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COMP SCI 3EA3 — Software Specification and Correctness

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Exercise 9.0 Co-transitivity

1. Recall that for a property 'P', it's `dual` is also known as 'co-P'.

   Recall that a relation $\mathcal{Z}$ is *transitive* precicely when —for any x, m, y—

   $$x \mathcal{Z} m \wedge m \mathcal{Z} y \quad \Rightarrow \quad x \mathcal{Z} y$$

   What does it mean for a relation to be *co-transitive*?

2. Duality is simply "changing your point of view".

   For example, for number-valued f and Boolean-valued p,

   $$f \text{ antimonotonic} \quad \equiv \quad (-f) \text{ monotonic} \qquad \text{and} \qquad p \text{ antimonotonic} \quad \equiv \quad (\neg p) \text{ monotonic}$$

   Likewise, verify that "co-transitive is the same as the complement is transitive":

   $$\mathcal{Z} \text{ co-transitive} \quad \equiv \quad (\neg \mathcal{Z}) \text{ transitive}$$

   where $x \; (\neg \mathcal{Z}) \; y \quad \equiv \quad \neg(x \mathcal{Z} y)$

3. Using the previous item, show that

   $$\neg \mathcal{Z} \text{ co-transitive} \qquad \text{for } \mathcal{Z} \text{ being any of } \; =, \wedge, \Rightarrow, \leq, <$$

4. As we've considered in previous algebraic settings, show that "the retract of a co-transitive relation is co-transitive":

   $$\mathcal{Z}_f \text{ co-transitive} \quad \Leftarrow \quad \mathcal{Z} \text{ co-transitive}$$

   where $x \; \mathcal{Z}_f \; y \equiv (f\,x) \; \mathcal{Z} \; (f\,y)$.

   (Notice that we are using the dual notion in the theorem: $\mathcal{Z}_f$ transitive $\Leftarrow$ $\mathcal{Z}$ transitive. Given any theorem, if we interchange the words "transitive" and "co-transitive" occurring in it, is the result still a theorem? What if we restrict ourselves to relational expressions?)

5. Notice —ie prove— that the above given relation constructors *commute*: $\neg(\mathcal{Z}_f) = (\neg\mathcal{Z})_f$;)

6. Use the previous items to show that the following $\mathcal{Z}$ is co-transtive —for any predicate b and transitive relation $\sim$.

   $$x \mathcal{Z} y \quad \equiv \quad \neg\,(b\,x \;\sim\; b\,y)$$

   Do so with reference to the "points" x and y, then present a more succinct proof by ignoring the points thereby presenting a "point-free / point-less proof":

   $$\begin{aligned}
   & \mathcal{Z} \text{ co-transitive} \\
   = \quad & \{ \; ? \; \} \\
   & \neg \sim_b \text{ co-transitive} \\
   = \quad & \{ \; ? \; \} \\
   & \sim_b \text{ transitive} \\
   \Leftarrow \quad & \{ \; ? \; \} \\
   & \sim \text{ transitive} \\
   = \quad & \{ \; ? \; \} \\
   & \mathit{true}
   \end{aligned}$$

7. Use the previous items, along with the "leap frog rule" $\neg(p \Rightarrow q) \;\equiv\; p \wedge \neg q$, to show that the following $\mathcal{Z}$ is co-transitive —for predicate b—:
$$x \,\mathcal{Z}\, y \quad \equiv \quad b\,x \wedge \neg b\,y$$

8. Recall that an informal understanding of relation properties can be obtained by construing a relation as adjacency '$\rightarrow$' in a graph setting.

   For example, "a relation is transitive precisely when for any two edges sharing a middle node $x \rightarrow m \rightarrow y$, there is an edge from the start to the end $x \rightarrow y$". That is, a relation is transitive precisely when finite paths $x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_{n-1}$ can be represented (internally in the graph) by edges $x_0 \rightarrow x_{n-1}$!

   What can be said for co-transtivity?

   Notice that in a graph, if two nodes are adjacent ($x \rightarrow y$), then the 'predecessors' ($p \rightarrow x$) of the first node can *reach* the second node via a 2-path ($p \rightarrow x \rightarrow y$). Is it the case that if two nodes are adjacent ($x \rightarrow y$), then the non-'successors' ($x \not\rightarrow m$) of the first node are necessarily adjacent to the second node ($m \rightarrow y$)?

   Finally, if you have a co-transitive relation $\rightarrow$ on a finite set, represented as a graph, and you draw a blue-coloured edge from x to y if there is no existing edge $x \rightarrow y$, then are finite paths consisting of blue-edges only representable by blue edges?

Exercise 9.1 — Report: Optional Assignment, 5% —due before the next quiz

Do the previous "Optional Assignment", at its original worth,

OR

Propose your own assignment to the instructor, possibly of greater worth,

OR

for 5%, write a report on "The Importance of Algebraic Properties" in which you select a few properties —eg commutativity, associtivity, symmetry, having an inverse, having an identity, representability of some sort— say two or three, and discuss their importance in calculation or in programming. For example, associtivity let's us avoid parentheses in expressions and is a part of the basis for concurrent algorithms such as map-reduce. Your report ought to

- give a colloquial and informal English presentation of each property
- discuss why it is useful and give an example of where its use is exemplified and discuss how things could have gone awry if the property did not hold for the operations under discussion.

  Alternatively, discuss why the property, or lack thereof, is *evil*, or undesired, in certain settings. For example, in concurrent programming, how undesirable is it that sequencing be non-commutative?
- The more examples given, the better.

Write and submit a report similar to the indications on Sheet 5.