

Information-Theoretic Measures

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1. (TV distance and Hockey-stick divergence. 30 points)

Solution. Recall that

$$\text{TV}(P, Q) = \frac{1}{2} \int |p(x) - q(x)| \, dx$$

and

$$\mathbb{E}_\gamma(P \| Q) = \int (p(x) - \gamma q(x))_+ \, dx, \quad \gamma \geq 1.$$

We will repeatedly use the following elementary identities, valid for all $a, b \geq 0$:

$$|a - b| = a + b - 2 \min\{a, b\} = 2 \max\{a, b\} - a - b,$$

$$a - \min\{a, b\} = (a - b)_+, \quad \max\{a, b\} - b = (a - b)_+,$$

and

$$(a - b)_+ = \frac{1}{2}(|a - b| + a - b).$$

- (a) Using the first identity with $a = p(x)$ and $b = q(x)$,

$$\begin{aligned} \text{TV}(P, Q) &= \frac{1}{2} \int |p(x) - q(x)| \, dx \\ &= \frac{1}{2} \int (p(x) + q(x) - 2 \min\{p(x), q(x)\}) \, dx \\ &= \frac{1}{2}(1 + 1) - \int \min\{p(x), q(x)\} \, dx \\ &= 1 - \int \min\{p(x), q(x)\} \, dx. \end{aligned}$$

- (b) Again using the first identity,

$$\begin{aligned} \text{TV}(P, Q) &= \frac{1}{2} \int |p(x) - q(x)| \, dx \\ &= \frac{1}{2} \int (2 \max\{p(x), q(x)\} - p(x) - q(x)) \, dx \\ &= \int \max\{p(x), q(x)\} \, dx - 1. \end{aligned}$$

- (c) Since

$$p(x) - \min\{p(x), \gamma q(x)\} = (p(x) - \gamma q(x))_+,$$

we have

$$\begin{aligned} \mathbb{E}_\gamma(P \| Q) &= \int (p(x) - \min\{p(x), \gamma q(x)\}) \, dx \\ &= 1 - \int \min\{p(x), \gamma q(x)\} \, dx. \end{aligned}$$

(d) Since

$$\max\{p(x), \gamma q(x)\} - \gamma q(x) = (p(x) - \gamma q(x))_+,$$

we obtain

$$\begin{aligned} \mathbb{E}_\gamma(P\|Q) &= \int (\max\{p(x), \gamma q(x)\} - \gamma q(x)) \, dx \\ &= \int \max\{p(x), \gamma q(x)\} \, dx - \gamma. \end{aligned}$$

(e) Using

$$(a - b)_+ = \frac{1}{2}(|a - b| + a - b)$$

with $a = p(x)$ and $b = \gamma q(x)$ gives

$$\begin{aligned} \mathbb{E}_\gamma(P\|Q) &= \frac{1}{2} \int |p(x) - \gamma q(x)| \, dx + \frac{1}{2} \int (p(x) - \gamma q(x)) \, dx \\ &= \frac{1}{2} \int |p(x) - \gamma q(x)| \, dx + \frac{1}{2}(1 - \gamma) \\ &= \frac{1}{2} \int |p(x) - \gamma q(x)| \, dx - \frac{1}{2}(\gamma - 1). \end{aligned}$$

2. (Rényi divergences. 10 points)

Solution. Recall that, for $\alpha \in (0, 1) \cup (1, \infty)$,

$$D_\alpha(P\|Q) = \frac{1}{\alpha - 1} \log \int p(x)^\alpha q(x)^{1-\alpha} \, dx.$$

Since $P = \prod_{i=1}^n P_i$ and $Q = \prod_{i=1}^n Q_i$, we can write

$$p(x) = \prod_{i=1}^n p_i(x_i), \quad q(x) = \prod_{i=1}^n q_i(x_i), \quad x = (x_1, \dots, x_n).$$

Hence, by factorization of the integral,

$$\begin{aligned} (\alpha - 1)D_\alpha(P\|Q) &= \log \int_{\mathbb{R}^n} \prod_{i=1}^n p_i(x_i)^\alpha q_i(x_i)^{1-\alpha} \, dx_1 \cdots dx_n \\ &= \log \prod_{i=1}^n \int_{\mathbb{R}} p_i(x_i)^\alpha q_i(x_i)^{1-\alpha} \, dx_i \\ &= \sum_{i=1}^n \log \int_{\mathbb{R}} p_i(x_i)^\alpha q_i(x_i)^{1-\alpha} \, dx_i. \end{aligned}$$

Therefore,

$$D_\alpha(P\|Q) = \sum_{i=1}^n D_\alpha(P_i\|Q_i).$$

It remains to compute $D_\alpha(P_i\|Q_i)$. For $P_i = \mathcal{N}(\mu_i, \sigma^2)$ and $Q_i = \mathcal{N}(\nu_i, \sigma^2)$,

$$p_i(x)^\alpha q_i(x)^{1-\alpha} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{1}{2\sigma^2} [\alpha(x - \mu_i)^2 + (1 - \alpha)(x - \nu_i)^2] \right\}.$$

The quadratic term satisfies

$$\alpha(x - \mu_i)^2 + (1 - \alpha)(x - \nu_i)^2 = (x - \alpha\mu_i - (1 - \alpha)\nu_i)^2 + \alpha(1 - \alpha)(\mu_i - \nu_i)^2.$$

Consequently,

$$\begin{aligned} \int p_i(x)^\alpha q_i(x)^{1-\alpha} dx &= \exp \left\{ -\frac{\alpha(1-\alpha)}{2\sigma^2} (\mu_i - \nu_i)^2 \right\} \\ &= \exp \left\{ \frac{\alpha(\alpha-1)}{2\sigma^2} (\mu_i - \nu_i)^2 \right\}, \end{aligned}$$

where the remaining Gaussian density integrates to one. Thus

$$D_\alpha(P_i \| Q_i) = \frac{\alpha}{2\sigma^2} (\mu_i - \nu_i)^2.$$

Finally,

$$D_\alpha(P \| Q) = \frac{\alpha}{2\sigma^2} \sum_{i=1}^n (\mu_i - \nu_i)^2, \quad \alpha \in (0, 1) \cup (1, \infty).$$

3. (Hockey-stick divergences. 20 points)

(a) **Solution.** Let $\gamma = e^\varepsilon$. Since

$$\delta = \frac{1}{1 + e^\varepsilon},$$

we have

$$1 - \delta = \frac{e^\varepsilon}{1 + e^\varepsilon} = \gamma\delta.$$

For either output symbol $y \in \{0, 1\}$, the two channel probabilities $P_{Y|X}(y|0)$ and $P_{Y|X}(y|1)$ are δ and $1 - \delta$ in some order. Therefore, for *any* input distribution P_X , the corresponding output distribution satisfies

$$\delta \leq P_Y(y) \leq 1 - \delta, \quad y \in \{0, 1\}.$$

The same is true for the output distribution Q_Y induced by any other input distribution Q_X . Hence, for every $y \in \{0, 1\}$,

$$P_Y(y) \leq 1 - \delta = \gamma\delta \leq \gamma Q_Y(y).$$

It follows that

$$E_\gamma(P_Y \| Q_Y) = \sum_{y \in \{0, 1\}} (P_Y(y) - \gamma Q_Y(y))_+ = 0$$

for every pair of input distributions P_X, Q_X . Therefore every ratio appearing in the definition of $\eta_\gamma(P_{Y|X})$ has numerator zero, and thus

$$\eta_\gamma(P_{Y|X}) = 0.$$

(b) This problem can be solved directly and indirectly (smarter :)) via contraction coefficient. We begin with the latter.

Solution 1. We will first show that $\eta_\gamma(P_{Y|X}) = 0$ for the conditional distribution given in the problem. For any $x \in \mathcal{A}$, we can write

$$P_{Y|X=x}(y) = \begin{cases} \frac{\gamma}{\gamma + k - 1}, & y = x, \\ \frac{1}{\gamma + k - 1}, & y \neq x. \end{cases}$$

Recall that

$$\eta_\gamma(P_{Y|X}) = \sup_{x_1, x_2 \in \mathcal{A}} \mathbb{E}_\gamma(P_{Y|X=x_1} \| P_{Y|X=x_2}).$$

If $x_1 = x_2$, then clearly

$$\mathbb{E}_\gamma(P_{Y|X=x_1} \| P_{Y|X=x_2}) = 0.$$

Now suppose that $x_1 \neq x_2$. Using

$$\mathbb{E}_\gamma(P \| Q) = \sum_y (P(y) - \gamma Q(y))_+,$$

we obtain

$$\begin{aligned} & \mathbb{E}_\gamma(P_{Y|X=x_1} \| P_{Y|X=x_2}) \\ &= \left(\frac{\gamma}{\gamma + k - 1} - \gamma \frac{1}{\gamma + k - 1} \right)_+ + \left(\frac{1}{\gamma + k - 1} - \gamma \frac{\gamma}{\gamma + k - 1} \right)_+ \\ & \quad + (k - 2) \left(\frac{1}{\gamma + k - 1} - \gamma \frac{1}{\gamma + k - 1} \right)_+ \\ &= 0 + \left(\frac{1 - \gamma^2}{\gamma + k - 1} \right)_+ + (k - 2) \left(\frac{1 - \gamma}{\gamma + k - 1} \right)_+ \\ &= 0, \end{aligned}$$

where the last equality follows from $\gamma \geq 1$. Hence

$$\eta_\gamma(P_{Y|X}) = 0.$$

Now let P_X and Q_X be arbitrary distributions on $\mathcal{A}$, and let P_Y and Q_Y be the corresponding output distributions. By the definition of the contraction coefficient,

$$\mathbb{E}_\gamma(P_Y \| Q_Y) \leq \eta_\gamma(P_{Y|X}) \mathbb{E}_\gamma(P_X \| Q_X) = 0.$$

Since the hockey-stick divergence is nonnegative,

$$\mathbb{E}_\gamma(P_Y \| Q_Y) = 0.$$

On the other hand,

$$\mathbb{E}_\gamma(P_Y \| Q_Y) = \sup_{A \subseteq \mathcal{A}} \{P_Y(A) - \gamma Q_Y(A)\}.$$

Therefore,

$$P_Y(A) - \gamma Q_Y(A) \leq 0 \quad \text{for every } A \subseteq \mathcal{A},$$

and thus

$$P_Y(A) \leq \gamma Q_Y(A), \quad \forall A \subseteq \mathcal{A}.$$

Solution 2. For every $y \in \mathcal{A}$,

$$\begin{aligned} P_Y(y) &= \sum_{x \in \mathcal{A}} P_X(x) P_{Y|X}(y|x) \\ &= \frac{\gamma P_X(y) + \sum_{x \neq y} P_X(x)}{\gamma + k - 1} \\ &= \frac{1 + (\gamma - 1) P_X(y)}{\gamma + k - 1}. \end{aligned}$$

Similarly,

$$Q_Y(y) = \frac{1 + (\gamma - 1)Q_X(y)}{\gamma + k - 1}.$$

Since $0 \leq P_X(y) \leq 1$ and $Q_X(y) \geq 0$,

$$1 + (\gamma - 1)P_X(y) \leq \gamma$$

and

$$\gamma(1 + (\gamma - 1)Q_X(y)) \geq \gamma.$$

Therefore

$$P_Y(y) \leq \gamma Q_Y(y) \quad \text{for every } y \in \mathcal{A}.$$

Summing this pointwise inequality over any $A \subseteq \mathcal{A}$ gives

$$\boxed{P_Y(A) \leq \gamma Q_Y(A), \quad A \subseteq \mathcal{A}.$$

4. (*Maximal coupling and TV. 20 points*)

Solution. We first prove that every coupling satisfies

$$\text{TV}(P, Q) \leq \Pr(X \neq Y).$$

Let P_{XY} be an arbitrary coupling of P and Q . For every $A \subseteq \mathcal{A}$,

$$\begin{aligned} P(A) - Q(A) &= \Pr(X \in A) - \Pr(Y \in A) \\ &= \Pr(X \in A, Y \notin A) - \Pr(X \notin A, Y \in A) \\ &\leq \Pr(X \in A, Y \notin A) \\ &\leq \Pr(X \neq Y). \end{aligned}$$

Taking the supremum over $A \subseteq \mathcal{A}$ and using

$$\text{TV}(P, Q) = \sup_{A \subseteq \mathcal{A}} (P(A) - Q(A))$$

gives

$$\text{TV}(P, Q) \leq \Pr(X \neq Y).$$

Since this is true for every coupling,

$$\text{TV}(P, Q) \leq \inf_{P_{XY}} \Pr(X \neq Y).$$

We now construct a coupling that attains equality. Define

$$\Delta := \sum_{x \in \mathcal{A}} \min\{P(x), Q(x)\}.$$

By the discrete analogue of Problem 1(a),

$$\text{TV}(P, Q) = 1 - \Delta.$$

If $\text{TV}(P, Q) = 0$, then $P = Q$, and the coupling $X = Y \sim P$ gives $\Pr(X \neq Y) = 0$. Thus suppose that $t := \text{TV}(P, Q) > 0$. Define

$$R(x) := \frac{(P(x) - Q(x))_+}{t}, \quad S(x) := \frac{(Q(x) - P(x))_+}{t}.$$

Since

$$\sum_x (P(x) - Q(x))_+ = \sum_x (Q(x) - P(x))_+ = t,$$

both R and S are probability distributions. Moreover, $R(x)S(x) = 0$ for every x .

Define the joint distribution

$$P_{XY}(x, y) = \min\{P(x), Q(x)\} \mathbf{1}\{x = y\} + tR(x)S(y).$$

Its first marginal is

$$\begin{aligned} \sum_y P_{XY}(x, y) &= \min\{P(x), Q(x)\} + tR(x) \\ &= \min\{P(x), Q(x)\} + (P(x) - Q(x))_+ \\ &= P(x), \end{aligned}$$

and similarly its second marginal is $Q(y)$. Hence P_{XY} is a coupling of P and Q .

Because $R(x)S(x) = 0$ for every x , the second term never places mass on the diagonal. Therefore

$$\Pr(X = Y) = \sum_{x \in \mathcal{A}} \min\{P(x), Q(x)\} = \Delta,$$

and hence

$$\Pr(X \neq Y) = 1 - \Delta = \text{TV}(P, Q).$$

Thus

$$\boxed{\text{TV}(P, Q) = \inf_{P_{XY}} \Pr(X \neq Y).}$$

5. (*Strong DPI. 20 points*)

Solution. Define

$$\eta^* := \sup_{x_1, x_2 \in \mathcal{X}} \mathbb{E}_\gamma(P_{Y|X=x_1} \| P_{Y|X=x_2}).$$

We prove the two inequalities separately. For notational simplicity, we write the proof using sums; the same argument holds with integrals whenever the alphabets are not discrete.

First, $\eta_\gamma(P_{Y|X}) \geq \eta^*$. Fix $x_1 \neq x_2$ and choose

$$P_X = \delta_{x_1}, \quad Q_X = \delta_{x_2}.$$

Then

$$\mathbb{E}_\gamma(P_X \| Q_X) = 1,$$

while the corresponding output distributions are $P_{Y|X=x_1}$ and $P_{Y|X=x_2}$. Hence

$$\eta_\gamma(P_{Y|X}) \geq \mathbb{E}_\gamma(P_{Y|X=x_1} \| P_{Y|X=x_2}).$$

Taking the supremum over x_1, x_2 gives

$$\eta_\gamma(P_{Y|X}) \geq \eta^*.$$

(The case $x_1 = x_2$ contributes zero and therefore does not affect the supremum.)

Next, $\eta_\gamma(P_{Y|X}) \leq \eta^*$. Fix arbitrary input distributions P_X, Q_X such that

$$t := \mathbb{E}_\gamma(P_X \| Q_X) > 0.$$

Following the positive/negative decomposition of $P_X - \gamma Q_X$, define

$$\phi^+(x) := (P_X(x) - \gamma Q_X(x))_+, \quad \phi^-(x) := (\gamma Q_X(x) - P_X(x))_+.$$

Then

$$P_X(x) - \gamma Q_X(x) = \phi^+(x) - \phi^-(x)$$

and

$$\sum_x \phi^+(x) = t.$$

Also,

$$\begin{aligned} \sum_x \phi^-(x) - \sum_x \phi^+(x) &= \sum_x (\gamma Q_X(x) - P_X(x)) \\ &= \gamma - 1. \end{aligned}$$

Therefore

$$\sum_x \phi^-(x) = t + \gamma - 1.$$

Since $\mathbb{E}_\gamma(P_X \| Q_X) \leq 1$, we also have $0 < t \leq 1$.

Define two probability distributions

$$U_X(x) := \frac{\phi^+(x)}{t}, \quad V_X(x) := \frac{\phi^-(x)}{t + \gamma - 1},$$

and let U_Y and V_Y denote their corresponding output distributions through the channel $P_{Y|X}$. Then, for every y ,

$$\begin{aligned} P_Y(y) - \gamma Q_Y(y) &= \sum_x P_{Y|X}(y|x) (P_X(x) - \gamma Q_X(x)) \\ &= t U_Y(y) - (t + \gamma - 1) V_Y(y) \\ &= t (U_Y(y) - \gamma V_Y(y)) - (\gamma - 1)(1 - t) V_Y(y) \\ &\leq t (U_Y(y) - \gamma V_Y(y)). \end{aligned}$$

Taking positive parts and summing over y yields

$$\mathbb{E}_\gamma(P_Y \| Q_Y) \leq t \mathbb{E}_\gamma(U_Y \| V_Y). \quad (1)$$

It remains to upper-bound $\mathbb{E}_\gamma(U_Y \| V_Y)$. We use the joint convexity of hockey-stick divergence. Indeed, if $\{\lambda_i\}$ are nonnegative weights summing to one, then

$$\begin{aligned} \mathbb{E}_\gamma \left(\sum_i \lambda_i P_i \left\| \sum_i \lambda_i Q_i \right\| \right) &= \sum_y \left(\sum_i \lambda_i (P_i(y) - \gamma Q_i(y)) \right)_+ \\ &\leq \sum_i \lambda_i \sum_y (P_i(y) - \gamma Q_i(y))_+ \\ &= \sum_i \lambda_i \mathbb{E}_\gamma(P_i \| Q_i). \end{aligned}$$

Now observe that

$$U_Y = \sum_{x, x'} U_X(x) V_X(x') P_{Y|X=x}$$

and

$$V_Y = \sum_{x, x'} U_X(x) V_X(x') P_{Y|X=x'}.$$

Therefore, by joint convexity,

$$\begin{aligned} \mathbb{E}_\gamma(U_Y \| V_Y) &\leq \sum_{x, x'} U_X(x) V_X(x') \mathbb{E}_\gamma(P_{Y|X=x} \| P_{Y|X=x'}) \\ &\leq \eta^*. \end{aligned}$$

Combining this with (1) and recalling that $t = \mathbb{E}_\gamma(P_X \| Q_X)$, we obtain

$$\mathbb{E}_\gamma(P_Y \| Q_Y) \leq \eta^* \mathbb{E}_\gamma(P_X \| Q_X).$$

Thus, for every admissible pair P_X, Q_X ,

$$\frac{\mathbb{E}_\gamma(P_Y \| Q_Y)}{\mathbb{E}_\gamma(P_X \| Q_X)} \leq \eta^*.$$

Taking the supremum gives

$$\eta_\gamma(P_{Y|X}) \leq \eta^*.$$

Combining the two directions,

$$\boxed{\eta_\gamma(P_{Y|X}) = \sup_{x_1, x_2 \in \mathcal{X}} \mathbb{E}_\gamma(P_{Y|X=x_1} \| P_{Y|X=x_2}).}$$

6. **[Bonus]** (*Maximal list coupling and hockey-stick. +25 points*)

Solution. We prove the lower bound first, and then construct a coupling that attains it.

Lower bound. Fix an arbitrary $P_{X, Y_1, \dots, Y_m} \in \Gamma_m(P, Q)$ and define the failure event

$$F := \{X \notin \{Y_1, \dots, Y_m\}\}.$$

For any $A \subseteq \mathcal{A}$, if $X \in A$ and F^c occurs, then $X = Y_i$ for at least one i , and hence $Y_i \in A$ for at least one i . Thus

$$\{X \in A\} \subseteq F \cup \bigcup_{i=1}^m \{Y_i \in A\}.$$

By the union bound and the fact that each $Y_i \sim Q$,

$$\begin{aligned} P(A) &\leq \Pr(F) + \sum_{i=1}^m \Pr(Y_i \in A) \\ &= \Pr(F) + mQ(A). \end{aligned}$$

Therefore

$$\Pr(F) \geq P(A) - mQ(A) \quad \text{for every } A \subseteq \mathcal{A}.$$

Taking the supremum over A and using

$$\mathbb{E}_m(P \| Q) = \sup_{A \subseteq \mathcal{A}} (P(A) - mQ(A))$$

gives

$$\Pr(F) \geq \mathbb{E}_m(P \| Q).$$

Since the coupling was arbitrary,

$$\inf_{P_{X,Y_1,\dots,Y_m} \in \Gamma_m(P,Q)} \Pr(F) \geq \mathbb{E}_m(P\|Q).$$

Achievability. For $x \in \mathcal{A}$, define

$$c_x := \min\{P(x), mQ(x)\}, \quad \Delta := \sum_{x \in \mathcal{A}} c_x.$$

By the discrete analogue of Problem 1(c),

$$\mathbb{E}_m(P\|Q) = 1 - \Delta.$$

We now construct a joint distribution in $\Gamma_m(P, Q)$ for which the failure probability is exactly $1 - \Delta$. Introduce an auxiliary random variable $J \in \{0, 1, \dots, m\}$ and define the joint distribution of (X, J) by

$$\Pr(X = x, J = 0) = P(x) - c_x$$

and, for $i = 1, \dots, m$,

$$\Pr(X = x, J = i) = \frac{c_x}{m}.$$

These numbers are nonnegative, and for every x ,

$$\Pr(X = x) = P(x) - c_x + \sum_{i=1}^m \frac{c_x}{m} = P(x),$$

so indeed $X \sim P$. Moreover,

$$\Pr(J = i) = \frac{\Delta}{m}, \quad i = 1, \dots, m,$$

and

$$\Pr(J = 0) = 1 - \Delta.$$

Except for the trivial case $m = 1$ and $\Delta = 1$, define the probability distribution

$$R(y) := \frac{Q(y) - c_y/m}{1 - \Delta/m}, \quad y \in \mathcal{A}.$$

This is well defined because $c_y \leq mQ(y)$, and

$$\sum_y \left(Q(y) - \frac{c_y}{m} \right) = 1 - \frac{\Delta}{m}.$$

Conditional on (X, J) , generate $Y_1, \dots, Y_m$ as follows:

- if $J = i \in \{1, \dots, m\}$, set $Y_i = X$, and generate each Y_j , $j \neq i$, according to R ;
- if $J = 0$, generate every Y_i according to R .

For definiteness, the variables generated according to R may be taken conditionally independent, although no independence is required by the problem.

We check the marginal of each Y_i . For every $y \in \mathcal{A}$,

$$\begin{aligned} \Pr(Y_i = y) &= \Pr(J = i, X = y) + \Pr(J \neq i)R(y) \\ &= \frac{c_y}{m} + \left(1 - \frac{\Delta}{m}\right) \frac{Q(y) - c_y/m}{1 - \Delta/m} \\ &= Q(y). \end{aligned}$$

Hence $Y_i \sim Q$ for every i , and therefore the constructed joint distribution belongs to $\Gamma_m(P, Q)$.

It remains to identify the failure event. If $J = i \geq 1$, then $Y_i = X$, so failure does not occur. On the other hand, suppose $J = 0$ and $X = x$. If this event has positive probability, then

$$P(x) - c_x > 0.$$

Hence $c_x < P(x)$, and by the definition $c_x = \min\{P(x), mQ(x)\}$ we must have

$$c_x = mQ(x).$$

Consequently,

$$R(x) = \frac{Q(x) - c_x/m}{1 - \Delta/m} = 0.$$

Thus, conditional on $J = 0$ and $X = x$, none of the variables $Y_1, \dots, Y_m$ can equal x . Therefore

$$F = \{J = 0\} \quad \text{almost surely,}$$

and

$$\Pr(F) = \Pr(J = 0) = 1 - \Delta = \mathbb{E}_m(P\|Q).$$

The only case excluded above is $m = 1$ and $\Delta = 1$. In that case $\mathbb{E}_1(P\|Q) = 0$, which implies $P = Q$, and taking $Y_1 = X \sim P$ gives failure probability zero.

Hence the lower bound is attainable, and

$$\boxed{\mathbb{E}_m(P\|Q) = \inf_{P_X, Y_1, \dots, Y_m \in \Gamma_m(P, Q)} \Pr(X \notin \{Y_1, \dots, Y_m\}).}$$