star$ and $-$ are interpreted as union, intersection, and complement, respectively.

We use the following notation throughout: $s, s_1, \dots$ for sorts, $v, v_1, \dots$ for sort elements, $sv, sv_1, sv_a, \dots$ for s-values, $dt, dt_1, dt_a, \dots$ for s-datums, and $a, b, x, y, a_1, x_1, \dots$ for s-datas. Unless otherwise specified, the sort corresponding to an attribute named 'attr' is denoted by S_{attr}.

We define the set of all possible s-values as

$$\mathit{SV} \overset{\text{def}}{=} \bigcup_{s \in \mathcal{U}} \{(s, v) \mid v \,.\, v \in s\}$$

To generate all possible well-formed s-datums, we define the special product operator $\bowtie\, : \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathit{SV})$. When it is clear from the context, we write $\bowtie T$ instead of $\bowtie(T)$. The $\bowtie$ operator is defined by induction, as follows, for any $s \in \mathcal{U}, T \subseteq \mathcal{U}, s \notin T$:

$$\begin{aligned} \bowtie\{\} &= \{\} \\ \bowtie(T \cup \{s\}) &= \bowtie T \,\cup \big\{ dt \cup \{(s, v)\} \mid dt, v \,.\, dt \in \bowtie T \,\wedge v \in s \big\} \end{aligned} \tag{1}$$

The set of s-datums is $SD \stackrel{\text{def}}{=} \bowtie\mathcal{U}$ and the carrier set of the cylindric algebra is $A = \mathcal{P}(SD)$. Due to the definitions of A, SD, and the powerset axiom ($x \in \mathcal{P}(S) \iff x \subseteq S$), it is immediate that, in a given DIS, for any $a \in A$ the following equation holds:

$$a \in A \iff a \subseteq \bowtie\mathcal{U} \tag{2}$$

The following proposition describe properties of the special product operator. Proposition 3.1 describes the sort exclusion in product closure, i.e., when a sort is not present in a given subset of sorts, its s-values cannot be present in the special product of the given subset. Proposition 3.2 describes the monotocity of the special product operator w.r.t. to set inclusion. Proposition 3.3 describes the sort preservation under the special product. Proposition 3.4 captures the fact that the special product operator distributes over intersection of sort sets.

Proposition 3.1. Let $\mathcal{U}$ be a DIS universe. For any $s \in \mathcal{U}, T \subseteq \mathcal{U}$:

1. $s \notin T \implies \bowtie\{s\} \cap \bowtie T = \{\}$
2. $\forall v \in s, dt \in SD.\ s \notin T \implies \big(dt \cup \{(s, v)\}\big) \notin \bowtie T$ $\square$

Proposition 3.2. Let $\mathcal{U}$ be a DIS universe and $\bowtie$ the special product operator defined in 1. The $\bowtie$ operator is montone, i.e., for any $T, T' \subseteq \mathcal{U}$ the following statement is true: $T' \subseteq T \implies \bowtie T' \subseteq \bowtie T$

Proposition 3.3. Let $\mathcal{U}$ be a DIS universe and $\bowtie$ the special product operator defined in 1. For any $s \in \mathcal{U}, T \subseteq \mathcal{U}$, $s \in T \implies \bowtie\{s\} \cap \bowtie T = \bowtie\{s\}$. $\square$

Proposition 3.4. Let $\mathcal{U}$ be a DIS universe and $\bowtie$ the special product operator defined in 1. The $\bowtie$ operator distributes over intersection, i.e., for any $T_1, T_2 \subseteq \mathcal{U}$, $\bowtie T_1 \cap \bowtie T_2 = \bowtie(T_1 \cap T_2)$. $\square$

3.2 Operators on Data and their Properties

To reason about structural properties of data in a DIS, we define several operators that act on s-datums and s-datas within the DDV. These operators allow us to manipulate the presence or absence of sorted values associated with specific sorts, enabling special kinds of projection and generalisation w.r.t. with a specified sort.

Let $\mathcal{I}$ be a DIS, and $\mathcal{A}$ its associated DDV with universe $\mathcal{U}$. Let $dt \in SD$ be an s-datum. For any sort $\kappa \in \mathcal{U}$, we say that "κ is part of dt", denoted by $\kappa \mathrel{\bar{\in}} dt$, if there exists a κ s-value in dt, i.e., if it is true that $\exists v \in \kappa.\ (\kappa, v) \in dt$. We define four key data operators: the reduction operator on s-datums, which is denoted by $dt{\downharpoonright}_{\kappa}$ and removes κ s-value from dt (if it exists); the corresponding reduction operator on s-datas, which is denoted by $a{\Downarrow}_{\kappa}$ and applies the reduction operator on each s-datum in a; the extension operator or s-datums, which is denoted by $dt{\upharpoonleft}^{\kappa}$ and generates all variants of dt extended with every possible κ s-value not already present; and the extension on s-datas, which is denoted by $a{\Uparrow}^{\kappa}$ and collects all extended s-datums obtained from each $dt \in a$. These operators form the foundation for manipulating sorted structures in a principled way and are defined formally in Equations3-6.

$$dt{\downarrow_{\kappa}} \stackrel{\text{def}}{=} \{(s,v) \mid s,v\,.\,(s,v) \in dt \land s \neq \kappa\} \tag{3}$$

$$a{\Downarrow_{\kappa}} \stackrel{\text{def}}{=} \{dt{\downarrow_{\kappa}} \mid dt\,.\,dt \in a\} \tag{4}$$

$$dt{\uparrow^{\kappa}} \stackrel{\text{def}}{=} \begin{cases} \{\big(dt{\downarrow_{\kappa}} \cup \{(\kappa,v)\}\big) \mid v\,.\,v \in \kappa\} & \text{if } \kappa \mathrel{\overline{\in}} dt \\ \{dt\} & \text{otherwise} \end{cases} \tag{5}$$

$$a{\Uparrow^{\kappa}} \stackrel{\text{def}}{=} \bigcup_{dt \in a} dt{\uparrow^{\kappa}} \tag{6}$$

By construction of s-datums, the following equation holds for any $dt \in SD, \kappa \in \mathcal{U}$, and $v \in \kappa$:

$$\kappa \mathrel{\overline{\in}} dt \iff \exists v \in \kappa.\; dt = \big(dt{\downarrow_{\kappa}} \cup \{(\kappa,v)\}\big) \tag{7}$$

The following propositions describe the properties of these operators. The behaviour of sort-reduction on a singleton s-datum is formalised in Proposition 3.5.

Proposition 3.5. Let $\mathcal{U}$ be a DIS universe. For any $\kappa, \lambda \in \mathcal{U}$ and $v \in \kappa$

$$\{(\kappa,v)\}{\downarrow_{\lambda}} = \begin{cases} \{(\kappa,v)\} & \text{if } \kappa \neq \lambda \\ \{\} & \text{otherwise} \end{cases}$$

$\square$

The κ-reduction removes all κ s-value from any s-datum or s-data it is applied to, while their κ-extension adds the missing κ s-value. Thus, the reduction and extension of any s-data are disjoint, a property formalised in Proposition 3.6.

Proposition 3.6. Let $\mathcal{A}$ be a DDV, with $\mathcal{U}$ its universe and A its carrier set. For any $\kappa \in \mathcal{U}$ and $a \in A$ the following holds: $(\forall dt \in a.\; \kappa \mathrel{\overline{\in}} dt) \implies a{\Downarrow_{\kappa}} \cap a{\Uparrow^{\kappa}} = \{\}$. $\square$

The operator $\downarrow$ distributes over both union and intersection. The operators $\Downarrow$ and $\Uparrow$ distribute over union; however, they do not distribute over intersection. These properties are formalised in Proposition 3.7.

Proposition 3.7. Let $\mathcal{A}$ be a DDV, with $\mathcal{U}$ its universe, SD its set of s-datums, and A its carrier set. For any $\kappa \in \mathcal{U}$, $dt_1, dt_2 \in SD$, and $a, b \in A$, the following properties hold:

1. $(dt_1 \cup dt_2){\downarrow_{\kappa}} = dt_1{\downarrow_{\kappa}} \cup dt_2{\downarrow_{\kappa}}$
2. $(dt_1 \cap dt_2){\downarrow_{\kappa}} = dt_1{\downarrow_{\kappa}} \cap dt_2{\downarrow_{\kappa}}$
3. $(a \cup b){\Downarrow_{\kappa}} = a{\Downarrow_{\kappa}} \cup b{\Downarrow_{\kappa}}$
4. $(a \cap b){\Downarrow_{\kappa}} \subseteq a{\Downarrow_{\kappa}} \cap b{\Downarrow_{\kappa}}$
5. $(a \cup b){\Uparrow^{\kappa}} = a{\Uparrow^{\kappa}} \cup b{\Uparrow^{\kappa}}$
6. $(a \cap b){\Uparrow^{\kappa}} \subseteq a{\Uparrow^{\kappa}} \cap b{\Uparrow^{\kappa}}$ $\square$

When applied repeatedly, the four data operators are idempotent, i.e., further applications have no effect, a property formalised in Proposition 3.8.

Proposition 3.8. Let $\mathcal{A}$ be a DDV, with $\mathcal{U}$ its universe, *SD* its set of s-datums, and A its carrier set. For any $\kappa \in \mathcal{U}$, $dt \in SD$, and $a \in A$, the data operators are idempotent w.r.t. to κ, i.e.,:

1. $(dt\downharpoonleft_\kappa)\downharpoonleft_\kappa = dt\downharpoonleft_\kappa$
2. $(dt\upharpoonleft^\kappa)\Uparrow^\kappa = dt\upharpoonleft^\kappa$
3. $(a\Downarrow_\kappa)\Downarrow_\kappa = a\Downarrow_\kappa$
4. $(a\Uparrow^\kappa)\Uparrow^\kappa = a\Uparrow^\kappa$ □

The reduction and extension operators commute over sorts, reflecting the fact that operations on distinct sorts are independent, and thus the order of their application has no effect. These properties are formalised in Proposition 3.8.

Proposition 3.9. Let $\mathcal{A}$ be a DDV, with $\mathcal{U}$ its universe, *SD* its set of s-datums, and A its carrier set. For any $\kappa, \lambda \in \mathcal{U}$, $dt \in SD$, and $a \in A$, the following equations hold:

1. $(dt\downharpoonleft_\kappa)\downharpoonleft_\lambda = (dt\downharpoonleft_\lambda)\downharpoonleft_\kappa$
2. $(a\Downarrow_\kappa)\Downarrow_\lambda = (a\Downarrow_\lambda)\Downarrow_\kappa$
3. $(a\Uparrow^\kappa)\Uparrow^\lambda = (a\Uparrow^\lambda)\Uparrow^\kappa$ □

To describe the internal structure of an s-data $a \in A$ w.r.t. a fixed sort $\kappa \in \mathcal{U}$, we define two disjoint component: the κ-floored component $\lfloor a \rfloor_\kappa$ that contains all s-datums in a without a κ s-value, and the κ-raised component $\lceil a \rceil^\kappa$ that contains the s-datums that include exactly one κ s-value. These two components are formalised as follows:

$$\lfloor a \rfloor_\kappa = \{dt \mid dt \in a \wedge \neg\exists v \in \kappa.\ (\kappa, v) \in dt\} \tag{8}$$
$$\lceil a \rceil^\kappa = \{dt \mid dt \in a \wedge \exists v \in \kappa.\ (\kappa, v) \in dt\} \tag{9}$$

These two operators partition a given s-data w.r.t. a given sort, as formalised in Proposition 3.10

Proposition 3.10. Let $\mathcal{A}$ be a DDV, with $\mathcal{U}$ its universe and A its carrier set. For any $\kappa \in \mathcal{U}$ and $a \in A$, the following properties hold:

1. $\lfloor a \rfloor_\kappa \cup \lceil a \rceil^\kappa = a$ (completeness: each s-datum in a is either κ-floored or κ-raised)
2. $\lceil a \rceil^\kappa \cap \lfloor a \rfloor_\kappa = \{\}$ (disjointness: no s-datum in a belongs to both components)) □

The interaction between these components and the s-data helper operators is formalised in Proposition 3.11. In particular, it establishes the following properties: preservation of the two components under intersection with extended s-data (Equations 1, 2, and Equation 3), disjointness between raised and reduced parts (Equation 4), and the structural equivalence between reduction and flooring (Equations 5 and 6).

Proposition 3.11. Let $\mathcal{A}$ be a DDV, with $\mathcal{U}$ its universe and A its carrier set. For any $\kappa \in \mathcal{U}$ and $a \in A$, the following equations hold:

1. $\lfloor a \rfloor_\kappa \cap a\Uparrow^\kappa = \lfloor a \rfloor_\kappa$
2. $\lceil a \rceil^\kappa \cap a\Uparrow^\kappa = \lceil a \rceil^\kappa$
3. $a \cap a\Uparrow^\kappa = a$
4. $\lceil a \rceil^\kappa \cap a\Downarrow_\kappa = \{\}$
5. $\lfloor a \rfloor_\kappa \cap a\Downarrow_\kappa = \lfloor a \rfloor_\kappa$
6. $a \cap a\Downarrow_\kappa = \lfloor a \rfloor_\kappa$ $\square$

Finally, Proposition 3.12 formalises the disjointness properties extended to any pair of s-datas.

Proposition 3.12. Let $\mathcal{A}$ be a DDV, with $\mathcal{U}$ its universe and A its carrier set. For any $\kappa \in \mathcal{U}$ and $a, b \in A$, the following equations hold:

1. $\lceil a \rceil^\kappa \cap \lfloor b \rfloor_\kappa = \{\}$
2. $\lceil a \rceil^\kappa \cap b\Downarrow_\kappa = \{\}$
3. $\lfloor a \rfloor_\kappa \cap b\Uparrow^\kappa = \lfloor a \rfloor_\kappa \cap \lfloor b \rfloor_\kappa$ $\square$

Together, these properties show that the structural components κ-flooring and κ-raising of an s-data are aligned with the algebraic behaviour of the data operators. The partitions they form are stable under various κ-modifications, and enable support for reasoning on data transformation in DIS.

3.3 Operators Interpretation

In the interpretation of DDV, the operators of the structure $\mathcal{M}_\mathcal{A}$ are defined using standard set-theoretic operations. The Boolean algebra is induced directly by interpreting union, intersection, complement, and identity elements over the set of s-data. The cylindrification operator is interpreted via the κ-extension operation defined earlier. Formally, for any $\kappa \in \mathcal{U}$ and $a, b \in A$, the operators are defined in Equations 10-15.

$$a + b \stackrel{\text{def}}{=} a \cup b \tag{10}$$

$$a \star b \stackrel{\text{def}}{=} a \cap b \tag{11}$$

$$\mathbf{0} \stackrel{\text{def}}{=} \{\} \tag{12}$$

$$\mathbf{1} \stackrel{\text{def}}{=} \bowtie\mathcal{U} \tag{13}$$

$$-a \stackrel{\text{def}}{=} \mathbf{1} \setminus a \tag{14}$$

$$\mathbf{c}_\kappa a \stackrel{\text{def}}{=} a\Uparrow^\kappa \tag{15}$$

Within this interpretation, we can establish several core properties of the cylindrification operator, formalised in the following propositions.

Proposition 3.13. Let $\mathcal{A}$ be a DDV, with $\mathcal{U}$ its universe and A its carrier set. The cylindrification operator distributes over the operator, i.e., for any $a, b \in A$ and any $\kappa \in \mathcal{U}$, $\mathbf{c}_\kappa(a + b) = \mathbf{c}_\kappa a + \mathbf{c}_\kappa b$. $\square$

Proposition 3.14. Let $\mathcal{A}$ be a DDV, with $\mathcal{U}$ its universe and A its carrier set. The cylindrification operator is idempotent w.r.t. to κ, i.e., for any $a \in A$ and any $\kappa \in \mathcal{U}$, $\mathbf{c}_\kappa(\mathbf{c}_\kappa a) = \mathbf{c}_\kappa a$. $\square$

In Section 4.2, we show that the $\mathcal{M}_\mathcal{A}$ structure satisfies the axioms of a diagonal-free cylindric algebra.

3.4 Domain Ontology Model

The model of a domain ontology $\mathcal{O}$ is denoted by $\mathcal{M}_\mathcal{O} = \big(\mathcal{M}_\mathcal{C}, \mathcal{M}_\mathcal{L}, \mathcal{M}_\mathcal{G}\big)$, corresponding to the concept monoid $\mathcal{M}_\mathcal{C} = \big(C_{At_\mathcal{C}}, \oplus, \mathbf{e}_C\big)$, the Boolean lattice $\mathcal{M}_\mathcal{L} = \big(L_{At_\mathcal{L}}, \oplus, \otimes, \ominus, \mathbf{e}_C, \top_\mathcal{L}\big)$, and the family of rooted graphs $\mathcal{M}_\mathcal{G} = \{G_{t_i}\}_{t_i \in L, i \in I}$, with $G_{t_i} = \big(C_i, R_i, t_i\big), C_i \subseteq C, R_i \subseteq C_i \times C_i, t_i \in L$.

The operators of a DOnt are formally defined for any $k_1, k_2 \in C$ and any $\kappa, \kappa_1, \kappa_2 \in L$, using set operators, as described in Equations 16 through 21.

$$k_1 \oplus k_2 \stackrel{\text{def}}{=} k_1 \cup k_2 \tag{16}$$
$$\mathbf{e}_C \stackrel{\text{def}}{=} \{\} \tag{17}$$
$$k_1 \sqsubseteq_C k_2 \stackrel{\text{def}}{=} k_1 \oplus k_2 = k_2 \tag{18}$$
$$\kappa_1 \otimes \kappa_2 \stackrel{\text{def}}{=} \kappa_1 \cap \kappa_2 \tag{19}$$
$$\top_\mathcal{L} \stackrel{\text{def}}{=} At_\mathcal{L} \tag{20}$$
$$\ominus\kappa \stackrel{\text{def}}{=} \top_\mathcal{L} \backslash \kappa \tag{21}$$

In Section 4.3, we we show that the $\mathcal{M}_\mathcal{O}$ structure satisfies the axioms of a domain ontology $\mathcal{O}$.

3.5 DIS Model: Mapping Operator Component

The final structural component of DIS is the mapping operator $\tau\colon A \to L$, which links data-level elements (from DDV) to concept-level elements (from DOnt). For an s-data $a \in A$, we define $\tau(a)$ as the concept formed by composing the atomic concepts corresponding to the sorts present in a.

To formalise this, we define two projection operators (given in postfix notation) on s-values: $.sort\colon SV \to \mathcal{U}$, which returns the sort component of a sorted value, and $.val\colon SV \to \mathcal{U}$, which returns its value component. Formally, for any $sv \in SV, sv = (s, v)$, $sv.sort \stackrel{\text{def}}{=} s$ and $sv.val \stackrel{\text{def}}{=} v$. A helper mapping $\eta\colon \mathcal{U} \to L$ establishes a one-to-one correspondence between the DDV sorts and atomic concepts in the DOnt Boolean lattice. Then, for any $a \in A$, $\tau(a)$ is formally defined in Equation 22.

$$\tau(a) = \{\eta(sv.sort) \ \mid\ sv, dt \,.\, sv \in dt \wedge dt \in a\} \tag{22}$$

The mapping operator preserves the composition, zero, and one operators between the two Boolean algebras, the $\mathcal{A}$ of DDV and the of the DOnt. These properties are described in Equations 23 to 25, for any $a, b \in A$. Proposition 3.15 expresses properties of the helper

η operator, and 3.16 formalise further properties of the mapping operator τ, including its behaviour w.r.t. the complement and $\star$ operators.

$$\tau(a+b) = \tau(a) \oplus \tau(b) \tag{23}$$
$$\tau(\mathbf{0}) = \mathbf{e}_C \tag{24}$$
$$\tau(\mathbf{1}) = \top_{\mathcal{L}} \tag{25}$$

Proposition 3.15. Let $\mathcal{I} = (\mathcal{O}, \mathcal{A}, \tau)$ be a DIS. For any $a \in A$, $\kappa \in \mathcal{U}$, the following properties hold:

1. $\exists dt \in a.\ \kappa \,\overline{\in}\, dt \implies \eta(\kappa) \sqsubseteq_C \tau(a{\Uparrow}^{\kappa})$
2. $\eta(\kappa) \not\sqsubseteq_C \tau(a{\Downarrow}_{\kappa})$ $\square$

Proposition 3.16. Let $\mathcal{I} = (\mathcal{O}, \mathcal{A}, \tau)$ be a DIS. For any $a, b \in A$, and $T \subseteq \mathcal{U}$:

1. $(a = \mathbf{0} \vee a = \mathbf{1}) \implies \tau(-a) = \ominus\tau(a)$
2. $\tau(a) \otimes \tau(b) = \mathbf{e}_C \implies \tau(a \star b) = \mathbf{e}_C$
3. $\tau(a \star b) \subseteq \tau(a) \otimes \tau(b)$
4. $a = \bowtie T_a \implies \big(\tau(a) = \bigcup\{\eta(s) \mid s\ .\ s \in T_a\}\big)$
5. $\big(a = \bowtie T_a \wedge b = \bowtie T_b\big) \implies \big(\tau(a \star b) = \tau(a) \otimes \tau(b)\big)$ $\square$

4 Domain Information System: Proofs

4.1 Operators on Data Properties

In this section, we present the proofs for the properties described in Section 3.2. We remind the reader that the set comprehension is given as $\{E \mid x\ .\ R\}$, where E, x, and R are the *expression*, (set of) *variable*(s), and *range*, respectively. In addition, all statements of the form $A \subseteq B$ are proved by showing that the equivalent equation $A \cap B = A$ holds.

For some of the proofs below, we use the following fact: for any $s \in \mathcal{U}, T_1, T_2 \subseteq \mathcal{U}$:

$$s \notin T_2 \implies \big(\{dt \cup \{(s,v)\} \mid dt, v\ .\ dt \in \bowtie T_1 \wedge v \in s\} \cap \bowtie T_2\big) = \{\} \tag{26}$$

Equation 26 is due to the fact that $\bowtie(T_2)$ can contain no $\{(s,v)\}$ s-values, as $s \notin T_2$. Proof for Equation 26 is done by induction on T_2:

Step 1: $T2 = \{\}$. The proof is trivial, using the definition of $\bowtie$, $\bowtie T_2 = \{\}$, and the annihilator for $\cap$ axiom.

Step 2: **IH**: $s \notin T_2 \implies \big(\{dt \cup \{(s,v)\} \mid dt, v\ .\ dt \in \bowtie T_1 \wedge v \in s\} \cap \bowtie T_2\big) = \{\}$. Show that for any $s' \in \mathcal{U}$, $s \notin (T_2 \cup \{s'\}) \implies \big(\{dt \cup \{(s,v)\} \mid dt, v\ .\ dt \in \bowtie T_1 \wedge v \in s\} \cap \bowtie (T_2 \cup \{s'\})\big) = \{\}$.

Case 1: $s' \in T_2$ is trivial, as it reduces to the **IH**.

Case2: $s' \notin T_2$

$\{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T_1 \;\wedge v \in s\} \cap \bowtie (T_2 \cup \{s'\})$

$==$ $\langle$ Definition of $\bowtie$ (1) $\rangle$

$\{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T_1 \;\wedge v \in s\} \cap$
$(\bowtie T_2 \cup \{dt' \cup \{(s',v')\} \mid dt', v' \,.\, dt' \in \bowtie T_2 \;\wedge v' \in s'\})$

$==$ $\langle$ Distributivity of $\cap$ over $\cup$ $\rangle$

$(\{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T_1 \;\wedge v \in s\} \cap \bowtie T_2) \cup$
$(\{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T_1 \;\wedge v \in s\} \cap$
$\{dt' \cup \{(s',v')\} \mid dt', v' \,.\, dt' \in \bowtie T_2 \;\wedge v' \in s'\})$

$==$ $\langle$ **IH** $\rangle$

$\{\} \cup$
$(\{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T_1 \;\wedge v \in s\} \cap$
$\{dt' \cup \{(s',v')\} \mid dt', v' \,.\, dt' \in \bowtie T_2 \;\wedge v' \in s'\})$

$==$ $\langle$ Annihilator for $\cup$ $\rangle$

$\{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T_1 \;\wedge v \in s\} \cap$
$\{dt' \cup \{(s',v')\} \mid dt', v' \,.\, dt' \in \bowtie T_2 \;\wedge v' \in s'\}$

$==$ $\langle\ \rangle$

$==$ $\langle\ \rangle$

$\{\}$

Proof for Proposition (3.2) is done by induction on T', with assumption $T' \subseteq T$:
Step 1: $T' = \{\}$

$\bowtie T' \cap \bowtie T$

$==$ $\langle$ Assumption $T' =$ and Definition of $\bowtie$ (1) $\rangle$

$\{\} \cap \bowtie T$

$==$ $\langle$ Annihilator of $\cap$ $\rangle$

$\{\}$

$==$ $\langle$ Assumption $T' =$ and Definition of $\bowtie$ (1) $\rangle$

$\bowtie T'$

Step 2: Induction hypothesis (**IH**): $T' \subseteq T \implies \bowtie T' \cap \bowtie T = \bowtie T'$. For any $s \in \mathcal{U}$, we show that $(T' \cup \{s\}) \subseteq T \implies \bowtie (T' \cup \{s\}) \subseteq \bowtie T$.

Case 1: $s \in T'$ is trivial, as it reduces to the **IH**.

Case 2: $s \notin T'$ and it is immediate that $s \in T$, due to assumption $(T' \cup \{s\}) \subseteq T$. The proof will use the additional fact:

$$\begin{aligned}&(\{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T' \;\wedge v \in s\}\\ &\cap \{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie (T \backslash \{s\}) \;\wedge v \in s\})\\ &= \{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T' \;\wedge v \in s\}\end{aligned} \tag{27}$$

Equation 27 is based on the assumptions $s \notin T'$ and $T' \subseteq T$, thus $T' \subseteq (T \backslash \{s\})$, and, due to **IH**, $\bowtie T' \subseteq \bowtie (T \backslash \{s\})$.

$\bowtie (T' \cup \{s\}) \cap \bowtie T$

$==$ $\langle$ Definition of $\bowtie$ (1) $\rangle$

$$
\begin{aligned}
& (\bowtie T' \cup \{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T' \wedge v \in s\}) \cap \bowtie T \\
== \quad & \langle \text{ Distributivty of } \cap \text{ over } \cup \, \rangle \\
& (\bowtie T' \cap \bowtie T\,) \cup (\{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T' \wedge v \in s\} \cap \bowtie T\,) \\
== \quad & \langle \, \mathbf{IH} \, \rangle \\
& \bowtie T' \cup (\{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T' \wedge v \in s\} \cap \bowtie T\,) \\
== \quad & \langle \text{ Assumption } s \in T \, \rangle \\
& \bowtie T' \cup (\{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T' \wedge v \in s\} \cap \bowtie ((T \backslash \{s\}) \cup \{s\})\,) \\
== \quad & \langle \text{ Definition of } \bowtie \ (1) \, \rangle \\
& \bowtie T' \cup \big(\{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T' \wedge v \in s\} \\
& \cap (\bowtie (T \backslash \{s\}) \cup \{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie (T \backslash \{s\}) \wedge v \in s\})\big) \\
== \quad & \langle \text{ Distributivty of } \cap \text{ over } \cup \, \rangle \\
& \bowtie T' \cup \big(\{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T' \wedge v \in s\} \cap \bowtie (T \backslash \{s\})\,\big) \\
& \cup \big(\{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T' \wedge v \in s\} \\
& \cap \{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie (T \backslash \{s\}) \wedge v \in s\}\big) \\
== \quad & \langle \text{ Equations (26), (27) } \rangle \\
& \bowtie T' \cup \{\} \cup \{dt \cup \{(s,v)\} \mid dt, v \,.\, dt \in \bowtie T' \wedge v \in s\} \\
== \quad & \langle \text{ Annihilator for } \cup \text{ and Definition of } \bowtie \ (1) \, \rangle \\
& \bowtie (T' \cup \{s\})
\end{aligned}
$$

Proof for Proposition (3.3), for any $s \in \mathcal{U}, T \subseteq \mathcal{U}$, with assumption $s \in T$, is immediate from $s \in T \iff \{s\} \subseteq T$ and Proposition 3.2.

Proof for Proposition (3.4) is done by induction on the first argument, $T_1 \subseteq \mathcal{U}$ and any $T_2 \subseteq \mathcal{U}$.
Step 1: $T_1 = \{\}$

$$
\begin{aligned}
& \bowtie (T_1 \cap T_2) \\
== \quad & \langle \text{ Assumption } T_1 = \{\} \, \rangle \\
& \bowtie (\{\} \cap T_2) \\
== \quad & \langle \text{ Annihilator of } \cap \, \rangle \\
& \bowtie \{\} \\
== \quad & \langle \text{ Definition of } \bowtie \ (1) \, \rangle \\
& \{\} \\
== \quad & \langle \text{ Annihilator of } \cap \, \rangle \\
& \{\} \cap \bowtie T_2 \\
== \quad & \langle \text{ Definition of } \bowtie \ (1) \, \rangle \\
& \bowtie \{\} \cap \bowtie T_2 \\
== \quad & \langle \text{ Assumption } T_1 = \{\} \, \rangle \\
& \bowtie T_1 \cap \bowtie T_2 \\
== \quad & \langle \text{ Definition of } \bowtie \ (1) \, \rangle \\
& \bowtie T_1 \cap \bowtie T_2
\end{aligned}
$$

Step 2: Induction hypothesis (**IH**): $T_1, T_2 \subseteq,\ \bowtie (T_1 \cap T_2) \ = \bowtie T_1 \cap \bowtie T_2$. We show that for any $s \in \mathcal{U},\ \bowtie \big((T_1 \cup \{s\}) \cap T_2\big) \ = \bowtie (T_1 \cup \{s\}) \cap \bowtie T_2$.

Case 1: $s \in T_1$ is trivial, as it reduces to the **IH**.

Case 2: $s \notin T_1 \wedge s \notin T_2$

LH: $\bowtie\big((T_1 \cup \{s\}) \cap T_2\big)$

$==$ $\langle$ Distributivity of $\cap$ over $\cup$ $\rangle$

$\bowtie\big((T_1 \cap T_2) \cup (\{s\} \cup T_2)\big)$

$==$ $\langle$ Assumption $s \notin T_2$ and $a \notin A \iff \{a\} \cap A = \{\}$ $\rangle$

$\bowtie\big((T_1 \cap T_2) \cup \{\}\big)$

$==$ $\langle$ Annihilator for $\cup$ $\rangle$

$\bowtie(T_1 \cap T_2)$

$==$ $\langle$ **IH** $\rangle$

$\bowtie T_1 \cap \bowtie T_2$

RH: $\bowtie(T_1 \cup \{s\}) \cap \bowtie T_2$

$==$ $\langle$ Definition of $\bowtie$ (1) $\rangle$

$\big(\bowtie T_1 \cup \big\{(dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie T_1 \wedge v \in s\big\}\big) \cap \bowtie T_2$

$==$ $\langle$ Distributivity of $\cap$ over $\cup$ $\rangle$

$(\bowtie T_1 \cap \bowtie T_2) \cup \big(\big\{(dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie T_1 \wedge v \in s\big\} \cap \bowtie T_2\big)$

$==$ $\langle$ Equation (26) $\rangle$

$(\bowtie T_1 \cap \bowtie T_2) \cup \{\}$

$==$ $\langle$ Annihilator for $\cup$ $\rangle$

$\bowtie T_1 \cap \bowtie T_2$

Case 3: $s \notin T_1 \wedge s \in T_2$ With these assumption, it is immediate that

$$T1 \cap (T_2 \backslash \{s\}) = T_1 \cap T_2 \tag{28}$$

$$\bowtie T_1 \cap \bowtie(T_2 \backslash \{s\}) = \bowtie T_1 \cap \bowtie T_2 \tag{29}$$

LH: $\bowtie\big((T_1 \cup \{s\}) \cap T_2\big)$

$==$ $\langle$ Distributivity of $\cap$ over $\cup$ $\rangle$

$\bowtie\big((T_1 \cap T_2) \cup (\{s\} \cup T_2)\big)$

$==$ $\langle$ Assumption $s \in T_2$ $\rangle$

$\bowtie\big((T_1 \cap T_2) \cup \{s\}\big)$

$==$ $\langle$ Definition of $\bowtie$ (1) $\rangle$

$\bowtie(T_1 \cap T_2) \cup \big\{(dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie(T_1 \cap T_2) \wedge v \in s\big\}$

RH: $\bowtie(T_1 \cup \{s\}) \cap \bowtie T_2$

$==$ $\langle$ Assumption $s \in T_2$ $\rangle$

$\bowtie(T_1 \cup \{s\}) \cap \bowtie\big((T_2 \backslash \{s\}) \cup \{s\}\big)$

$==$ $\langle$ Definition of $\bowtie$ (1) $\rangle$

$\big(\bowtie T_1 \cup \big\{(dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie T_1 \wedge v \in s\big\}\big) \cap$
$\big(\bowtie(T_2 \backslash \{s\}) \cup \big\{(dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie(T_2 \backslash \{s\}) \wedge v \in s\big\}\big)$

$==$ $\langle$ Distributivity of $\cap$ over $\cup$ $\rangle$

$\big(\bowtie T_1 \cap \bowtie(T_2 \backslash \{s\})\big) \cup$
$\big(\bowtie T_1 \cap \big\{(dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie(T_2 \backslash \{s\}) \wedge v \in s\big\}\big) \cup$
$\big(\big\{(dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie T_1 \wedge v \in s\big\} \cap \bowtie(T_2 \backslash \{s\})\big) \cup$
$\big(\big\{(dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie T_1 \wedge v \in s\big\} \cap$
$\big\{(dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie(T_2 \backslash \{s\}) \wedge v \in s\big\}\big)$

$==$ $\langle$ Equation (29) $\rangle$

$\big(\bowtie T_1 \cap \bowtie T_2\big) \cup$
$\big(\bowtie T_1 \cap \big\{(dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie(T_2 \backslash \{s\}) \wedge v \in s\big\}\big) \cup$
$\big(\big\{(dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie T_1 \wedge v \in s\big\} \cap \bowtie(T_2 \backslash \{s\})\big) \cup$
$\big(\big\{(dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie T_1 \wedge v \in s\big\} \cap$
$\big\{(dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie(T_2 \backslash \{s\}) \wedge v \in s\big\}\big)$

$$
\begin{aligned}
= & \quad \langle \text{ Assumption } s \notin T_1 \text{ and Equation (26) } \rangle \\
& \big(\bowtie T_1 \cap \bowtie T_2 \big) \cup \{\} \cup \\
& \Big(\big\{ (dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie T_1 \ \wedge v \in s \big\} \cap \bowtie (T_2 \backslash \{s\}) \Big) \cup \\
& \Big(\big\{ (dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie T_1 \ \wedge v \in s \big\} \cap \\
& \big\{ (dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie (T_2 \backslash \{s\}) \ \wedge v \in s \big\} \Big) \\
= & \quad \langle \ s \notin (T_2 \backslash \{s\} \text{ and Equation (26) } \rangle \\
& \big(\bowtie T_1 \cap \bowtie T_2 \big) \cup \{\} \cup \{\} \cup \\
& \Big(\big\{ (dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie T_1 \ \wedge v \in s \big\} \cap \\
& \big\{ (dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie (T_2 \backslash \{s\}) \ \wedge v \in s \big\} \Big) \\
= & \quad \langle \text{ Distributivity of } \cap \text{ over set comprehension } \rangle \\
& \big(\bowtie T_1 \cap \bowtie T_2 \big) \cup \{\} \cup \{\} \cup \big\{ (dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in (\bowtie T_1 \cap \bowtie (T_2 \backslash \{s\})) \wedge v \in s \big\} \\
= & \quad \langle \text{ Annihilator for } \cup \ \rangle \\
& \big(\bowtie T_1 \cap \bowtie T_2 \big) \cup \big\{ (dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in (\bowtie T_1 \cap \bowtie (T_2 \backslash \{s\})) \wedge v \in s \big\} \\
= & \quad \langle \text{ Equation (29) } \rangle \\
& \big(\bowtie T_1 \cap \bowtie T_2 \big) \cup \big\{ (dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in (\bowtie T_1 \cap \bowtie T_2) \wedge v \in s \big\} \\
= & \quad \langle \ \mathbf{IH} \ \rangle \\
& \bowtie (T_1 \cap T_2) \cup \big\{ (dt \cup \{(s,v)\}) \mid dt, v \,.\, dt \in \bowtie (T_1 \cap T_2) \ \wedge v \in s \big\}
\end{aligned}
$$

Thus, **LH** $=$ **RH** for all cases of the induction hypothesis step.

Proof for Proposition (3.5), for any $\kappa, \lambda \in$ and $v \in \kappa$. Case 1: $\kappa \neq \lambda$:

$$
\begin{aligned}
& \{(\kappa, v)\}\big\downarrow_{\lambda} \\
= & \quad \langle \text{ Definition of } \Downarrow (3) \ \rangle \\
& \{(s,v) \mid s \,.\, (s,v) \in \{(\kappa,v)\} \wedge s \neq \lambda\} \\
= & \quad \langle \ x \in \{y\} \iff x = y, \text{ Variable replacing, } s = \kappa \ \rangle \\
& \{(\kappa,v) \mid \kappa \neq \lambda\} \\
= & \quad \langle \text{ Assumption } \kappa \neq \lambda \ \rangle \\
& \{(\kappa,v)\}
\end{aligned}
$$

Proof for Proposition (3.5), for any any $\kappa, \lambda \in$ and $v \in \kappa$. Case 2: $\kappa = \lambda$:

$$
\begin{aligned}
& \{(\kappa, v)\}\big\downarrow_{\lambda} \\
= & \quad \langle \text{ Definition of } \Downarrow (3) \ \rangle \\
& \{(s,v) \mid s \,.\, (s,v) \in \{(\kappa,v)\} \wedge s \neq \lambda\} \\
= & \quad \langle \ x \in \{y\} \iff x = y, \text{ Variable replacing, } s = \kappa \ \rangle \\
& \{(\kappa,v) \mid \kappa \neq \lambda\} \\
= & \quad \langle \text{ Assumption } \kappa = \lambda, \text{ Contradiction, } P \wedge \neg P = False \ \rangle \\
& \{(\kappa,v) \mid False\} \\
= & \quad \langle \ \{x \mid False\} = \{\} \ \rangle \\
& \{\}
\end{aligned}
$$

Proof for Proposition (3.6), for any $\kappa \in \mathcal{U}$, $a \in A$, and $\forall dt \in a.\ \kappa \,\overline{\in}\, dt$:

$$
\begin{aligned}
& a\Downarrow_{\kappa} \cap\, a\Uparrow^{\kappa} \\
= & \quad \langle \text{ Definition of } \Downarrow (4) \ \rangle \\
& \{dt\big\downarrow_{\kappa} \mid dt \,.\, dt \in a\} \cap a\Uparrow^{\kappa}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \langle \text{ Definition of } \Uparrow (6), \text{ with } \kappa \in \mathcal{U},\ a \in A, \text{ and } \forall dt \in a.\ \kappa \,\overline{\in}\, dt \ \rangle \\
& \{dt{\downarrow}_{\kappa} \ \mid \ dt \ .\ dt \in a\} \cap \big\{\big(dt{\downarrow}_{\kappa} \cup \{(\kappa, v)\}\big) \ \mid \ dt, v \ .\ dt \in a \wedge v \in \kappa\big\} \\
= \quad & \langle \text{ Variable renaming, both sets } \rangle \\
& \{x \ \mid \ x, dt \ .\ x = dt{\downarrow}_{\kappa} \wedge dt \in a\} \cap \\
& \{x \ \mid \ x, dt, v \ .\ x = \big(dt{\downarrow}_{\kappa} \cup \{(\kappa, v)\}\big) \wedge dt \in a \wedge v \in \kappa\} \\
= \quad & \langle\ \{E \ \mid \ R_1 \wedge R_2\} \iff \{E \ \mid \ R_1\} \cap \{E \ \mid \ R_2\} \ \rangle \\
& \{x \ \mid \ x, dt, v \ .\ x = dt{\downarrow}_{\kappa} \wedge x = \big(dt{\downarrow}_{\kappa} \cup \{(\kappa, v)\}\big) \wedge dt \in a \wedge v \in \kappa\} \\
= \quad & \langle \text{ Contradiction: by Definition (3), } x = dt{\downarrow}_{\kappa} \text{ contains no } \kappa \text{ s-values } \rangle \\
& \{x \ \mid \ False\} \\
= \quad & \langle \text{ Set Comprehension } \rangle \\
& \{\}
\end{aligned}
$$

Proof for Proposition (3.7), Equation (1), for any $\kappa \in \mathcal{U}$ and any $dt_1, dt_2 \in SD^{\mathcal{M}}$:

$$
\begin{aligned}
& (dt_1 \cup dt_2){\downarrow}_{\kappa} \\
= \quad & \langle \text{ Definition of } \downarrow (3) \ \rangle \\
& \{(s, v) \ \mid \ s, v \ .\ (s, v) \in (dt_1 \cup dt_2) \wedge s \neq \kappa\} \\
= \quad & \langle \text{ Axiom, Union, } x \in X_1 \cup X_2 \iff x \in X_1 \vee x \in X_2 \ \rangle \\
& \{(s, v) \ \mid \ s, v \ .\ \big((s, v) \in dt_1 \vee (s, v) \in dt_2\big) \wedge s \neq \kappa\} \\
= \quad & \langle \text{ Distributivity of } \wedge \text{ over } \vee \ \rangle \\
& \{(s, v) \ \mid \ s, v \ .\ \big((s, v) \in dt_1 \wedge s \neq \kappa\big) \vee \big((s, v) \in dt_2 \wedge s \neq \kappa\big)\} \\
= \quad & \langle\ \{E \ \mid \ R_1 \vee R_2\} \iff \{E \ \mid \ R_1\} \cup \{E \ \mid \ R_2\} \ \rangle \\
& \{(s, v) \ \mid \ s, v \ .\ (s, v) \in dt_1 \wedge s \neq \kappa\} \cup \{(s, v) \ \mid \ s, v \ .\ (s, v) \in dt_2 \wedge s \neq \kappa\} \\
= \quad & \langle \text{ Definition of } \downarrow (3), \text{ twice } \rangle \\
& dt_1{\downarrow}_{\kappa} \cup dt_2{\downarrow}_{\kappa}
\end{aligned}
$$

Proof for Proposition (3.7), Equation (2), for any $\kappa \in \mathcal{U}$ and any $dt_1, dt_2 \in SD^{\mathcal{M}}$:

$$
\begin{aligned}
& (dt_1 \cap dt_2){\downarrow}_{\kappa} \\
= \quad & \langle \text{ Definition of } \downarrow (3) \ \rangle \\
& \{(s, v) \ \mid \ s, v \ .\ (s, v) \in (dt_1 \cap dt_2) \wedge s \neq \kappa\} \\
= \quad & \langle \text{ Axiom, Intersection, } x \in X_1 \cap X_2 \iff x \in X_1 \wedge x \in X_2 \ \rangle \\
& \{(s, v) \ \mid \ s, v \ .\ \big((s, v) \in dt_1 \wedge (s, v) \in dt_2\big) \wedge s \neq \kappa\} \\
= \quad & \langle \text{ Commutativity and Idempotency of } \wedge \text{ (for } s \neq \kappa) \ \rangle \\
& \{(s, v) \ \mid \ s, v \ .\ \big((s, v) \in dt_1 \wedge s \neq \kappa\big) \wedge \big((s, v) \in dt_2 \wedge s \neq \kappa\big)\} \\
= \quad & \langle\ \{E \ \mid \ R_1 \wedge R_2\} \iff \{E \ \mid \ R_1\} \cap \{E \ \mid \ R_2\} \ \rangle \\
& \{(s, v) \ \mid \ s, v \ .\ (s, v) \in dt_1 \wedge s \neq \kappa\} \cap \{s, v \ \mid \ (s, v) \ .\ (s, v) \in dt_2 \wedge s \neq \kappa\} \\
= \quad & \langle \text{ Definition of } \downarrow (3), \text{ twice } \rangle \\
& dt_1{\downarrow}_{\kappa} \cap dt_2{\downarrow}_{\kappa}
\end{aligned}
$$

Proof for Proposition (3.7), Equation (3), for any $\kappa \in \mathcal{U}$ and any $a, b \in A$:

$$
\begin{aligned}
& (a \cup b){\Downarrow}_{\kappa} \\
= \quad & \langle \text{ Definition of } \Downarrow (4) \ \rangle \\
& \{dt{\downarrow}_{\kappa} \ \mid \ dt \ .\ dt \in (a \cup b)\} \\
= \quad & \langle \text{ Axiom, Union, } x \in X_1 \vee x \in X_2 \iff x \in X_1 \cup X_2 \ \rangle
\end{aligned}
$$

$$
\begin{array}{ll}
& \{dt{\downharpoonright}_{\kappa} \ \mid \ dt \,.\, dt \in a \vee dt \in b)\} \\
= & \qquad \langle\ \{E \ \mid \ R_1 \vee R_2\} \iff \{E \ \mid \ R_1\} \cup \{E \ \mid \ R_2\}\ \rangle \\
& \{dt{\downharpoonright}_{\kappa} \ \mid \ dt \,.\, dt \in a\} \cup \{dt{\downharpoonright}_{\kappa} \ \mid \ dt \,.\, dt \in b\} \\
= & \qquad \langle\ \text{Definition of} \Downarrow (4)\text{, twice}\ \rangle \\
& a{\Downarrow}_{\kappa} \cup b{\Downarrow}_{\kappa}
\end{array}
$$

Proof for Proposition (3.7), Equation (4), for any $\kappa \in \mathcal{U}$ and any $a, b \in A$ is done by showing that $(a \cap b){\Downarrow_\kappa} \cap (a{\Downarrow_\kappa} \cap b{\Downarrow_\kappa}) = (a \cap b){\Downarrow_\kappa}$

$$
\begin{array}{ll}
& (a \cap b){\Downarrow_\kappa} \cap (a{\Downarrow_\kappa} \cap b{\Downarrow_\kappa}) \\
= & \quad \langle\ A = A \cup (A \cap B)\ \rangle \\
& (a \cap b){\Downarrow_\kappa} \cap \big((a \cup (a \cap b)){\Downarrow_\kappa} \cap (b \cup (a \cap b))){\Downarrow_\kappa}\big) \\
= & \quad \langle\ \text{Associativity of } \cap\ \rangle \\
& (a \cap b){\Downarrow_\kappa} \cap (a \cup (a \cap b)){\Downarrow_\kappa} \cap (b \cup (a \cap b))){\Downarrow_\kappa} \\
= & \quad \langle\ \text{Proposition (3.7), Equation (3)}\ \rangle \\
& (a \cap b){\Downarrow_\kappa} \cap (a{\Downarrow_\kappa} \cup (a \cap b){\Downarrow_\kappa}) \cap (b{\Downarrow_\kappa} \cup (a \cap b){\Downarrow_\kappa}) \\
= & \quad \langle\ A \cap (A \cup B) = A, \text{ twice}\ \rangle \\
& (a \cap b){\Downarrow_\kappa}
\end{array}
$$

Thus, $(a \cap b){\Downarrow_\kappa} \subseteq (a{\Downarrow_\kappa} \cap b{\Downarrow_\kappa})$.

Proof for Proposition (3.7), Equation (5), for any $\kappa \in \mathcal{U}$ and any $a, b \in A$:

$$
\begin{array}{ll}
& (a \cup b){\Uparrow^\kappa} \\
= & \quad \langle\ \text{Definition of} \Uparrow (6)\ \rangle \\
& \bigcup\{dt{\uparrow^\kappa} \mid dt \,.\, dt \in (a \cup b)\} \\
= & \quad \langle\ \text{Axiom, Union, } x \in X_1 \vee x \in X_2 \iff x \in X_1 \cup X_2\ \rangle \\
& \bigcup\{dt{\uparrow^\kappa} \mid dt \,.\, dt \in a \vee dt \in b\} \\
= & \quad \langle\ \{E \mid R_1 \vee R_2\} \iff \{E \mid R_1\} \cup \{E \mid R_2\}\ \rangle \\
& \bigcup\big(\{dt{\uparrow^\kappa} \mid dt \,.\, dt \in a\} \cup \{dt{\uparrow^\kappa} \mid dt \,.\, dt \in b\}\big) \\
= & \quad \langle\ \text{Union composition}\ \rangle \\
& \bigcup\{dt{\uparrow^\kappa} \mid dt \,.\, dt \in a\} \cup \bigcup\{dt{\uparrow^\kappa} \mid dt \,.\, dt \in b\} \\
= & \quad \langle\ \text{Definition of} \Uparrow (6), \text{ twice}\ \rangle \\
& a{\Uparrow^\kappa} \cup b{\Uparrow^\kappa}
\end{array}
$$

Proof for Proposition (3.7), Equation (6), for any $\kappa \in \mathcal{U}$ and any $a, b \in A$ is done by showing that $(a \cap b){\Uparrow^\kappa} \cap (a{\Uparrow^\kappa} \cap b{\Uparrow^\kappa}) = (a \cap b){\Uparrow^\kappa}$

$$
\begin{array}{ll}
& (a \cap b){\Uparrow^\kappa} \cap (a{\Uparrow^\kappa} \cap b{\Uparrow^\kappa}) \\
= & \quad \langle\ A = A \cup (A \cap B)\ \rangle \\
& (a \cap b){\Uparrow^\kappa} \cap ((a \cup (a \cap b)){\Uparrow^\kappa} \cap (b \cup (a \cap b)){\Uparrow^\kappa}) \\
= & \quad \langle\ \text{Associativity of } \cap\ \rangle \\
& (a \cap b){\Uparrow^\kappa} \cap ((a \cup (a \cap b)){\Uparrow^\kappa}) \cap ((b \cup (a \cap b)){\Uparrow^\kappa}) \\
= & \quad \langle\ \text{Proposition(3.7), Equation (5)}\ \rangle \\
& (a \cap b){\Uparrow^\kappa} \cap (a{\Uparrow^\kappa} \cup (a \cap b){\Uparrow^\kappa}) \cap (b{\Uparrow^\kappa} \cup (a \cap b){\Uparrow^\kappa}) \\
= & \quad \langle\ A \cap (A \cup B) = A, \text{ twice}\ \rangle \\
& (a \cap b){\Uparrow^\kappa}
\end{array}
$$

Thus, $(a \cap b){\Uparrow^\kappa} \subseteq a{\Uparrow^\kappa} \cap b{\Uparrow^\kappa}$.

Proof for Proposition (3.8), Equation (1), for any $\kappa \in \mathcal{U}$ and any $dt \in SD^{\mathcal{M}}$:

$$
\begin{array}{ll}
& (dt{\downarrow_\kappa}){\downarrow_\kappa} \\
= & \quad \langle\ \text{Definition of} \downarrow (3)\ \rangle \\
& \{(s, v) \mid s, v \,.\, (s, v) \in dt{\downarrow_\kappa} \wedge s \neq \kappa \wedge v \in s\}
\end{array}
$$

$=$ ⟨ Definition of $\downarrow$ (3) ⟩
$\{(s,v) \mid s,v \,.\, [(s,v) \in \{(s',v') \mid s',v' \,.\, (s',v') \in dt \land s' \neq \kappa \land v' \in s'\} \land$
$s \neq \kappa \land v \in s]\}$
$=$ ⟨ $x \in \{x \mid x \,.\, R\} \iff R$ ⟩
$\{(s,v) \mid s,v \,.\, (s,v) \in dt \land s \neq \kappa \land s \neq \kappa\}$
$=$ ⟨ Idempotency of $\land$ ⟩
$\{(s,v) \mid s,v \,.\, (s,v) \in dt \land s \neq \kappa\}$
$=$ ⟨ Definition of $\downarrow$ (3) ⟩
$dt{\downarrow}_{\kappa}$

Proof for Proposition (3.8), Equation (2), for any $\kappa \in \mathcal{U}$ and $dt \in SD$
Case 1: $\kappa \mathrel{\overline{\notin}} dt$

$(dt{\uparrow}^{\kappa}){\Uparrow}^{\kappa}$
$=$ ⟨ Definition of $\Uparrow$ (6) ⟩
$\bigcup\{dt'{\uparrow}^{\kappa} \mid dt' \,.\, dt' \in dt{\uparrow}^{\kappa}\}$
$=$ ⟨ Definition of $\uparrow$ (5) ⟩
$\bigcup\{dt'{\uparrow}^{\kappa} \mid dt' \,.\, dt' \in \{dt\}\}$
$=$ ⟨ Set comprehension $\{Ex \mid x \,.\, x \in \{y\}\} = \{Ey\}$ ⟩
$\bigcup\{dt{\uparrow}^{\kappa}\}$
$=$ ⟨ Union axiom $\bigcup\{A\} = A$ ⟩
$dt{\uparrow}^{\kappa}$

Case 2: $\kappa \mathrel{\overline{\in}} dt$

$(dt{\uparrow}^{\kappa}){\Uparrow}^{\kappa}$
$=$ ⟨ Definition of $\Uparrow$ (6) ⟩
$\bigcup\{dt'{\uparrow}^{\kappa} \mid dt' \,.\, dt' \in dt{\uparrow}^{\kappa}\}$
$=$ ⟨ Definition of $\uparrow$ (5) ⟩
$\bigcup\{dt'{\uparrow}^{\kappa} \mid dt' \,.\, dt' \in \{(dt{\downarrow}_{\kappa} \cup \{(\kappa,v)\}) \mid v \,.\, v \in \kappa\}\}$
$=$ ⟨ Variable replacing: $x \in \{y\} \iff x = y$ ⟩
$\bigcup\{(dt{\downarrow}_{\kappa} \cup \{(\kappa,v)\}){\uparrow}^{\kappa} \mid v \,.\, v \in \kappa\}$
$=$ ⟨ Definition of $\uparrow$ (5) ⟩
$\bigcup\{\{(dt{\downarrow}_{\kappa} \cup \{(\kappa,v)\}){\downarrow}_{\kappa} \cup \{(\kappa,v')\} \mid v' \,.\, v' \in \kappa\} \mid v \,.\, v \in \kappa\}$
$=$ ⟨ Definition of $\downarrow$ (3) ⟩
$\bigcup\{\{(dt{\downarrow}_{\kappa}){\downarrow}_{\kappa} \cup \{(\kappa,v')\} \mid v' \,.\, v' \in \kappa\} \mid v \,.\, v \in \kappa\}$
$=$ ⟨ Set comprehension, unbound variable $v \in \kappa$ and $\kappa \in \mathcal{U} \implies \kappa \neq \{\}$ ⟩
$\bigcup\{\{(dt{\downarrow}_{\kappa}){\downarrow}_{\kappa} \cup \{(\kappa,v')\} \mid v' \,.\, v' \in \kappa\}\}$
$=$ ⟨ Proposition (3.8), Equation (1) ⟩
$\bigcup\{\{dt{\downarrow}_{\kappa} \cup \{(\kappa,v')\} \mid v' \,.\, v' \in \kappa\}\}$
$=$ ⟨ Definition of $\uparrow$ (5) ⟩
$\bigcup\{dt{\uparrow}^{\kappa}\}$
$=$ ⟨ Union axiom $\bigcup\{A\} = A$ ⟩
$dt{\uparrow}^{\kappa}$

Proof for Proposition (3.8), Equation (3), for any $\kappa \in \mathcal{U}$ and any $a \in A$:

$$
\begin{aligned}
& (a{\Downarrow_\kappa}){\Downarrow_\kappa} \\
= \quad & \langle \text{ Definition of } \Downarrow (4) \ \rangle \\
& \{dt{\downarrow_\kappa} \ \mid \ dt \ . \ dt \in a{\Downarrow_\kappa}\} \\
= \quad & \langle \text{ Definition of } \Downarrow (4) \ \rangle \\
& \{dt{\downarrow_\kappa} \ \mid \ dt \ . \ dt \in \{dt'{\downarrow_\kappa} \ \mid \ dt' \ . \ dt' \in a\}\} \\
= \quad & \langle \ x \in \{x \ \mid \ x \ . \ R\} \iff R \ \rangle \\
& \{dt{\downarrow_\kappa} \ \mid \ dt \ . \ dt = dt'{\downarrow_\kappa} \wedge dt' \in a\} \\
= \quad & \langle \text{ Variable replacing, } dt = dt'{\downarrow_\kappa} \ \rangle \\
& \{(dt'{\downarrow_\kappa}){\downarrow_\kappa} \ \mid \ dt' \ . \ dt' \in a\} \\
= \quad & \langle \text{ Idempotency of } \downarrow (3.9), (1) \ \rangle \\
& \{dt'{\downarrow_\kappa} \ \mid \ dt' \ . \ dt' \in a\} \\
= \quad & \langle \text{ Definition of } \Downarrow (4) \ \rangle \\
& a{\Downarrow_\kappa}
\end{aligned}
$$

Proof for Proposition (3.8), Equation (4), for any $\kappa \in \mathcal{U}$ and any $a \in A$:

$$
\begin{aligned}
& (a{\Uparrow^\kappa}){\Uparrow^\kappa} \\
= \quad & \langle \text{ Definition of } \Uparrow (6) \ \rangle \\
& \big(\bigcup\{dt{\uparrow^\kappa} \ \mid \ dt \ . \ dt \in a\}\big){\Uparrow^\kappa} \\
= \quad & \langle \text{ Proposition (3.7), Equation (5) } \rangle \\
& \bigcup\big(\{dt{\uparrow^\kappa} \ \mid \ dt \ . \ dt \in a\}\big){\Uparrow^\kappa} \\
= \quad & \langle \text{ Applying a function on a set } \rangle \\
& \bigcup\{(dt{\uparrow^\kappa}){\Uparrow^\kappa} \ \mid \ dt \ . \ dt \in a\} \\
= \quad & \langle \text{ Proposition (3.8), Equation (2) } \rangle \\
& \bigcup\{dt{\uparrow^\kappa} \ \mid \ dt \ . \ dt \in a\} \\
= \quad & \langle \text{ Definition of } \Uparrow (6) \ \rangle \\
& a{\Uparrow^\kappa}
\end{aligned}
$$

Proof for Proposition (3.9), Equation (1), for any $\kappa, \lambda \in \mathcal{U}$ and $dt \in SD^{\mathcal{M}}$:

$$
\begin{aligned}
& (dt{\downarrow_\kappa}){\downarrow_\lambda} \\
= \quad & \langle \text{ Definition of } \downarrow (3), \text{ applied to } \lambda \ \rangle \\
& \{(s, v) \ \mid \ s, v \ . \ (s, v) \in dt{\downarrow_\kappa} \wedge s \neq \lambda\} \\
= \quad & \langle \text{ Definition of } \downarrow (3), \text{ applied to } \kappa \ \rangle \\
& \{(s, v) \ \mid \ s, v \ . \ (s, v) \in \{(s', v') \ \mid \ s', v' \ . \ (s', v') \in dt \wedge s' \neq \kappa\} \wedge s \neq \lambda\} \\
= \quad & \langle \ x \in \{x \ \mid \ x \ . \ R\} \iff R \ \rangle \\
& \{(s, v) \ \mid \ s, v \ . \ (s, v) \in dt \wedge s \neq \kappa \wedge s \neq \lambda\} \\
= \quad & \langle \text{ Commutativity of } \wedge \ \rangle \\
& \{(s, v) \ \mid \ s, v \ . \ (s, v) \in dt \wedge s \neq \lambda \wedge s \neq \kappa\} \\
= \quad & \langle \ x \in \{x \ \mid \ x \ . \ R\} \iff R \ \rangle \\
& \{(s, v) \ \mid \ s, v \ . \ (s, v) \in \{(s', v') \ \mid \ s', v' \ . \ (s', v') \in dt \wedge s' \neq \lambda\} \wedge s \neq \kappa\} \\
= \quad & \langle \text{ Definition of } \downarrow (3), \text{ applied to } \lambda \ \rangle \\
& \{(s, v) \ \mid \ s, v \ . \ (s, v) \in dt{\downarrow_\lambda} \wedge s \neq \kappa\} \\
= \quad & \langle \text{ Definition of } \downarrow (3), \text{ applied to } \kappa \ \rangle \\
& (dt{\downarrow_\lambda}){\downarrow_\kappa}
\end{aligned}
$$

Proof for Proposition (3.9), Equation (2), for any $\kappa, \lambda \in \mathcal{U}$ and $a \in A$:

$$
\begin{array}{ll}
& (a{\Downarrow_\kappa}){\Downarrow_\lambda} \\
= & \quad \langle \text{ Definition of } \Downarrow \text{ (4), applied to } \lambda \ \rangle \\
& \{dt{\downarrow_\lambda} \ | \ dt \,.\, dt \in a{\Downarrow_\kappa}\} \\
= & \quad \langle \text{ Definition of } \Downarrow \text{ (4), applied to } \kappa \ \rangle \\
& \{dt{\downarrow_\lambda} \ | \ dt \,.\, dt \in \{dt'{\downarrow_\kappa} \ | \ dt' \,.\, dt' \in a\}\} \\
= & \quad \langle \ x \in \{x \ | \ x \,.\, R\} \iff R, \text{ replace variable } dt \text{ with } dt'{\downarrow_\kappa} \ \rangle \\
& \{(dt'{\downarrow_\kappa}){\downarrow_\lambda} \ | \ dt' \,.\, dt' \in a\} \\
= & \quad \langle \text{ Associativity of } \downarrow \text{ (3.9), (1) } \rangle \\
& \{(dt'{\downarrow_\lambda}){\downarrow_\kappa} \ | \ dt' \,.\, dt' \in a\} \\
= & \quad \langle \text{ Variable renaming, } dt'{\downarrow_\lambda} = dt \ \rangle \\
& \{dt{\downarrow_\kappa} \ | \ dt \,.\, dt = dt'{\downarrow_\lambda} \wedge dt' \in a\} \\
= & \quad \langle \ x \in \{x \ | \ x \,.\, R\} \iff R \ \rangle \\
& \{dt{\downarrow_\kappa} \ | \ dt \,.\, dt \in \{dt'{\downarrow_\lambda} \ | \ dt' \,.\, dt' \in a\}\} \\
= & \quad \langle \text{ Definition of } \Downarrow \text{ (4), applied to } \lambda \ \rangle \\
& \{dt{\downarrow_\kappa} \ | \ dt \,.\, dt \in a{\Downarrow_\lambda}\} \\
= & \quad \langle \text{ Definition of } \Downarrow \text{ (4), applied to } \kappa \ \rangle \\
& (a{\Downarrow_\lambda}){\Downarrow_\kappa}
\end{array}
$$

Proof for Proposition (3.9), Equation (3), for $\kappa, \lambda \in \mathcal{U}, \kappa \neq \lambda$ and $a \in A$.
Case 1: $\kappa \neq \lambda$:

$$
\begin{aligned}
& (a{\Uparrow}^{\kappa}){\Uparrow}^{\lambda} \\
= \quad & \langle \text{ Definition of} \Uparrow (6), \text{ applied to } \lambda \ \rangle \\
& \big\{ \big(dt{\downarrow}_{\lambda} \cup \{(\lambda, v)\}\big) \ \mid \ dt, v \ . \ dt \in a{\Uparrow}^{\kappa} \wedge v \in \lambda \big\} \\
= \quad & \langle \text{ Definition of} \Uparrow (6), \text{ applied to } \kappa \ \rangle \\
& \big\{ \big(dt{\downarrow}_{\lambda} \cup \{(\lambda, v)\}\big) \ \mid \ dt, v \ . \ dt \in \big\{ \big(dt'{\downarrow}_{\kappa} \cup \{(\kappa, v')\}\big) \ \mid \ dt', v' \ . \ dt' \in a \wedge v' \in \kappa \big\} \wedge v \in \lambda \big\} \\
= \quad & \langle \ x \in \{x \ \mid \ x \ . \ R\} \iff R \ \rangle \\
& \big\{ \big(dt{\downarrow}_{\lambda} \cup \{(\lambda, v)\}\big) \ \mid \ dt, v, dt', v' \ . \ dt = \big(dt'{\downarrow}_{\kappa} \cup \{(\kappa, v')\}\big) \wedge dt' \in a \wedge v' \in \kappa \wedge v \in \lambda \big\} \\
= \quad & \langle \text{ Variable replacing, } dt = \big(dt'{\downarrow}_{\kappa} \cup \{(\kappa, v')\}\big) \ \rangle \\
& \big\{ \big(\big(dt'{\downarrow}_{\kappa} \cup \{(\kappa, v')\}\big){\downarrow}_{\lambda} \cup \{(\lambda, v)\}\big) \ \mid \ v, dt', v' \ . \ dt' \in a \wedge v' \in \kappa \wedge v \in \lambda \big\} \\
= \quad & \langle \downarrow \text{ preserves } \cup \ (3.7), (1) \ \rangle \\
& \big\{ \big[(dt'{\downarrow}_{\kappa}){\downarrow}_{\lambda} \cup \{(\kappa, v')\}{\downarrow}_{\lambda} \cup \{(\lambda, v)\}\big] \ \mid \ dt', v, v' \ . \ dt' \in a \wedge v' \in \kappa \wedge v \in \lambda \big\} \\
= \quad & \langle \text{ Assumption } \kappa \neq \lambda, \text{ Proposition (3.5) } \rangle \\
& \big\{ \big[(dt'{\downarrow}_{\kappa}){\downarrow}_{\lambda} \cup \{(\kappa, v')\} \cup \{(\lambda, v)\}\big] \ \mid \ dt', v, v' \ . \ dt' \in a \wedge v' \in \kappa \wedge v \in \lambda \big\} \\
= \quad & \langle \text{ Associativity of} \downarrow (3.9), 1 \ \rangle \\
& \big\{ \big[(dt'{\downarrow}_{\lambda}){\downarrow}_{\kappa} \cup \{(\kappa, v')\} \cup \{(\lambda, v)\}\big] \ \mid \ dt', v, v' \ . \ dt' \in a \wedge v' \in \kappa \wedge v \in \lambda \big\} \\
= \quad & \langle \text{ Commutativity of } \cup \ \rangle \\
& \big\{ \big[(dt'{\downarrow}_{\lambda}){\downarrow}_{\kappa} \cup \{(\lambda, v)\} \cup \{(\kappa, v')\}\big] \ \mid \ dt', v, v' \ . \ dt' \in a \wedge v' \in \kappa \wedge v \in \lambda \big\} \\
= \quad & \langle \text{ Assumption } \kappa \neq \lambda, \text{ Proposition (3.5) } \rangle \\
& \big\{ \big[(dt'{\downarrow}_{\lambda}){\downarrow}_{\kappa} \cup \{(\lambda, v)\}{\downarrow}_{\kappa} \cup \{(\kappa, v')\}\big] \ \mid \ dt', v, v' \ . \ dt' \in a \wedge v' \in \kappa \wedge v \in \lambda \big\} \\
= \quad & \langle \downarrow \text{ preserves } \cup \ (3.7), (1) \ \rangle \\
& \big\{ \big[\big(dt'{\downarrow}_{\lambda} \cup \{(\lambda, v)\}\big){\downarrow}_{\kappa} \cup \{(\kappa, v')\}\big] \ \mid \ dt', v, v' \ . \ dt' \in a \wedge v' \in \kappa \wedge v \in \lambda \big\} \\
= \quad & \langle \text{ Variable replacing, } dt = \big(dt'{\downarrow}_{\lambda} \cup \{(\lambda, v)\}\big) \ \rangle \\
& \big\{ \big(dt{\downarrow}_{\kappa} \cup \{(\kappa, v')\}\big) \ \mid \ dt, dt', v, v' \ . \ dt = \big(dt'{\downarrow}_{\lambda} \cup \{(\lambda, v)\}\big) \wedge dt' \in a \wedge v' \in \kappa \wedge v \in \lambda \big\} \\
= \quad & \langle \text{ Commutativity of } \wedge \ \rangle \\
& \big\{ \big(dt{\downarrow}_{\kappa} \cup \{(\kappa, v')\}\big) \ \mid \ dt, dt', v, v' \ . \ dt = \big(dt'{\downarrow}_{\lambda} \cup \{(\lambda, v)\}\big) \wedge dt' \in a \wedge v \in \lambda \wedge v' \in \kappa \big\} \\
= \quad & \langle \ x \in \{x \ \mid \ x \ . \ R\} \iff R \ \rangle \\
& \big\{ \big(dt{\downarrow}_{\kappa} \cup \{(\kappa, v')\}\big) \ \mid \ dt, v' \ . \ dt \in \big\{ \big(dt'{\downarrow}_{\lambda} \cup \{(\lambda, v)\}\big) \ \mid \ dt', v \ . \ dt' \in a \wedge v \in \lambda \big\} \wedge v' \in \kappa \big\} \\
= \quad & \langle \text{ Definition of} \Uparrow (6), \text{ applied to } \lambda \ \rangle \\
& \big\{ \big(dt{\downarrow}_{\kappa} \cup \{(\kappa, v')\}\big) \ \mid \ dt, v' \ . \ dt \in a{\Uparrow}^{\lambda} \wedge v' \in \kappa \big\} \\
= \quad & \langle \text{ Definition of} \Uparrow (6), \text{ applied to } \kappa \ \rangle \\
& (a{\Uparrow}^{\lambda}){\Uparrow}^{\kappa}
\end{aligned}
$$

Proof for Proposition (3.9), Equation (3). Case 2, for $\kappa = \lambda$ is immediate, as both sides of the equation reduce to $a{\Uparrow}^{\kappa}$, due to associativity of $\Uparrow$ operator.

Proof for Proposition (3.10), Equation (1) , for any $\kappa \in \mathcal{U}$ and $a \in A$:

$$\begin{aligned}
& \lfloor a \rfloor_\kappa \cup \lceil a \rceil^\kappa \\
== & \qquad \langle \text{ Definitions for } \bigsqcup \text{ and } \bigsqcap \text{ operators (8), (9) } \rangle \\
& \{dt \mid dt \in a \wedge \neg\exists v \in \kappa.\ (\kappa, v) \in dt\} \cup \{dt \mid dt \in a \wedge \exists v \in \kappa.\ (\kappa, v) \in dt\} \\
== & \qquad \langle\ \{E \mid R_1 \vee R_2\} \iff \{E \mid R_1\} \cup \{E \mid R_2\}\ \rangle \\
& \{dt \mid dt \in a \wedge \neg\exists v \in \kappa.\ (\kappa, v) \in dt \vee dt \in a \wedge \exists v \in \kappa.\ (\kappa, v) \in dt\} \\
== & \qquad \langle \text{ Distributivity of } \cap \text{ over } \cup \rangle \\
& \{dt \mid dt \in a \wedge (\neg\exists v \in \kappa.\ (\kappa, v) \in dt \vee \exists v \in \kappa.\ (\kappa, v) \in dt)\} \\
== & \qquad \langle \text{ Law of excluded middle, } P \vee \neg P \iff \mathit{True}\ \rangle \\
& \{dt \mid dt \in a \vee \mathit{True}\} \\
== & \qquad \langle\ P \vee \mathit{True} \iff P\ \rangle \\
& \{dt \mid dt \in a\} \\
== & \qquad \langle \text{ Extensionality of sets } \rangle \\
& a
\end{aligned}$$

Proof for Proposition (3.10), Equation (2), for any $\kappa \in \mathcal{U}$ and $a \in A$:

$$\begin{aligned}
& \lfloor a \rfloor_\kappa \cap \lceil a \rceil^\kappa \\
== & \qquad \langle \text{ Definitions for } \bigsqcup \text{ and } \bigsqcap \text{ operators (8), (9) } \rangle \\
& \{dt \mid dt \in a \wedge \neg\exists v \in \kappa.\ (\kappa, v) \in dt\} \cap \{dt \mid dt \in a \wedge \exists v \in \kappa.\ (\kappa, v) \in dt\} \\
== & \qquad \langle\ \{E \mid R_1 \wedge R_2\} \iff \{E \mid R_1\} \cap \{E \mid R_2\}\ \rangle \\
& \{dt \mid dt \in a \wedge \neg\exists v \in \kappa.\ (\kappa, v) \in dt \wedge dt \in a \wedge \exists v \in \kappa.\ (\kappa, v) \in dt\} \\
== & \qquad \langle \text{ Contradiction, } P \wedge \neg P \iff \mathit{False}\ \rangle \\
& \{dt \mid dt \in a \wedge \mathit{False}\} \\
== & \qquad \langle \text{ Zero of } \wedge \rangle \\
& \{dt \mid \mathit{False}\} \\
== & \qquad \langle\ \{x \mid \mathit{False}\} = \{\}\ \rangle \\
& \{\}
\end{aligned}$$

Proof for Proposition (3.11), we use a few helper equations. Based on Definitions (6) and (5), we have

$a{\Uparrow}^\kappa = \{dt \mid dt \in a \wedge \neg(\kappa \mathrel{\overline{\in}} dt)\} \cup$
$\big\{\big(dt{\downharpoonright}_\kappa \cup \{(\kappa, v)\}\big) \mid dt, v \ .\ dt \in a \wedge \kappa \mathrel{\overline{\in}} dt \wedge v \in \kappa\big\}$

It is immediate that we can re-write the following:

$\big\{\big(dt{\downharpoonright}_\kappa \cup \{(\kappa, v)\}\big) \mid dt, v \ .\ dt \in a \wedge \kappa \mathrel{\overline{\in}} dt \wedge v \in \kappa\big\} = \lceil a \rceil^\kappa \cup X$, where

$X = \big\{\big(dt{\downharpoonright}_\kappa \cup \{(\kappa, v)\}\big) \mid dt, v \ .\ dt \in a \wedge \kappa \mathrel{\overline{\in}} dt \wedge v \in \kappa \wedge \big(dt{\downharpoonright}_\kappa \cup \{(\kappa, v)\}\big) \notin a\big\}$. Thus, we have the following helper equations:

$$a{\Uparrow}^\kappa = \lfloor a \rfloor_\kappa \cup \lceil a \rceil^\kappa \cup X \tag{30}$$

$$X \cap \lfloor a \rfloor_\kappa = \{\} \tag{31}$$

$$X \cap \lceil a \rceil^\kappa = \{\} \tag{32}$$

Proof for Proposition (3.11), Equation (1), for any any $\kappa \in \mathcal{U}$ and $a \in A$:

$$\lfloor a \rfloor_\kappa \cap a{\Uparrow}^\kappa$$
$$= \qquad \langle \text{ Equation (30) } \rangle$$
$$\lfloor a \rfloor_\kappa \cap (\lfloor a \rfloor_\kappa \cup \lceil a \rceil^\kappa \cup X)$$
$$= \qquad \langle \text{ Distributivity of } \cap \text{ over } \cup \text{ } \rangle$$
$$(\lfloor a \rfloor_\kappa \cap \lfloor a \rfloor_\kappa) \cup (\lfloor a \rfloor_\kappa \cap \lceil a \rceil^\kappa) \cup (\lfloor a \rfloor_\kappa \cap X)$$
$$= \qquad \langle \text{ Idempotency of } \cap\text{, Proposition (3.10)(2), Equation (31) } \rangle$$
$$\lfloor a \rfloor_\kappa$$

Proof for Proposition (3.11), Equation (2), for any any $\kappa \in \mathcal{U}$ and $a \in A$:

$$\lceil a \rceil^\kappa \cap a{\Uparrow}^\kappa$$
$$= \qquad \langle \text{ Equation (30) } \rangle$$
$$\lceil a \rceil^\kappa \cap (\lfloor a \rfloor_\kappa \cup \lceil a \rceil^\kappa \cup X)$$
$$= \qquad \langle \text{ Distributivity of } \cap \text{ over } \cup \text{ } \rangle$$
$$(\lceil a \rceil^\kappa \cap \lfloor a \rfloor_\kappa) \cup (\lceil a \rceil^\kappa \cap \lceil a \rceil^\kappa) \cup (\lceil a \rceil^\kappa \cap X)$$
$$= \qquad \langle \text{ Idempotency of } \cap\text{, Proposition (3.10)(2), Equation (32) } \rangle$$
$$\lceil a \rceil^\kappa$$

Proof for Proposition (3.11), Equation (3), for any any $\kappa \in \mathcal{U}$ and $a \in A$:

$$a \cap a{\Uparrow}^\kappa$$
$$= \qquad \langle \text{ Proposition (3.10)(1) } \rangle$$
$$(\lfloor a \rfloor_\kappa \cup \lceil a \rceil^\kappa) \cap a{\Uparrow}^\kappa$$
$$= \qquad \langle \text{ Distributivity of } \cap \text{ over } \cup \text{ } \rangle$$
$$(\lfloor a \rfloor_\kappa \cap a{\Uparrow}^\kappa) \cup (\lceil a \rceil^\kappa \cap a{\Uparrow}^\kappa)$$
$$= \qquad \langle \text{ Proposition (3.11), Equations (1), (2) } \rangle$$
$$\{\} \cup \lceil a \rceil^\kappa$$
$$= \qquad \langle\ \{\} \text{ neutral element for } \cup \text{ } \rangle$$
$$\lceil a \rceil^\kappa$$

The proofs for Proposition (3.11), Equations (4), (5), and (6) are similar, as they are the duals of the above equations, and they have been omitted.

Proof for Proposition (3.12), Equation (1) for $\kappa \in \mathcal{U}$ and $a, b \in A$:

$$\lceil a \rceil^\kappa \cap \lfloor b \rfloor_\kappa$$
$$= \qquad \langle \text{ Definition of } \lfloor\rfloor \text{ (8) } \rangle$$
$$\lceil a \rceil^\kappa \cap \{dt \mid dt \in b \wedge \neg\exists v \in \kappa.\ (\kappa, v) \in dt\}$$
$$= \qquad \langle \text{ Definition of } \lceil\rceil \text{ (9) } \rangle$$
$$\{dt \mid dt \in a \wedge \exists v \in \kappa.\ (\kappa, v) \in dt\} \cap \{dt \mid dt \in b \wedge \neg\exists v \in \kappa.\ (\kappa, v) \in dt\}$$
$$= \qquad \langle\ \{E \mid R_1 \wedge R_2\} \iff \{E \mid R_1\} \cap \{E \mid R_2\}\ \rangle$$
$$\{dt \mid dt \in a \wedge \exists v \in \kappa.\ (\kappa, v) \in dt \wedge dt \in b \wedge \neg\exists v \in \kappa.\ (\kappa, v) \in dt\}$$
$$= \qquad \langle \text{ Contradiction } R \wedge \neg R = \mathit{False}\ \rangle$$
$$\{dt \mid dt \in a \wedge dt \in b \wedge \mathit{False}\}$$
$$= \qquad \langle \text{ Zero of } \wedge \text{ } \rangle$$
$$\{dt \mid \mathit{False}\}$$
$$= \qquad \langle\ \{x \mid \mathit{False}\} = \{\}\ \rangle$$
$$\{\}$$

Proof for Proposition (3.12), Equation (2) for $\kappa \in \mathcal{U}$ and $a, b \in A$:

$$
\begin{array}{ll}
& \lceil a \rceil^{\kappa} \cap b\Downarrow_{\kappa} \\
= & \qquad \langle \text{ Definition of } \lceil\,\rceil \text{ (9) } \rangle \\
& \{dt \mid dt \,.\, dt \in a \wedge \exists v \in \kappa.\ (\kappa, v) \in dt\} \cap b\Downarrow_{\kappa} \\
= & \qquad \langle \text{ Definition of } \Downarrow \text{ (4) } \rangle \\
& \{dt \mid dt \,.\, dt \in a \wedge \exists v \in \kappa.\ (\kappa, v) \in dt\} \cap \{dt\downarrow_{\kappa} \mid dt \,.\, dt \in b\} \\
= & \qquad \langle \text{ Variable renaming, } dt = dt'\downarrow_{\kappa} \rangle \\
& \{dt \mid dt \,.\, dt \in a \wedge \exists v \in \kappa.\ (\kappa, v) \in dt\} \cap \{dt \mid dt, dt' \,.\, dt = dt'\downarrow_{\kappa} \wedge dt' \in a\} \\
= & \qquad \langle \text{ Definition of } \downarrow \text{ (3) } \rangle \\
& \{dt \mid dt \,.\, dt \in a \wedge \exists v \in \kappa.\ (\kappa, v) \in dt\} \cap \\
& \{dt \mid dt, dt' \,.\, dt = \{(s, v) \mid s, v \,.\, (s, v) \in dt' \wedge s \neq \kappa\} \wedge dt' \in a\} \\
= & \qquad \langle\ \{E \mid R_1 \wedge R_2\} \iff \{E \mid R_1\} \cap \{E \mid R_2\}\ \rangle \\
& \{dt \mid dt, dt' \,.\, dt \in a \wedge \exists v \in \kappa.\ (\kappa, v) \in dt \wedge \\
& dt = \{(s, v) \mid s, v \,.\, (s, v) \in dt' \wedge s \neq \kappa\} \wedge dt' \in a\} \\
= & \qquad \langle \text{ Contradiction } dt = \{(s, v) \mid s, v \,.\, (s, v) \in dt' \wedge s \neq \kappa\} \text{ and} \\
& \qquad\ \ \exists v \in \kappa.\ (\kappa, v) \in dt\ \rangle \\
& \{dt \mid dt, dt' \,.\, dt' \in a \wedge dt \in a \wedge False\} \\
= & \qquad \langle \text{ Zero of } \wedge \rangle \\
& \{dt \mid False\} \\
= & \qquad \langle\ \{x \mid False\} = \{\}\ \rangle \\
& \{\}
\end{array}
$$

4.2 Domain Data View Model

In this Section, we show that $\mathcal{M}_{\mathcal{A}} = \big(A, +, \star, -, \mathbf{0}, \mathbf{1}, \{\mathbf{c}_{\kappa}\}_{\kappa \in \mathcal{U}}\big)$ is a diagonal-free cylindric algebra. For that, we need to prove the following axioms within $\mathcal{M}_{\mathcal{A}}$:

(C1) $\big(A, +, \star, -, \mathbf{0}, \mathbf{1}\big)$ is a Boolean algebra

(C2) $\mathbf{c}_{\kappa}\mathbf{0} = \mathbf{0}$

(C3) $a \leq \mathbf{c}_{\kappa} a$

(C4) $\mathbf{c}_{\kappa}(a \star \mathbf{c}_{\kappa} b) = \mathbf{c}_{\kappa} a \star \mathbf{c}_{\kappa} b$

(C5) $\mathbf{c}_{\kappa}\mathbf{c}_{\lambda} a = \mathbf{c}_{\lambda}\mathbf{c}_{\kappa} a$

For Axiom (C1), we need to show the following axioms hold in our model:

(C1.1) The binary operators $+, \star$ are associative, commutative, and distribute over each other

(C1.2) Identity axiom for $\mathbf{0}$: $a + \mathbf{0} = a$

(C1.3) Annihilator axiom for $\mathbf{0}$: $a \star \mathbf{0} = \mathbf{0}$

(C1.4) Identity axiom for $\mathbf{1}$: $a \star \mathbf{1} = a$

(C1.5) Annihilator axiom for $\mathbf{1}$: $a + \mathbf{1} = \mathbf{1}$

(C1.6) 1st Complement axiom: $a + (-a) = \mathbf{1}$

(C1.7) 2nd Complement axiom: $a \star (-a) = \mathbf{0}$

Due to the definitions of + (10) and $\star$ (11), it is immediate that Axiom (C1.1) holds in this model. Both $\cup$ and $\cap$ are associative and commutative, and distribute over each other. Similarly, showing the Axioms (C1.2) and (C1.3) hold is trivial. This is done using the definitions for $\mathbf{0}$ (12), + (10), $\star$ (11), and the facts that $\{\}$ is the neutral element for $\cup$ and annihilator for $\cap$.

Axiom (C1.4) holds, for any $a \in A$:

$$
\begin{array}{ll}
\quad a \star \mathbf{1} & \\
= & \langle \text{ Definition of } \star \text{ (11) } \rangle \\
\quad a \cap \mathbf{1} & \\
= & \langle \text{ Definition of } \mathbf{1} \text{ (13) } \rangle \\
\quad a \cap \bowtie\mathcal{U} & \\
= & \langle\ A \subseteq B \text{ iff } A \cap B = A, \text{ Equation (2) } \rangle \\
\quad a &
\end{array}
$$

Axiom (C1.5) holds, for any $a \in A$:

$$
\begin{array}{ll}
\quad a + \mathbf{1} & \\
= & \langle \text{ Definition of } + \text{ (10) } \rangle \\
\quad a \cup \mathbf{1} & \\
= & \langle \text{ Definition of } \mathbf{1} \text{ (13) } \rangle \\
\quad a \cup \bowtie\mathcal{U} & \\
= & \langle\ A \subseteq B \text{ iff } A \cup B = B, \text{ Equation (2) } \rangle \\
\quad \bowtie\mathcal{U} & \\
= & \langle \text{ Definition of } \mathbf{1} \text{ (13) } \rangle \\
\quad \mathbf{1} &
\end{array}
$$

Using the definitions of + (10) and $-$ (14); and of $\star$ (11) and $-$ (14), the complement Axioms (C1.6) and (C1.7) are trivial to show they hold for $a \in A$. Thus, Axiom (C1) holds.

Axiom (C2) holds, for any $\kappa \in \mathcal{U}$:

$$
\begin{array}{ll}
\quad \mathbf{c}_\kappa \mathbf{0} & \\
= & \langle \text{ (Definition of } \mathbf{c}_\kappa \text{ 15) } \rangle \\
\quad \mathbf{0} \cup \{dt\Uparrow^{\kappa} \mid dt \,.\, dt \in \mathbf{0}\} & \\
= & \langle \text{ Definition of } \mathbf{0} \text{ (12) } \rangle \\
\quad \mathbf{0} \cup \{dt\Uparrow^{\kappa} \mid dt \,.\, dt \in \{\}\} & \\
= & \langle\ x \in \{\} \iff \mathit{False}\ \rangle \\
\quad \mathbf{0} \cup \{dt\Uparrow^{\kappa} \mid \mathit{False}\} & \\
= & \langle\ \{x \mid \mathit{False}\} = \{\}\ \rangle \\
\quad \mathbf{0} \cup \{\} & \\
= & \langle \text{ Identity of } \cup) \rangle \\
\quad \mathbf{0} &
\end{array}
$$

For Axiom (C3), we use the classic definition of $\leq$, and to show that $a \leq \mathbf{c}_\kappa a$ holds, we need to show that $a + \mathbf{c}_\kappa a = \mathbf{c}_\kappa a$ for $a \in A$:

$$
\begin{array}{ll}
& a + \mathbf{c}_\kappa a \\
\iff & \langle \text{ Definition of } + \text{ (10) } \rangle \\
& a \cup \mathbf{c}_\kappa a \\
\iff & \langle \text{ Definition of } \mathbf{c}_\kappa \text{ (15) } \rangle \\
& a \cup a \cup \{dt{\Uparrow}^\kappa \mid dt \,.\, dt \in a\} \\
\iff & \langle \text{ Idempotency of } \cup \rangle \\
& a \cup \{dt{\Uparrow}^\kappa \mid dt \,.\, dt \in a\} \\
\iff & \langle \text{ Definition of } \mathbf{c}_\kappa \text{ (15) } \rangle \\
& \mathbf{c}_\kappa a
\end{array}
$$

Axiom (C4) holds for any $a, b \in A$ and $\kappa \in \mathcal{U}$:

$$
\begin{array}{ll}
& \mathbf{c}_\kappa(a \star \mathbf{c}_\kappa b) \\
= & \langle \text{ Definition of } \mathbf{c}_\kappa \text{ (15) } \rangle \\
& (a \star \mathbf{c}_\kappa b) \cup (a \star \mathbf{c}_\kappa b){\Uparrow}^\kappa \\
= & \langle \text{ Definition of } \star \text{ (11) } \rangle \\
& (a \cap \mathbf{c}_\kappa b) \cup (a \cap \mathbf{c}_\kappa b){\Uparrow}^\kappa \\
= & \langle \text{ Distributivity of } \Uparrow \text{ over } \cap \text{ (3.7), (5) } \rangle \\
& (a \cap \mathbf{c}_\kappa b) \cup (a{\Uparrow}^\kappa \cap (\mathbf{c}_\kappa b){\Uparrow}^\kappa) \\
= & \langle \text{ Definition of } \mathbf{c}_\kappa \text{ (15) } \rangle \\
& (a \cap (b \cup b{\Uparrow}^\kappa)) \cup (a{\Uparrow}^\kappa \cap (b \cup b{\Uparrow}^\kappa){\Uparrow}^\kappa) \\
= & \langle \text{ Distributivity of } \Uparrow \text{ over } \cap \text{ (3.7), (5) } \rangle \\
& (a \cap (b \cup b{\Uparrow}^\kappa)) \cup (a{\Uparrow}^\kappa \cap (b{\Uparrow}^\kappa \cup (b{\Uparrow}^\kappa){\Uparrow}^\kappa)) \\
= & \langle \text{ Idempotency of } \Uparrow \text{ (3.8), (4) } \rangle \\
& (a \cap (b \cup b{\Uparrow}^\kappa)) \cup (a{\Uparrow}^\kappa \cap (b{\Uparrow}^\kappa \cup b{\Uparrow}^\kappa)) \\
= & \langle \text{ Idempotency of } \cup \rangle \\
& (a \cap (b \cup b{\Uparrow}^\kappa)) \cup (a{\Uparrow}^\kappa \cap b{\Uparrow}^\kappa) \\
= & \langle\ b{\Uparrow}^\kappa = b \cup b{\Uparrow}^\kappa \text{, from Axiom (C3) } \rangle \\
& (a \cap (b \cup b{\Uparrow}^\kappa)) \cup (a{\Uparrow}^\kappa \cap (b \cup b{\Uparrow}^\kappa)) \\
= & \langle \text{ Distributivity of } \cap \text{ over } \cup \rangle \\
& (a \cup a{\Uparrow}^\kappa) \cap (b \cup b{\Uparrow}^\kappa) \\
= & \langle \text{ Definition of } \mathbf{c}_\kappa \text{ (15) } \rangle \\
& \mathbf{c}_\kappa a \cap \mathbf{c}_\kappa b \\
= & \langle \text{ Definition of } \star \text{ (11) } \rangle \\
& \mathbf{c}_\kappa a \star \mathbf{c}_\kappa b
\end{array}
$$

Axiom (C5) holds, for any $\kappa, \lambda \in \mathcal{U}$ and $a \in A$

$$
\begin{array}{ll}
& \mathbf{c}_\kappa \mathbf{c}_\lambda a \\
= & \langle \text{ Definition of } \mathbf{c}_\kappa \text{ (15), applied to } \kappa \rangle \\
& \mathbf{c}_\lambda a \cup (\mathbf{c}_\lambda a){\Uparrow}^\kappa \\
= & \langle \text{ Definition of } \mathbf{c}_\kappa \text{ (15), applied to } \lambda \rangle \\
& (a \cup a{\Uparrow}^\lambda) \cup (a \cup a{\Uparrow}^\lambda){\Uparrow}^\kappa \\
= & \langle \text{ Distributivity of } \Uparrow \text{ over } \cup \text{ (3.7),(5) } \rangle
\end{array}
$$

$$
\begin{aligned}
& a \cup a{\Uparrow}^{\lambda} \cup a{\Uparrow}^{\kappa} \cup (a{\Uparrow}^{\lambda}){\Uparrow}^{\kappa} \\
= \quad & \langle \text{ Associativity of } \Uparrow (3), \text{ Commutativity of } \cup \rangle \\
& a \cup a{\Uparrow}^{\kappa} \cup a{\Uparrow}^{\lambda} \cup (a{\Uparrow}^{\kappa}){\Uparrow}^{\lambda} \\
= \quad & \langle \text{ Distributivity of } \Uparrow \text{ over } \cup \text{ (3.7),(5) } \rangle \\
& (a \cup a{\Uparrow}^{\kappa}) \cup (a \cup a{\Uparrow}^{\kappa}){\Uparrow}^{\lambda} \\
= \quad & \langle \text{ Definition of } \mathbf{c}_\kappa \text{ (15), applied to } \kappa \rangle \\
& \mathbf{c}_\kappa a \cup (\mathbf{c}_\kappa a){\Uparrow}^{\kappa} \\
= \quad & \langle \text{ Definition of } \mathbf{c}_\kappa \text{ (15), applied to } \lambda \rangle \\
& \mathbf{c}_\lambda \mathbf{c}_\kappa a
\end{aligned}
$$

Thus, the structure $\mathcal{M}_\mathcal{A} = \big(A, +, \star, -, \mathbf{0}, \mathbf{1}, \{\mathbf{c}_\kappa\}_{\kappa \in \mathcal{U}}\big)$ is a model for the DDV.

4.3 Domain Ontology Model

We remind the reader that the model for the DOnt component of a DIS consists of three mathematical structures: $\mathcal{M}_\mathcal{C} = \big(C_{At_\mathcal{C}}, \oplus, \mathbf{e}_C\big)$, $\mathcal{M}_\mathcal{L} = \big(L_{At_\mathcal{L}}, \oplus, \otimes, \ominus, \mathbf{e}_C, \top_\mathcal{L}\big)$, and $\mathcal{M}_\mathcal{G} = \{G_{t_i}\}_{t_i \in L, i \in I}$, with the model for DOnt given by $\mathcal{M}_\mathcal{O} = \big(\mathcal{M}_\mathcal{C}, \mathcal{M}_\mathcal{L}, \mathcal{M}_\mathcal{G}\big)$. In this section, we show that $\mathcal{M}_\mathcal{C}$ is a commutative idempotent monoid, and $\mathcal{M}_\mathcal{L}$ is a Boolean algebra.

To show that $\mathcal{M}_\mathcal{C}$ is a commutative idempotent monoid, we need to show that $\oplus$ is commutative and idempotent. This is immediate, from the definition of $\oplus$ (16), with $\cup$ a commutative, idempotent operator. In addition, the $\mathbf{e}_C$ is the neutral element for $\oplus$, due to the definition of $\mathbf{e}_C$ (17) and the fact that $\{\}$ is the neutral element for $\cup$. Thus, $\mathcal{M}_\mathcal{C}$ is a commutative idempotent monoid.

Similar to the proofs for the DDV model, showing that $\mathcal{M}_\mathcal{L}$ is a Boolean algebra is immediate. In addition, by construction, the graphs in $\mathcal{M}_\mathcal{G}$ are rooted in concepts of the Boolean algebra, thus they are rooted graphs. Therefore, the structure $\mathcal{M}_\mathcal{O}$ is a model for the DOnt component of the DIS theory.

4.4 Mapping operator

In this section, we show the properties of the type operator τ hold in the chosen model. For that, we show that the type operator preserves the zero and one between the two Boolean algebras of the DDV and DOnt (Axioms ((A15)), ((A16))), as well as the plus operators (Axiom ((A17))). In addition, we provide the proofs for the type operator results in Proposition (3.16).

Axiom (A15) holds for any $a \in A$:

$$
\begin{aligned}
& \tau(\mathbf{0}) \\
= \quad & \langle \text{ Definition of } \tau \text{ (22) } \rangle \\
& \{\eta(sv.sort) \mid sv, dt \,.\, sv \in dt \wedge dt \in \mathbf{0}\} \\
= \quad & \langle \text{ Definition of } \mathbf{0} \text{ (12) } \rangle \\
& \{\eta(sv.sort) \mid sv, dt \,.\, sv \in dt \wedge dt \in \{\}\} \\
= \quad & \langle\, x \in \{\} \iff False \,\rangle \\
& \{\eta(sv.sort) \mid False\}
\end{aligned}
$$

$$
\begin{aligned}
&= \quad \langle \{x \mid False\} = \{\} \rangle \\
&\{\} \\
&= \quad \langle \text{Definition of } \mathbf{e}_{C} \text{ (17)} \rangle \\
&\mathbf{e}_{C}
\end{aligned}
$$

Axiom (A16) holds for any $a \in A$:

$$
\begin{aligned}
&\tau(\mathbf{1}) \\
&= \quad \langle \text{Definition of } \tau \text{ (22)} \rangle \\
&\{\eta(sv.sort) \mid sv, dt \;.\; sv \in dt \wedge dt \in \mathbf{1}\} \\
&= \quad \langle \text{Definition of } \mathbf{1} \text{ (13)} \rangle \\
&\{\eta(sv.sort) \mid sv, dt \;.\; sv \in dt \wedge dt \in \bowtie \mathcal{U}\} \\
&= \quad \langle \text{Here I am lost completely on how to approach it} \quad \rangle \\
&\{\eta(sv.sort) \mid sv \;.\; sv.sort \in \mathcal{U}\} \\
&= \quad \langle \text{By construction of } A \rangle \\
&\top_{\mathcal{L}}
\end{aligned}
$$

Axiom (A17) holds for any $a, b \in A$:

$$
\begin{aligned}
&\tau(a+b) \\
&= \quad \langle \text{Definition of } \tau \text{ (22)} \rangle \\
&\{\eta(sv.sort) \mid sv, dt \;.\; sv \in dt \wedge dt \in (a+b)\} \\
&= \quad \langle \text{Definition of } + \text{ (10)} \rangle \\
&\{\eta(sv.sort) \mid sv, dt \;.\; sv \in dt \wedge dt \in (a \cup b)\} \\
&= \quad \langle \{E \mid R_1 \vee R_2\} \iff \{E \mid R_1\} \cup \{E \mid R_2\} \rangle \\
&\{\eta(sv.sort) \mid sv, dt \;.\; sv \in dt \wedge (dt \in a \vee dt \in b)\} \\
&= \quad \langle \text{Distributivity of } \wedge \text{ over } \vee \rangle \\
&\{\eta(sv.sort) \mid sv, dt \;.\; (sv \in dt \wedge dt \in a) \vee ((s, v) \in dt \wedge dt \in b)\} \\
&= \quad \langle \{E \mid R_1 \vee R_2\} \iff \{E \mid R_1\} \cup \{E \mid R_2\} \rangle \\
&\{\eta(sv.sort) \mid sv, dt \;.\; sv \in dt \wedge dt \in a\} \cup \{\eta(sv.sort) \mid sv, dt \;.\; sv \in dt \wedge dt \in b\} \\
&= \quad \langle \text{Definition of } \tau \text{ (22)} \rangle \\
&\tau(a) \cup \tau(b) \\
&= \quad \langle \text{Definition of } \oplus \text{ (16)} \rangle \\
&\tau(a) \oplus \tau(b)
\end{aligned}
$$

Proof for Proposition (3.15), Equation (1), for $\kappa \in \mathcal{U}$ and $a \in A$:

$$
\begin{aligned}
&\eta(\kappa) \in \tau(a{\Uparrow}^{\kappa}) \\
&\iff \quad \langle \text{Definition of } \Uparrow \text{ (6)} \rangle \\
&\eta(\kappa) \in \tau(\{(dt\downharpoonleft_{\kappa} \cup \{(\kappa, v)\}) \mid dt, v \;.\; dt \in a \wedge v \in \kappa\}) \\
&\iff \quad \langle \text{Set comprehension} \rangle \\
&\eta(\kappa) \in \tau\Big(\bigcup_{dt \in a} \bigcup_{v \in \kappa} (dt\downharpoonleft_{\kappa} \cup \{(\kappa, v)\})\Big) \\
&\iff \quad \langle \text{Distrbutivity of } \tau \text{ over } \cup \rangle \\
&\eta(\kappa) \in \bigcup_{dt \in a} \bigcup_{v \in \kappa} \tau\Big((dt\downharpoonleft_{\kappa} \cup \{(\kappa, v)\})\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\Longleftrightarrow \qquad \langle \text{ Distrbutivity of } \tau \text{ over } \cup \rangle \\
&\quad \eta(\kappa) \in \bigcup_{dt \in a} \bigcup_{v \in \kappa} [\tau(dt\rfloor_{\downarrow\kappa}) \cup \tau(\{(\kappa, v)\})] \\
&\Longleftrightarrow \qquad \langle \text{ Definition of } \tau \text{ (22) } \rangle \\
&\quad \eta(\kappa) \in \bigcup_{dt \in a} \bigcup_{v \in \kappa} [\tau(dt\rfloor_{\downarrow\kappa}) \cup \{\eta(\kappa)\}] \\
&\Longleftrightarrow \qquad \langle \text{ Set membership } \rangle \\
&\quad True
\end{aligned}
$$

Proof for Proposition (3.15), Equation (2), for $\kappa \in \mathcal{U}$ and $a \in A$:

$$
\begin{aligned}
& \tau(a\Downarrow_\kappa) \\
= \quad & \langle \text{ Definition of } \tau \text{ (22) } \rangle \\
& \{\eta(sv.sort) \ \mid \ sv, dt \ . \ sv \in dt \wedge dt \in a\Downarrow_\kappa\} \\
= \quad & \langle \text{ Definition of } \Downarrow \text{ (4) } \rangle \\
& \{\eta(sv.sort) \ \mid \ sv, dt \ . \ sv \in dt\downarrow_\kappa \wedge dt \in a\} \\
= \quad & \langle \text{ Definition of } \downarrow \text{(3) } \rangle \\
& \{\eta(sv.sort) \ \mid \ sv, dt \ . \ sv \in dt\downarrow_\kappa \wedge dt \in a \wedge sv.sort \neq \kappa\}
\end{aligned}
$$

Thus, $\eta(\kappa)$ cannot be a part of $\tau(a\Downarrow_\kappa)$.

For Proposition (3.16), we make the following notations, for any $a, b \in A$

$$a_{a\backslash b} = \{dt \mid dt \in a \wedge dt \notin b\} \tag{33}$$
$$a_{ab} = \{dt \mid dt \in a \wedge dt \in b\} \tag{34}$$

It is immediate that

$$a = a_{a\backslash b} \cup a_{ab} \tag{35}$$
$$a_{a\backslash b} \cap a_{ab} = \{\} \tag{36}$$
$$a_{a\backslash b} \cap b_{b\backslash a} = \{\} \tag{37}$$
$$a_{ab} = b_{ba} \tag{38}$$

We show that, for any $a, b \in A$, the following equation holds:

$$a \star b = a_{ab} \tag{39}$$

$$\begin{array}{ll}
& a \star b \\
= & \quad \langle \text{ Definition of } \otimes \text{ (19) } \rangle \\
& a \cap b \\
= & \quad \langle \text{ Notation (35), applied twice } \rangle \\
& (a_{a\backslash b} \cup a_{ab}) \cap (b_{b\backslash a} \cup b_{ba}) \\
= & \quad \langle \text{ Distributivity of } \cap, \cup \text{ over each other } \rangle \\
& (a_{a\backslash b} \cap b_{b\backslash a}) \cup (a_{a\backslash b} \cap b_{ba}) \cup (a_{ab} \cap b_{b\backslash a}) \cup (a_{ab} \cap b_{ba}) \\
= & \quad \langle \text{ Equation (38), three times } \rangle \\
& (a_{a\backslash b} \cap b_{b\backslash a}) \cup (a_{a\backslash b} \cap a_{ab}) \cup (b_{ba} \cap b_{b\backslash a}) \cup (a_{ab} \cap a_{ab}) \\
= & \quad \langle \text{ Equations (37), (36) } \rangle \\
& \{\} \cup \{\} \cup \{\} \cup (a_{ab} \cap a_{ab}) \\
= & \quad \langle \text{ Zero of } \cup \rangle \\
& a_{ab} \cap a_{ab} \\
= & \quad \langle \text{ Idempotency of } \cap \rangle \\
& a_{ab}
\end{array}$$

We show the following equation holds, for any $a, b \in A$:

$$\tau(a) \otimes \tau(b) = \big(\tau(a_{a\backslash b}) \cap \tau(b_{b\backslash a})\big) \cup \big(\tau(a_{a\backslash b}) \cap \tau(a_{ab})\big) \cup \big(\tau(a_{ab}) \cap \tau(b_{b\backslash a})\big) \cup \tau(a_{ab}) \tag{40}$$

$$\begin{array}{ll}
& \tau(a) \otimes \tau(b) \\
= & \quad \langle \text{ Equation (35) } \rangle \\
& \tau(a_{a\backslash b} \cup a_{ab}) \otimes \tau(b_{b\backslash a} \cup b_{ba}) \\
= & \quad \langle \text{ Definition of + (10) } \rangle \\
& \tau(a_{a\backslash b} + a_{ab}) \otimes \tau(b_{b\backslash a} + b_{ba}) \\
= & \quad \langle\ \tau \text{ preserves + } ((A17)) \rangle \\
& \big(\tau(a_{a\backslash b}) \oplus \tau(a_{ab})\big) \otimes \big(\tau(b_{b\backslash a}) \oplus \tau(b_{ba})\big) \\
= & \quad \langle\ \oplus, \otimes \text{ distribute over each other } \rangle \\
& \big(\tau(a_{a\backslash b}) \otimes \tau(b_{b\backslash a})\big) \oplus \big(\tau(a_{a\backslash b}) \otimes \tau(b_{ba})\big) \oplus \big(\tau(a_{ab}) \otimes \tau(b_{b\backslash a})\big) \oplus \big(\tau(a_{ab}) \otimes \tau(b_{ba})\big) \\
= & \quad \langle \text{ Equation (38) } \rangle \\
& \big(\tau(a_{a\backslash b}) \otimes \tau(b_{b\backslash a})\big) \oplus \big(\tau(a_{a\backslash b}) \otimes \tau(b_{ba})\big) \oplus \big(\tau(a_{ab}) \otimes \tau(b_{b\backslash a})\big) \oplus \big(\tau(a_{ab}) \otimes \tau(b_{ba})\big)
\end{array}$$

$=$ $\langle$ Definitions of $\oplus$ (16), $\cap$ (19) $\rangle$
$\big(\tau(a_{a\backslash b}) \cap \tau(b_{b\backslash a})\big) \cup \big(\tau(a_{a\backslash b}) \cap \tau(b_{ba})\big) \cup \big(\tau(a_{ab}) \cap \tau(b_{b\backslash a})\big) \cup \big(\tau(a_{ab}) \cap \tau(b_{ba})\big)$
$=$ $\langle$ Equation (38) $\rangle$
$\big(\tau(a_{a\backslash b}) \cap \tau(b_{b\backslash a})\big) \cup \big(\tau(a_{a\backslash b}) \cap \tau(b_{ba})\big) \cup \big(\tau(a_{ab}) \cap \tau(b_{b\backslash a})\big) \cup \big(\tau(a_{ab}) \cap \tau(b_{ba})\big)$
$=$ $\langle$ Idempotency of $\cap$ $\rangle$
$\big(\tau(a_{a\backslash b}) \cap \tau(b_{b\backslash a})\big) \cup \big(\tau(a_{a\backslash b}) \cap \tau(a_{ab})\big) \cup \big(\tau(a_{ab}) \cap \tau(b_{b\backslash a})\big) \cup \tau(a_{ab})$

Proof for Proposition (3.16), Equation (2), for any $a, b \in A$:

$\tau(a) \otimes \tau(b) = \mathbf{e}_C$
$\iff$ $\langle$ Equation (40) $\rangle$
$\big(\tau(a_{a\backslash b}) \cap \tau(b_{b\backslash a})\big) \cup \big(\tau(a_{a\backslash b}) \cap \tau(a_{ab})\big) \cup \big(\tau(a_{ab}) \cap \tau(b_{b\backslash a})\big) \cup \tau(a_{ab}) = \mathbf{e}_C$
$\iff$ $\langle$ Definition of $\mathbf{e}_C$ (17) $\rangle$
$\big(\tau(a_{a\backslash b}) \cap \tau(b_{b\backslash a})\big) \cup \big(\tau(a_{a\backslash b}) \cap \tau(a_{ab})\big) \cup \big(\tau(a_{ab}) \cap \tau(b_{b\backslash a})\big) \cup \tau(a_{ab}) = \{\}$
$\implies$ $\langle$ $S \cup T = \{\} \implies S = \{\}$, with $S = \tau(a_{ab})$ $\rangle$
$\tau(a_{ab}) = \{\}$
$\iff$ $\langle$ Equation (39) $\rangle$
$\tau(a \otimes b) = \{\}$
$\iff$ $\langle$ Definition of $\mathbf{e}_C$ (17) $\rangle$
$\tau(a \otimes b) = \mathbf{e}_C$

Proof for Proposition (3.16), Equation (3), for any $a, b \in A$, using $S \subseteq T \iff S \cap T = S$, for $S = \tau(a \star b)$ and $T = \tau(a) \otimes \tau(b)$:

$$
\begin{aligned}
& \tau(a \star b) \cap \big(\tau(a) \otimes \tau(b)\big) \\
= \quad & \langle \text{ Equation (39) } \rangle \\
& \tau(a_{ab}) \cap \big(\tau(a) \otimes \tau(b)\big) \\
= \quad & \langle \text{ Equation (40) } \rangle \\
& \tau(a_{ab}) \cap \big[\big(\tau(a_{a\setminus b}) \cap \tau(b_{b\setminus a})\big) \cup \big(\tau(a_{a\setminus b}) \cap \tau(a_{ab})\big) \cup \big(\tau(a_{ab}) \cap \tau(b_{b\setminus a})\big) \cup \tau(a_{ab})\big] \\
= \quad & \langle\ S \cap (S \cup T) = S, \text{ with } S = \tau(a_{ab}),\ T = \big(\tau(a_{a\setminus b}) \cap \tau(b_{b\setminus a})\big) \cup \big(\tau(a_{a\setminus b}) \cap \\
& \quad \tau(a_{ab})\big) \cup \big(\tau(a_{ab}) \cap \tau(b_{b\setminus a})\big)\ \rangle \\
& \tau(a_{ab}) \\
= \quad & \langle \text{ Equation (39) } \rangle \\
& \tau(a \star b)
\end{aligned}
$$

Proof for Proposition (3.16), Equation (4) is done by induction on $T \subseteq \mathcal{U}$, for $a \in A$.

Step 1: $T = \{\}$

$$
\begin{aligned}
& a = \bowtie T \\
\iff \quad & \langle \text{ Assumption } T = \{\} \ \rangle \\
& a = \bowtie(\{\}) \\
\iff \quad & \langle \text{ Definition of } \bowtie \text{ (1) } \rangle \\
& a = \{\} \\
\implies \quad & \langle \text{ Application of operator } \tau \ \rangle \\
& \tau(a) = \tau(\{\}) \\
\iff \quad & \langle \text{ Equation ((A15)) } \rangle \\
& \tau(a) = \{\} \\
\iff \quad & \langle \text{ Set comprehension } \rangle \\
& \tau(a) = \{\eta(s) \mid s \,.\, s \in \{\}\} \\
\iff \quad & \langle \text{ Assumption } T = \{\} \ \rangle \\
& \tau(a) = \{\eta(s) \mid s \,.\, s \in T\}
\end{aligned}
$$

Step 2: $T = \{s\}$, with $s \in \mathcal{U}$,

$$
\begin{aligned}
& a = \bowtie T \\
\iff \quad & \langle \text{ Assumption } T = \{s\} \ \rangle \\
& a = \bowtie \{s\} \\
\iff \quad & \langle \text{ Definition of } \bowtie \text{ (1) } \rangle \\
& a = \big\{\{(s, v)\} \mid v \,.\, v \in s\big\} \\
\implies \quad & \langle \text{ Application of operator } \tau \ \rangle \\
& \tau(a) = \tau(\big\{\{(s, v)\} \mid v \,.\, v \in s\big\}) \\
\iff \quad & \langle \text{ Definition of } \tau \text{ (22) } \rangle \\
& \tau(a) = \{\eta(s) \mid s \,.\, s \in T\}
\end{aligned}
$$

Step 3: Induction hypothesis $a = \bowtie T \implies \tau(a) = \{\eta(s) \mid s \in T\}$, $T \subseteq \mathcal{U}, a \in A$. We show that for any $s' \in \mathcal{U}, s' \notin T, a = \bowtie(T \cup \{s'\}) \implies \tau(a) = \{\eta(s) \mid s \in (T \cup \{s'\})\}$.

$$a = \bowtie(T \cup \{s'\})$$

$\Longleftrightarrow \qquad \langle$ Definition of $\bowtie$ (1) $\rangle$

$a = \bowtie T \cup \big\{ dt \cup \{(s', v)\} \mid dt, v \,.\, dt \in \bowtie T \wedge v \in s' \big\}$

$\Longrightarrow \qquad \langle$ Application of operator τ $\rangle$

$\tau(a) = \tau\Big(\bowtie T \cup \big\{ dt \cup \{(s', v)\} \mid dt, v \,.\, dt \in \bowtie T \wedge v \in s' \big\}\Big)$

$\Longleftrightarrow \qquad \langle$ Definition of $\oplus$ (16), τ operator preserves $\oplus$ ((A17)) $\rangle$

$\tau(a) = \tau(\bowtie T) \cup \tau(\big\{ dt \cup \{(s', v)\} \mid dt, v \,.\, dt \in \bowtie T \wedge v \in s' \big\})$

$\Longleftrightarrow \qquad \langle$ Inductive hypothesis $\rangle$

$\tau(a) = \{\eta(s) \mid s \,.\, s \in T\} \cup \tau(\big\{ dt \cup \{(s', v)\} \mid dt, v \,.\, dt \in \bowtie T \wedge v \in s' \big\})$

$\Longleftrightarrow \qquad \langle$ Definition of τ (22) $\rangle$

$\tau(a) = \{\eta(s) \mid s \,.\, s \in T\} \cup$

$\{\eta(s) \mid s, v \,.\, (s, v) \in \big(dt \cup \{(s', v)\}\big) \wedge v \in s \wedge dt \in \bowtie T \wedge v \in s'\}$

$\Longleftrightarrow \qquad \langle\ x \in X_1 \cup X_2 \iff x \in X_1 \vee x \in X_2\ \rangle$

$\tau(a) = \{\eta(s) \mid s \,.\, s \in T\} \cup$

$\{\eta(s) \mid s, v \,.\, \big((s, v) \in dt \vee (s, v) \in \{(s', v)\}\big) \wedge v \in s \wedge dt \in \bowtie T \wedge v \in s'\}$

$\Longleftrightarrow \qquad \langle$ Distributivity of $\wedge$, $\vee$ over each other $\rangle$

$\tau(a) = \{\eta(s) \mid s \,.\, s \in T\} \cup$

$\big\{\eta(s) \mid s \,.\, \big((s, v) \in dt \wedge v \in s \wedge dt \in \bowtie T \wedge v \in s'\big) \vee$

$\big((s, v) \in \{(s', v)\} \wedge v \in s \wedge dt \in \bowtie T \wedge v \in s'\big)\}$

$\Longrightarrow \qquad \langle\ (s, v) \in dt \wedge dt \in \bowtie T \implies s \in T\ \rangle$

$\tau(a) = \{\eta(s) \mid s \,.\, s \in T\} \cup$

$\{\eta(s) \mid s \,.\, \big(s \in T \vee \big((s, v) \in \{(s', v)\} \wedge v \in s\big)\}$

$\Longrightarrow \qquad \langle\ (s, v) \in \{(s', v)\} \implies s \in \{s'\}\ \rangle$

$\tau(a) = \{\eta(s) \mid s \,.\, s \in T\} \cup \{\eta(s) \mid s \,.\, s \in T \vee s \in \{s'\}\}$

$\Longleftrightarrow \qquad \langle\ \{E \mid R_1 \vee R_2\} \iff \{E \mid R_1\} \cup \{E \mid R_2\}\ \rangle$

$\tau(a) = \{\eta(s) \mid s \,.\, s \in T \vee s \in T \vee s \in \{s'\}\}$

$\Longleftrightarrow \qquad \langle\ x \in X_1 \cup X_2 \iff x \in X_1 \vee x \in X_2\ \rangle$

$\tau(a) = \{\eta(s) \mid s \,.\, s \in (T \cup T \cup s \in \{s'\}\}$

$\Longleftrightarrow \qquad \langle$ Idempotency of $\cup$ $\rangle$

$\tau(a) = \big\{\eta(s) \mid s \,.\, s \in (T \cup s \in \{s'\})\big\}$

Proof for Proof for Proposition (3.16), Equation (5), for any $a, b \in A$, and $T_a, T_b \subseteq \mathcal{U}$, with $a = \bowtie T_a \wedge b = \bowtie T_b$.

$\tau(a \star b)$

$= \qquad \langle$ Definition of $\star$ (11) $\rangle$

$\tau(a \cap b)$

$= \qquad \langle$ Assumptions $a = \bowtie T_a \wedge b = \bowtie T_b$ $\rangle$

$\tau(\bowtie T_a \cap \bowtie T_b)$

$= \qquad \langle$ Proposition (3.4) $\rangle$

$\tau(\bowtie (T_a \cap T_b))$

$= \qquad \langle$ Proposition (3.16), (4) $\rangle$

$\{\eta(s) \mid s \,.\, s \in T_a \cap T_b\}$

$= \qquad \langle$ Set comprehension $\rangle$

$\{\eta(s) \mid s \,.\, s \in T_a\} \cap \{\eta(s) \mid s \,.\, s \in T_b\}$

$= \qquad \langle$ Assumptions, and Proposition (3.16), (4) $\rangle$

$\tau(a) \cap \tau(b)$

$$= \quad \langle \text{ Definition of } \otimes \text{ (19) } \rangle$$
$$\tau(a) \otimes \tau(b)$$

5 Conclusion

Modern knowledge systems face increasing demands for consistency, scalability, and adaptability, particularly in integrating real-world data with structured domain knowledge. This report addressed these concerns by formalising the DIS, a heterogeneous, modular formalism that separates the semantics of data from those of domain conceptualisation. Rather than introducing DIS as a new construct, this work focused on its formal semantics and theoretical foundation, demonstrating how it enables the construction of verifiable and adaptable knowledge systems grounded in data semantics.

Central to this contribution is the rigorous specification of DIS as a layered structure comprising a data theory and a domain knowledge theory, each capturing a distinct semantic level. The data theory, implemented through cylindric algebra, encodes structured datasets in the DDV. In parallel, the domain theory defines concept hierarchies, relationships, and contextual knowledge through a monoid of concepts, a Boolean lattics and a family of rooted graphs in the DOnt. A formal and automatable mapping operator bridges the two layers, ensuring that every DIS instance reflects the structure of the data within a well-defined conceptual framework. This guarantees that each instance of a DIS is not merely an application-specific construct, but a concrete model of the general theory, preserving modularity, semantic alignment, and proof-based correctness.

The modular separation of concerns and the clarity of the semantic layers naturally support the semi-automation of ontology specification. This facilitates engineering pipelines where formal correctness and domain alignment are achieved with minimal manual intervention, making DIS an efficient and scalable foundation for data-grounded knowledge system development.

Glossary

D

DDV Domain Data View. 1–8, 26, 34

DIS Domain Information System. 1–5, 7–9, 26, 34

DOnt Domain Ontology. 1–3, 8, 26, 34

References

[DGLL+18] Giuseppe De Giacomo, Domenico Lembo, Maurizio Lenzerini, Antonella Poggi, and Riccardo Rosati. Using Ontologies for Semantic Data Integration, pages 187–202. Springer International Publishing, Cham, 2018.

[KVS20] Konstantinos I Kotis, George A Vouros, and Dimitris Spiliotopoulos. Ontology engineering methodologies for the evolution of living and reused ontologies: status, trends, findings and recommendations. The Knowledge Engineering Review, 35, 2020.

[Mar16] Alicia Marinache. On the structural link between ontologies and organised data sets. Master's thesis, McMaster University, Hamilton, ON, Canada, 2016.

[THM71] A. Tarski, L. Henkin, and J.D. Monk. Cylindric Algebras. North-Holland, 1971.