

Preliminaries of Probability:

- Random Variables are denoted by upper-case letters, e.g., X , & its Supp set ^{or alphabet} denoted by calligraphic letters: $\mathcal{X}$.
- we write $X \sim P_X$ to denote X is distributed according the dist P_X
- In this course, we are only concerned with either discrete (X is a discrete set) continuous r.v (X is a continuum).
- Discrete r.v are usually characterized by their pmf: $X \sim P_X \Leftrightarrow \Pr(X=x) = P_X(x)$
- continuous r.v " " " " pdf: $X \sim P_X \Leftrightarrow P_X$ has density p_X
- cdf: $F_X(x) := \Pr(X \leq x)$
- Expected value: $X \sim P_X$: $E[X] := \sum x P_X(x)$
or $E[X] := \int x p_X(x) dx$ (pdf)

- k^{th} Moment: $E[X^k]$.

- For example: 2nd moment: $E[X^2] = \sum x^2 p_X(x)$ or $E[X^2] = \int x^2 p_X(x) dx$

- Variance: $\text{Var}(X) := E[(X - E[X])^2] = E[X^2] - (E[X])^2$

* Examples of discrete r.v:

1. Bernoulli: $X \sim \text{Bernoulli}(p)$

$0 \leq p \leq 1$

$X = \{0, 1\}$

$$E[X] = p$$

$$\text{Var}(X) = p \cdot (1-p)$$

pmf

$$p_X(x) := \Pr(X=x)$$

$x \in \{0, 1\}$

$$= p^x (1-p)^{1-x}$$

$$= \begin{cases} p & x=1 \\ 1-p & x=0 \end{cases}$$

2- Uniform: $X \sim \mathcal{U}([k])$ for some integer k : $p_X(x) = \begin{cases} \frac{1}{k} & x \in [k] \\ 0 & \text{o.w.} \end{cases}$

$$E[X] = \frac{k+1}{2}$$

$$\text{Var}(X) = \frac{k^2 - 1}{12}$$

$$= \frac{1}{k} \mathbb{1}_{\{x \in [k]\}}$$

3. Binomial : $X \sim \text{Bin}(n, p)$ integer: n
 & $p \in [0, 1]$

Let $B_1, \dots, B_n \stackrel{\text{iid}}{\sim} \text{Ber}(p)$. Then $X = B_1 + B_2 + \dots + B_n$

It can be easily shown that

$$\Pr(X=k) = \begin{cases} \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} & 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = np$$

$$\text{Var}(X) = n \cdot p \cdot (1-p)$$

Example of continuous dist:

1- Exponential: $X \sim \text{Exp}(\lambda)$ with $\lambda > 0$:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{on} \end{cases} \quad X = \mathbb{R}^+$$

verify that $\lambda \int_0^{\infty} e^{-\lambda x} dx = 1 \Rightarrow$ the above is a valid pdf.

Also, verify that:

$$E[X] = \frac{1}{\lambda} \quad \& \quad \text{Var}(X) = \frac{1}{\lambda^2}$$

2- Laplace $X \sim L(\mu, b)$ ↙ scale parameter $\mu \in \mathbb{R} \quad \& \quad b > 0$

pdf: $f(x) = \frac{1}{2b} e^{\frac{-|x-\mu|}{b}} \quad X = \mathbb{R}$

Verify that $\frac{1}{2b} \int_{-\infty}^{\infty} e^{-\frac{|x-\mu|}{b}} dx = 1$.

$$E[X] = \mu$$

$$\text{Var}(X) = 2b^2$$

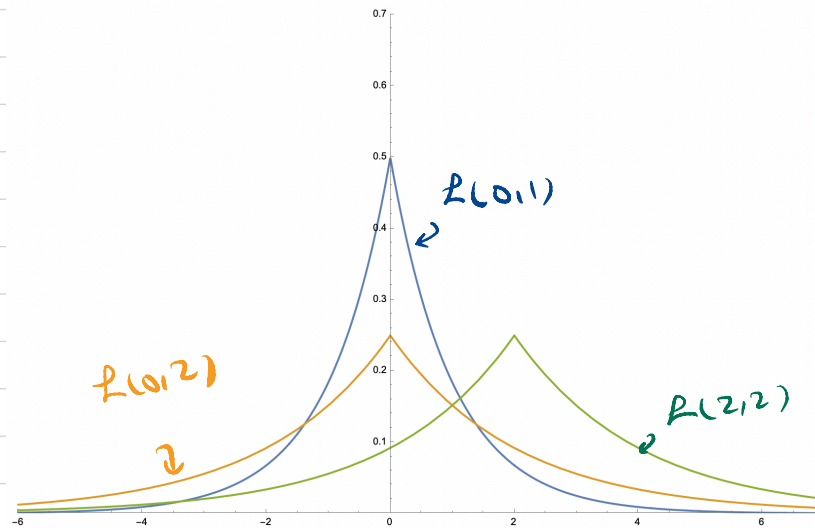
Exercise: Let $X \sim L(\mu, b)$

Show $\Pr(|X - \mu| \geq t) = e^{-t/b}$

Approach I: Directly

Approach II: First verify that:

$$|X - \mu| \sim \text{Exp}(b^{-1})$$



3 - Gaussian $X \sim \mathcal{N}(\mu, \sigma^2)$

$$\mu \in \mathbb{R}$$

$$\sigma > 0$$

$$X \in \mathbb{R}$$

$$\text{pdf: } f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Verify that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$

Note that Gaussian has a faster decay than Lap

$$E[X] = \mu \quad \& \quad \text{Var}(X) = \sigma^2$$

$$E[(X-\mu)^k] = \begin{cases} 0 & k \text{ odd} \\ \sigma^k (k-1)!! & k \text{ even} \end{cases}$$

central moment

Exercise: $\Pr(|X-\mu| \geq t) = 2Q(t/\sigma)$

where

$$Q(a) := \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{-x^2/2} dx$$

double factorial
 $n!! = n \cdot (n-2) \cdot (n-4) \cdots$