Ramsey problem on Multiplicities of Complete Subgraphs in Nearly Quasirandom Graphs.

F. Franek[1*] and V. Rödl[2]**

[1] Department of Compter Science and Systems, McMaster University, Hamilton, Ontario, L8S 4K1, Canada

[2] Department of Mathematatcis and Computer Science, Emory University, Atlanta, Georgia, 30322 U.S.A.

Abstract

Let $k_t(G)$ be the number of cliques of order t in the graph G. For a graph G with n vertices let $c_t(G) = \frac{k_t(G)+k_t(\bar{G})}{\binom{n}{t}}$. Let $c_t(n) = Min\{c_t(G) : |G| = n\}$ and let $c_t = \lim_{n\to\infty} c_t(n)$. An old conjecture of Erdös [2] related to Ramsey's theorem states that $c_t = 2^{1-\binom{t}{2}}$. Recently it was shown to be false by A. Thomason [12]. It is known that $c_t(G) \sim 2^{1-\binom{t}{2}}$ whenever G is a pseudorandom graph. Pseudorandom graphs - the graphs "which behave like random graphs" - were introduced and studied in [1] and [13]. The aim of this paper is to show that for $t = 4$, $c_t(G) \geq 2^{1-\binom{t}{2}}$ if G is a graph arising from pseudorandom by a small perturbation.

1 Introduction.

Denote by $k_t(G)$ the number of cliques of order t in the graph G. For a graph G with n vertices let $c_t(G) = \frac{k_t(G)+k_t(\bar{G})}{\binom{n}{t}}$. Let $c_t(n) = Min\{c_t(G) : |G| = n\}$, and let $c_t = \lim_{n\to\infty} c_t(n)$. Thus $c_t(n)$ denotes the minimum proportion of monochromatic K_t's in a coloring of the edges of K_n with two colors. An old conjecture of Erdös [2], related to Ramsey's theorem, states that $c_t = 2^{1-\binom{t}{2}}$. It follows from [6], that the conjecture is true for $t = 3$. For a graph H let $k_H(G)$ denote the number of (not necessarily induced) subgraphs of G which are isomorphic to H. Set $c_H(G) = \frac{k_H(G)+k_H(\bar{G})}{\binom{n}{t}}$ where t is the order of H, and $c_H(n) = Min\{c_H(G) : |G| = n\}$. Finally let e denote the number of edges of

*Research supported by NSERC grant OGP0025112

**Research supported by NSF grant DMS 9011850

H. One may ask in general for which graphs H $\lim_{n\to\infty} c_H(n) = \frac{t!}{|Aut(H)|}2^{1-e}$, i.e. for which graphs H the asymptotic minimum of $c_H(G)$ over all graphs G is the same as $c_H(G)$ when G is a random graph. This has been shown to be true for H complete bipartite by Erdös and Moon [3]. Sidorenko [11] showed that it is also true whenever H is a cycle and not true for certain incomplete graph (K_4 less two incident edges). Thomason proved [12] it false for H a complete graph K_t, $t \geq 4$ (i.e. disproved Erdös's conjecture) by constructing for every t an infinite sequence from a single underlying graph, leading to a limit smaller than what the conjecture stipulates. As far as the lower bound, Giraud [7] showed that $c_4 > \frac{1}{46}$.

We shall write $g_1(n) \sim g_2(n)$ in place of $\frac{g_1(n)}{g_2(n)} = 1 + o(1)$. If G is a graph, then $V(G)$ denotes its vertex set, while $E(G)$ denotes the set of its edges. A *neighborhood* $N(u)$ of a vertex $u \in V(G)$ is the set of all vertices of G adjacent to u. The *degree* $d(u)$ of u is the size of its neighborhood.

In [13] and [1] pseudorandom graphs are defined as graphs with the property that $|N(v)| \sim \frac{1}{2}|V|$, and $|N(u) \cap N(v)| \sim \frac{1}{4}|V|$ for almost all $v \in V$ and almost all pairs $u, v \in V$. It was established in [13] and [1] (see also [5], and [8]) that for any fixed t, $k_t(R) + k_t(\bar{R}) \sim 2^{1-\binom{t}{2}}\binom{|V|}{t}$ for any sufficiently large pseudorandom graph R with vertex set V.

Definition 1. *A sequence of graphs $\mathcal{R} = \{R_n\}_{n=0}^{\infty}$ is a* **pseudorandom sequence** *iff for all but $o(|V(R_n)|$ vertices $u \in V(R_n)$, $d(u) = |N(u)|$ satisfies $\left|d(u) - \frac{|V(R_n)|}{2}\right| < o(|V(R_n)|)$, and for all but $o\left(\binom{|V(R_n)|}{2}\right)$ pairs of vertices $u, v \in V(R_n)$, the size $d(u,v)$ of their common neighborhood $N(u) \cap N(v)$ satisfies $\left|d(u,v) - \frac{|V(R_n)|}{4}\right| < o(|V(R_n)|)$.*

Pseudorandom graphs have the following property (cf. [5], [13], [8],[1]):

Theorem 2. *Let $\mathcal{R} = \{R_n\}$ be a pseudorandom sequence of graphs, then there exists a sequence of positive reals $\{\varepsilon_n\}$ so that $\varepsilon_n \to 0$ as $n \to \infty$ and so that for every $V \subset V(R_n)$, $|V| \geq \varepsilon_n |V(R_n)|$, $\left(\frac{1}{2} - \varepsilon_n\right)\binom{|V|}{2} < e < \left(\frac{1}{2} + \varepsilon_n\right)\binom{|V|}{2}$, where the e is the number of edges of R_n induced on a set V.*

For a graph $D = (V, E)$ and $U \subset V$ let $\delta_D(U) = \frac{E \cap [U]^2}{\binom{|U|}{2}}$ denote the edge density of the subgraph induced on U. For a sequence $\mathcal{D} = \{D_n\}$ and $0 < p \leq 1$ let $p\mathcal{D} = \{pD_n\}$ be any sequence with the following property: $V_n = V(pD_n) = V(D_n)$, and there exists $\varepsilon_n \to 0$ such that $\left|\delta_{pD_n}(U) - p\delta_{D_n}(U)\right| < \varepsilon_n$ as $n \to \infty$ for any $U \subset V_n$, $|U| > \varepsilon_n |V_n|$. We can think of pD as a graph obtained from the graph D by flipping a p-biased coin (i.e. the probability of the heads coming up is p, while the probability of the tails coming up is $1-p$) for each edge of D, if the heads shows up the edge is left there, otherwise the edge is removed.

Let G be a graph and let H be a graph on 4 vertices and 5 edges (i.e. K_4 less one edge), then $d(G)$ denotes $c_H(G)$.

For a sequence $\mathcal{G} = \{G_n\}$ of graphs with $|V(G_n)| \to \infty$ as $n \to \infty$, let $d(\mathcal{G}) = lim\ inf\ d(G_n)$.

Answering a question of Erdös (private communication) we proved that (see Theorem 12) $d(\mathcal{G}) \geq \frac{3}{8}$ for any sequence $\mathcal{G}$ of graphs, and the equality holds if and only if $\mathcal{G}$ is a pseudorandom sequence.

We shall employ the following notation: if G and H are graphs such that $V = V(G) = V(H)$ then $G \cap H$ denotes the graph with vertex set V and edge set $E(G) \cap E(H)$, while $G - H$ denotes the graph with vertex set V and edge set $E(G) - E(H)$.

As mentioned above, disproving the conjecture of Erdös, Thomason [12] constructed sequences of graphs $\mathcal{H} = \{H_n\}$ with $c_4(\mathcal{H}) = \lim_{n\to\infty} c_4(H_n) < \frac{1}{32}$. The main puprose of this note is to establish a result which goes in some sense in the opposite direction and prove that for sequences arising from pseudorandom ones by certain small perturbations Erdös's conjecture is valid:

Let $\mathcal{H} = \{H_n\}$ be an arbitrary sequence of graphs and let $\mathcal{R} = \{R_n\}$ be a pseudorandom sequence with $V(R_n) = V(H_n) = V_n$ for all n. Let $D_n = R_n \div H_n$ be a graph whose edges are formed by all pairs one needs to change to obtain H_n from R_n (i.e. $E(D_n)$ is formed by symmetric difference $E(H_n) \div E(R_n)$). It follows that $H_n = R_n \div D_n$ as well. Suppose that we will not carry all the "changes" corresponding to D_n to obtain H_n from R_n but only "changes" on a "random" subgraph pD_n of D_n. This way we obtain a graph sequence $\{p(R_n, D_n)\} = \{R_n \div pD_n\}$. More formally $p(R_n, D_n)$ is a graph sequence that satisfies:

- there exists a sequence $\{\varepsilon_n\}$ of positive reals such that $\varepsilon_n \to 0$ and for

every $U \subset V_n$, $|U| > \varepsilon_n |V_n|$,
$|\delta_{p(R_n,D_n)}(U) - \delta_{R_n - D_n}(U) - (1-p)\delta_{R_n \cap D_n}(U) - p\delta_{D_n - R_n}(U)| < \varepsilon_n$.

Figure 1 shows the relative position of edge sets of R, D, pD, and $p(R, D)$ respectively.

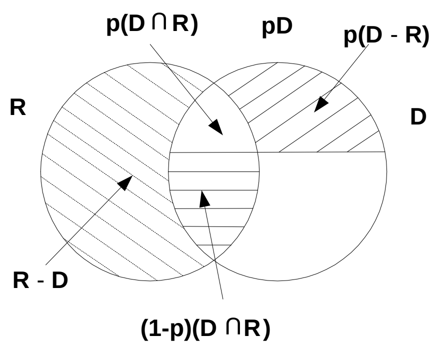

Figure 1:

Now we are ready to formulate our main result (see Theorem 16):

For every $\lambda > \frac{3}{8}$ there exists p_λ, $0 < p_\lambda \leq 1$, such that for every pseudorandom sequence of graphs $\mathcal{R} = \{R_n\}$, and for every sequence of graphs $\mathcal{D} = \{D_n\}$ with $d(\mathcal{R} \div \mathcal{D}) \geq \lambda$, if $c_4(p(\mathcal{R}, \mathcal{D}))$ exists, then $c_4(p(\mathcal{R}, \mathcal{D})) \geq \frac{1}{32} + \frac{1}{8}(\lambda - \frac{3}{8})p^4$ whenever $0 < p \leq p_\lambda$.
Loosely speaking this means that counterexamples to Erdös's conjecture have to differ essentially from pseudorandom graphs.

2 Further Definitions.

Definition 3. *If V, W are disjoint sets of vertices of G, then $e(V, W)$ denotes the number of edges of G with one endpoint in V and the other in W. $\delta(V, W) = \frac{e(V,W)}{|V| \cdot |W|}$ is the edge-density between V and W. If $\varepsilon > 0$, we say that V,W is an ε-uniform pair if $|\delta(V, W) - \delta(V', W')| < \varepsilon$ whenever $V' \subset V$ and $|V'| \geq \varepsilon \cdot |V|$, and $W' \subset W$ and $|W'| \geq \varepsilon \cdot |W|$.*

Definition 4. *Let t be a positive integer. $\vec{x}$ is a t-vector if it is a vector with t^2 real valued entries $x_{i,j}$, $1 \leq i, j \leq t$ and so that $x_{i,j} = x_{j,i}$. $B_t = \{\vec{x} \in R^{t^2} : \vec{x}$ is a t-vector & $|x_{i,j}| \leq 1$ for all $1 \leq i, j \leq t\}$.*

Definition 5. *Let G be a graph. Let $\varepsilon > 0$, and let t be a positive integer. We say that a t-vector $\vec{x}$ ε-represents graph G iff the vertex set of G can be partitioned into t disjoint classes $A_1, ..., A_t$ so that $\big||A_i| - |A_j|\big| \leq 1$ for all $1 \leq i, j \leq t$, and all but $t^2\varepsilon$ pairs $\{A_i, A_j\}$, are ε-uniform, and where $\delta(A_i, A_j) = \frac{1}{2}(1+x_{i,j})$ for all $1 \leq i, j \leq t$, $i \neq j$, and $\delta(A_i, A_i) = \delta(A_i)$ for all $1 \leq i \leq t$. If $\mathcal{G}$ is an infinite sequence of graphs and $\vec{x}$ is a t-vector, we say that $\vec{x}$ represents sequence $\mathcal{G}$ iff there is a sequence of positive reals $\{\varepsilon_n\}$ so that $\varepsilon_n \to 0$ and $\vec{x}$ ε_n-represents G_n, for every n.*

(We use t-vectors as representatives of sequences of graphs. For technical reasons the coordinates of t-vectors are not edge-densities directly, but edge-densities transformed by $p_{i,j} = \frac{1}{2}(1 + x_{i,j})$. Henceforth B_t defined above is the part of R^{t^2} which is meaningful for us. Note also that the origin then represents pseudorandom graphs as $p_{i,j} = \frac{1}{2}$ corresponds to $x_{i,j} = 0$.)

Now we can reformulate Theorem 2 in our language as follows:

Theorem 5. *A t-vector $\vec{x}$ represents a pseudorandom sequence iff $\vec{x} = \vec{o}$.*

At this point we introduce a few polynomials in t^2 variables.

Definition 7.: *Let $\vec{x}$ be t-vector.*

$$C_4(\vec{x}) = \frac{1}{2^6 \cdot t^4} \sum_{1 \leq i,j,k,l \leq t} [(1+x_{i,j})(1+x_{i,k})(1+x_{i,l})(1+x_{j,k})(1+x_{j,l})(1+x_{k,l}) \\ + (1-x_{i,j})(1-x_{i,k})(1-x_{i,l})(1-x_{j,k})(1-x_{j,l})(1-x_{k,l})] \tag{7.1}$$

$$D(\vec{x}) = \frac{6}{2^5 \cdot t^4} \sum_{1 \leq i,j,k,l \leq t} [(1+x_{i,j})(1+x_{i,k})(1+x_{i,l})(1+x_{j,k})(1+x_{j,l}) \\ + (1-x_{i,j})(1-x_{i,k})(1-x_{i,l})(1-x_{j,k})(1-x_{j,l})] \tag{7.2}$$

$$c(\vec{x}) = \frac{3}{2^5 \cdot t^4} \left(4t \sum_{1 \leq i,j,k \leq t} x_{i,j}x_{j,k} + \sum_{1 \leq i,j,k,l \leq t} x_{i,j}x_{k,l} \right) \tag{7.3}$$

$$b(\vec{x}) = \frac{3}{2^5 \cdot t^4} \left(\sum_{1 \leq i,j,k,l \leq t} x_{i,j}x_{i,l}x_{j,k}x_{k,l} + 4 \sum_{1 \leq i,j,k,l \leq t} x_{i,j}x_{i,l}x_{j,l}x_{k,l} \right) \tag{7.4}$$

$$a(\vec{x}) = \frac{1}{2^5 \cdot t^4} \sum_{1 \leq i,j,k,l \leq t} x_{i,j}x_{i,k}x_{i,l}x_{j,k}x_{j,l}x_{k,l} \tag{7.5}$$

Lemma 8.

(a) Let $\{\varepsilon_n\}$ be an infinite sequence of positive reals so that $\varepsilon_n \to 0$. Let $\{t_n\}$ be an infinite sequence of positive integers so that $t_n \to \infty$. Let $\{G_n\}$ be an infinite sequence of graphs. Let for each n, $\vec{x}_n$ be a t_n-vector such that it ε_n-represents graph G_n. Then $\lim_{n\to\infty} c_4(G_n) = \lim_{n\to\infty} C_4(\vec{x}_n)$, and $\lim_{n\to\infty} d(G_n) = \lim_{n\to\infty} D(\vec{x}_n)$.

(b) Let a t-vector $\vec{x}$ represent a graph sequence $\mathcal{G}$. Then $d(\mathcal{G}) = D(\vec{x})$.

Proof We omit the somehow tedious but not difficult proof. For the method see [9]. $\square$

3 Methods and Results.

Lemma 9. *For any t-vector $\vec{x}$,*

$$C_4(\vec{x}) = \tfrac{1}{32} + c(\vec{x}) + b(\vec{x}) + a(\vec{x}) \tag{9.1}$$

$$D(\vec{x}) = \tfrac{3}{8} + 4\Big(2c(\vec{x}) + b(\vec{x})\Big) \tag{9.2}$$

Proof The tedious although straightforward calculations to prove the claim are left to the reader. $\square$

Lemma 10. *For any t-vector $\vec{x} \in B_t$, $|a(\vec{x})| \le \frac{1}{32}$.*

Proof By Eq. 7.5

$$|a(\vec{x})| = \tfrac{1}{2^5 \cdot t^4}\left| \sum_{1\le i,j,k,l\le t} x_{i,j}x_{i,k}x_{i,k}x_{j,k}x_{j,l}x_{k,l}\right| \le$$
$$\tfrac{1}{2^5 \cdot t^4} \sum_{1\le i,j,k,l\le t} |x_{i,j}||x_{i,k}||x_{i,k}||x_{j,k}||x_{j,l}||x_{k,l}| \le \tfrac{1}{2^5 \cdot t^4}\textstyle\sum_{1\le i,j,k,l\le t} 1 =$$
$$\tfrac{1}{2^5 \cdot t^4} t^4 = \tfrac{1}{2^5} = \tfrac{1}{32}.$$

$\square$

Lemma 11. *For any t-vector $\vec{x}$, $c(\vec{x}) \ge 0$.*

Proof $c(\vec{x}) = \frac{3}{32\cdot t^4}\left(4t\sum_{1\le i,j,k\le t} x_{i,j}x_{i,k} + \sum_{1\le i,j,k,l\le t} x_{i,j}x_{k,l}\right)$. First observe that $\sum_{1\le j,k\le t} x_{i,j}x_{i,k} = \left(\sum_{1\le j\le t} x_{i,j}\right)^2$ for any fixed i. Hence $\sum_{1\le i\le t}\left(\sum_{1\le j\le t} x_{i,j}\right)^2 = \sum_{1\le i,j,k\le t} x_{i,j}x_{i,k}$. Then observe that $\left(\sum_{1\le i,j\le t} x_{i,j}\right)^2 = \sum_{1\le i,j\le t}\sum_{1\le k,l\le t} x_{i,j}x_{k,l} = \sum_{1\le i,j,k,l\le t} x_{i,j}x_{k,l}$. Therefore $c(\vec{x}) =$

$\frac{3}{32\cdot t^4}\left(4t\sum_{1\le i\le t}\left(\sum_{1\le j\le t} x_{i,j}\right)^2 + \left(\sum_{1\le i\le t}\sum_{1\le j\le t} x_{i,j}\right)^2\right) \ge 0.$ $\square$

Theorem 12. *Let $\mathcal{G}$ be a sequence of graphs. Then $d(\mathcal{G}) \ge \frac{3}{8}$ and equality holds if and only if $\mathcal{G}$ is a pseudorandom sequence.*

Proof Fix a graph $G \in \mathcal{G}$ of order m. For a pair of vertices $\{v,w\} \in [V(G)]^2$ define $b(v,w)$ as the number of vertices $u \in V(G)$ such that $\{v,u\},\{w,u\} \in E(G)$ provided $\{v,w\} \in V(G)$, or the number of vertices $u \in V(G)$ such that neither $\{v,u\} \in V(G)$, nor $\{w,u\} \in V(G)$ provided $\{v,w\} \notin V(G)$.
Let $q(G) = k_3(G) +\ k_3(\bar{G})$ and set $q = q(G)$. Then $\sum_{\{v,w\}\in[V(G)]^2} b(v,w) = 3q$. Set

$$b(v,w) = \frac{3q}{\binom{m}{2}} + \Delta(v,w) \qquad (12.0)$$

Then
$3q = \sum_{\{v,w\}\in[V(G)]^2} b(v,w) = \sum_{\{v,w\}\in[V(G)]^2}\left(\frac{3q}{\binom{m}{2}} + \Delta(v,w)\right) = 3q + \sum_{\{v,w\}\in[V(G)]^2} \Delta(v,w)$
and hence $\sum_{\{v,w\}\in[V(G)]^2} \Delta(v,w) = 0$. On the other hand the number of non-induced subgraphs on 4 vertices and 5 edges in G and its complement $\bar{G}$ equals $\sum_{\{v,w\}\in[V(G)]^2} \binom{b(v,w)}{2} = \frac{1}{2}\sum_{\{v,w\}\in[V(G)]^2}\left(b^2(v,w) - b(v,w)\right) \ge \frac{1}{2}\left[\frac{9q^2}{\binom{m}{2}} - 3q + \sum_{\{v,w\}\in[V(G)]^2} \Delta^2(v,w)\right]$.

Set $q_0(G) = \binom{\lfloor \frac{|V(G)|}{2} \rfloor}{3} + \binom{\lceil \frac{|V(G)|}{2} \rceil}{3}$ and set $q_0 = q_0(G)$. As $q \geq q_0$ (cf. [6]) and the function $f(q) = \frac{1}{2}\left[\frac{9q^2}{\binom{m}{2}} - 3q\right]$ is increasing for $q \geq q_0 \left(\text{ as } q_0 \geq \frac{1}{6}\binom{m}{2}\right)$, we can conclude that $d(G) \geq \frac{f(q_0)}{\binom{m}{4}}$. Since $f(q_0) \sim \frac{1}{64}m^4$, we can infer that $\lim_{n\to\infty} d(G_n) \geq \frac{3}{8}$, and hence $d(\mathcal{G}) \geq \frac{3}{8}$.

Conversely, let $d(\mathcal{G}) = \frac{3}{8}$. This means that for $G_n \in \mathcal{G}$ both $f(q(G_n)) \left(\text{and hence also } q(G_n)\right)$ and $\sum_{\{v,w\}\in[V(G_n)]^2} \Delta^2(v,w)$ have to be asymptotically minimal. More precisely

$$q(G_n) \sim q_0(G_n) \tag{12.1}$$

and

$$\lim_{n\to\infty} \frac{1}{\binom{|V(G_n)|}{4}} \sum_{\{v,w\}\in[V(G_n)]^2} \Delta^2(v,w) = 0 \tag{12.2}$$

Fix a graph $G \in \mathcal{G}$ of order m, and let d_1, ... ,d_m be the degree sequence of G. Using the argument of [6] one can show that Eq. 12.1 implies that

$$\text{for all but } o(m) \text{ vertices of } G \ |d_i - \tfrac{m}{2}| < o(m) \tag{12.3}$$

Indeed, the number of induced subgraphs of G which have 3 vertices and one or two edges equals to

$$\binom{m}{3} - q(G) = \frac{1}{2}\sum_{i=1}^{m} d_i(m-1-d_i) = \frac{1}{4}\sum_{i=1}^{m}\left((m-1)^2 - d_i^2 - (m-1-d_i)^2\right)$$

and thus is asymptotically maximized when (12.3) holds.

It follows from Eq. 12.0 that

$$\frac{1}{\binom{m}{2}} \sum_{\{v,w\}\in[V(G)]^2} b(v,w) = \frac{3q_0(G)}{\binom{m}{2}} \sim \frac{m}{4} \tag{12.4}$$

On the other hand $\sum_{\{v,w\}\in[V(G)]^2} b^2(v,w) = \frac{9q_0^2(G)}{\binom{m}{2}} + \sum_{\{v,w\}\in[V(G)]^2} \Delta^2(v,w)$ and hence

$$\frac{1}{\binom{m}{2}} \sum_{\{v,w\}\in[V(G)]^2} b^2(v,w) \sim \frac{m^2}{16} \tag{12.5}$$

Combining Eqs. 12.4 and 12.5 we conclude that $b(v,w) \sim \frac{m}{4}$ for all but $o\left(\binom{m}{2}\right)$ pairs $v, w \in V(G)$. Whence $\mathcal{G}$ is a pseudorandom sequence. $\square$

Lemma 13. *$D(\vec{x})$ is strictly minimal for $\vec{x} = \vec{o}$.*

Proof Follows from Lemma 8 (b), Theorem 5, and Theorem 12. $\square$

Corollary 14. *For any t-vector $\vec{x}$, $2c(\vec{x})+b(\vec{x}) \geq 0$. The equality is attained if and only if $\vec{x} = \vec{o}$.*

Proof Follows directly from Lemma 13 using Eq. 9.2. $\square$

Lemma 15. *For any $\lambda > \frac{3}{8}$ there is μ_λ, $0 < \mu_\lambda \leq 1$, so that for any positive integer t and for any $\vec{u} \in B_t$ with $D(\vec{u}) \geq \lambda$, $f_{\vec{u}}(\mu) = a(\vec{u})\mu^6 + b(\vec{u})\mu^4 + c(\vec{u})\mu^2 \geq \frac{1}{8}(\lambda - \frac{3}{8})\mu^4$ for any $\mu \in [0, \mu_\lambda]$.*

Proof We have in view of Eq. 9.2 and abbreviating $a(\vec{u})$ as a, $b(\vec{u})$ as b, and $c(\vec{u})$ as c $D(\vec{u}) = \frac{3}{8}+4(2c+b) \geq \lambda$ which means that $2c+b \geq \lambda_0 = \frac{1}{4}(\lambda - \frac{3}{8}) > 0$, and so $b \geq \lambda_0 - 2c$. Set $\mu_\lambda = min\left\{4\sqrt{\lambda_0}, \frac{1}{\sqrt{2}}\right\}$, and let $\mu \in [0, \mu_\lambda]$.

$$f_{\vec{u}}(\mu) = a\mu^6 + b\mu^4 + c\mu^2 = \mu^2(a\mu^4 + b\mu^2 + c).$$

Since $|a| \leq \frac{1}{32}$, and since $b \geq \lambda_0 - 2c$, $a\mu^4 + b\mu^2 + c \geq -\frac{1}{32}\mu^4 + (\lambda_0 - 2c)\mu^2 + c = (-\frac{1}{32}\mu^4 + \lambda_0\mu^2) + (c - 2c\mu^2) = (*)$
Since $\mu \leq 4\sqrt{\lambda_0}$, $(-\frac{1}{32}\mu^4 + \lambda_0\mu^2) \geq (-\frac{1}{32}\mu^2 16\lambda_0 + \lambda_0\mu^2) = (-\frac{1}{2}\lambda_0\mu^2 + \lambda_0\mu^2) = \frac{1}{2}\lambda_0\mu^2$.
Since $\mu \leq \frac{1}{\sqrt{2}}$, $(c - 2c\mu^2) \geq (c - 2c\frac{1}{2}) = 0$.
Thus $(*) \geq \frac{1}{2}\lambda_0\mu^2 = \frac{1}{8}(\lambda - \frac{3}{8})\mu^2$. It follows that $f_{\vec{u}}(\mu) = \mu^2(a\mu^4 + b\mu^2 + c) \geq \frac{1}{8}(\lambda - \frac{3}{8})\mu^4$. $\square$

Theorem 16. *For every $\lambda > \frac{3}{8}$ there exists p_λ, $0 < p_\lambda \leq 1$, such that for every pseudorandom sequence of graphs $\mathcal{R} = \{R_n\}$, and for every sequence of graphs $\mathcal{D} = \{D_n\}$ with $d(\mathcal{R} \div \mathcal{D}) \geq \lambda$, if $c_4(p(\mathcal{R}, \mathcal{D}))$ exists, then $c_4(p(\mathcal{R}, \mathcal{D})) \geq \frac{1}{32} + \frac{1}{8}(\lambda - \frac{3}{8})p^4$ whenever $0 < p \leq p_\lambda$.*

In the proof of this theorem we shall need the following very powerful theorem:

Szemerédi's Uniformity Lemma. [10] *Given $\varepsilon > 0$, and a positive integer l. Then there exist positive integers $m = m(\varepsilon, l)$ and $n = n(\varepsilon, l)$ with the property that the vertex set of every graph G of order $\geq n$ can be partitioned*

into t disjoint classes $A_1, ..., A_t$ such that
(a) $l \le t \le m$,
(b) $\big||A_i| - |A_j|\big| \le 1$ for all $1 \le i, j \le t$,
(c) All but at most $t^2\varepsilon$ pairs A_i, A_j, $1 \le i, j \le t$, are ε-uniform.

Note that if we set $\frac{1}{2}(1 + u_{i,j}) = \delta(A_i, A_j)$ for all $1 \le i, j \le t$, then the t-vector $\vec{u}$ with the entries $u_{i,j}$ ε-represents the graph G.

Proof of Theorem 16. Let $V_n = V(R_n) = V(D_n)$ for every n. Since $\{R_n\}$ is a pseudorandom sequence, there exists a sequence $\{\varepsilon_n^{(0)}\}$ of positive reals so that $\varepsilon_n^{(0)} \to 0$ as $n \to \infty$ and such that $|\delta_{R_n}(U) - \frac{1}{2}| < \varepsilon_n^{(0)}$ whenever $U \subset V_n$, $|U| > \varepsilon_n^{(0)}|V_n|$. It follows that there must exist a sequence $\{\varepsilon_n^{(1)}\}$ of positive reals so that $\varepsilon_n^{(1)} \to 0$ as $n \to \infty$ and such that:
(1) $|\delta_{R_n}(U, U') - \frac{1}{2}| < \varepsilon_n^{(1)}$ whenever $U, U' \subset V_n$, $U \cap U' = \emptyset$, $|U|, |U'| > \varepsilon_n^{(1)}|V_n|$.
By the definition of $p(R_n, D_n)$, there exists a sequence $\{\varepsilon_n^{(2)}\}$ of positive reals so that $\varepsilon_n^{(2)} \to 0$ as $n \to \infty$ and such that $|\delta_{p(R_n,D_n)}(U) - \delta_{R_n - D_n}(U) - (1-p)\delta_{R_n \cap D_n}(U) - p\delta_{D_n - R_n}(U)| < \varepsilon_n^{(2)}$ whenever $U \subset V_n$, $|U| > \varepsilon_n^{(2)}|V_n|$. It follows that there must exist a sequence $\{\varepsilon_n^{(3)}\}$ of positive reals so that $\varepsilon_n^{(3)} \to 0$ as $n \to \infty$ and such that
(2) $|\delta_{p(R_n,D_n)}(U, U') - \delta_{R_n - D_n}(U, U') - (1-p)\delta_{R_n \cap D_n}(U, U') - p\delta_{D_n - R_n}(U, U')| < \varepsilon_n^{(3)}$ whenever $U, U' \subset V_n$, $U \cap U' = \emptyset$, $|U|, |U'| > \varepsilon_n^{(3)}|V_n|$.
Take an arbitray sequence of positive reals $\{\varepsilon_s^{(4)}\}$ so that $\varepsilon_s^{(4)} \to 0$ as $s \to \infty$. Let $\{l_s\}$ be an arbitrary increasing sequence of positive integers. Let $n(\varepsilon_s^{(4)}, l_s)$ and $m(\varepsilon_s^{(4)}, l_s)$ are from Szemerédi's Uniformity Lemma. Choose an increasing sequence $\{n_s\}_{s=0}^{\infty}$ so that

(a) $|V_{n_s}| \ge n(\varepsilon_s^{(4)}, l_s)$,
(b) $\varepsilon_{n_s}^{(1)} \le \frac{\varepsilon_s^{(4)}}{m(\varepsilon_s^{(4)}, l_s)}$,
(c) $\varepsilon_{n_s}^{(3)} \le \frac{\varepsilon_s^{(4)}}{m(\varepsilon_s^{(4)}, l_s)}$.

Fix an s, and set $n = n_s$. For $\varepsilon = \varepsilon_s^{(4)}$ and $l = l_s$ apply Szemerédi's Uniformity Lemma to the graph $R_n \div D_n$ to obtain a partition of V_n into almost equal classes A_1,...,A_{t_s}, where t_s satisfies $l_s \le t_s \le m(\varepsilon_s^{(4)}, l_s)$ and so that
(3) $\frac{1}{2}(1+u_{i,j}) - \varepsilon_s^{(4)} < \delta_{R_n \div D_n}(U_i, U_j) < \frac{1}{2}(1+u_{i,j}) + \varepsilon_s^{(4)}$ whenever $U_i \subset A_i$, $|U_i| > \varepsilon_s^{(4)}|A_i|$, $U_j \subset A_j$, $|U_j| > \varepsilon_s^{(4)}|A_j|$, for all but $t_s^2\varepsilon_s^{(4)}$ pairs A_i,A_j, and where $\frac{1}{2}(1+u_{i,j}) = \delta_{R_n \div D_n}(A_i, A_j)$ for all $1 \le i, j \le t_s$.

(a) Note that $|A_i| \geq \frac{|V_n|}{m(\varepsilon_s^{(4)},l_s)}$ for every $1 \leq i \leq t_s$.
Also note that (3) means that $\vec{u}_s$ (the t_s-vector with entries $u_{i,j}$, $1 \leq i,j \leq t_s$) $\varepsilon_s^{(4)}$-represents the graph $R_n \div D_n$.
It follows from (1) and (3a) that
(4) $|\delta_{R_n}(U_i,U_j) - \frac{1}{2}| < \varepsilon_n^{(1)}$ whenever $U_i \subset A_i$, $|U_i| > \varepsilon_s^{(4)}|A_i|$, $U_j \subset A_j$, $|U_j| > \varepsilon_s^{(4)}|A_j|$,
as according to (b) $|U_i| > \varepsilon_s^{(4)}|A_i| \geq \frac{\varepsilon_s^{(4)}|V_n|}{m(\varepsilon_s^{(4)},l_s)} \geq \varepsilon_n^{(1)}|V_n|$, and similarly $|U_j| > \varepsilon_s^{(4)}|A_j| \geq \varepsilon_n^{(1)}|V_n|$.
It follows from (2) and (3a) that
(5) $|\delta_{p(R_n,D_n)}(U_i,U_j) - \delta_{R_n-D_n}(U_i,U_j) - (1-p)\delta_{R_n\cap D_n}(U_i,U_j) - p\delta_{D_n-R_n}(U_i,U_j)| < \varepsilon_n^{(3)}$ whenever $U_i \subset A_i$, $|U_i| > \varepsilon_s^{(4)}|A_i|$, $U_j \subset A_j$, $|U_j| > \varepsilon_s^{(4)}|A_j|$,
as according to (c) $|U_i| > \varepsilon_s^{(4)}|A_i| \geq \frac{\varepsilon_s^{(4)}|V_n|}{m(\varepsilon_s^{(4)},l_s)} \geq \varepsilon_n^{(3)}|V_n|$, and similarly $|U_j| > \varepsilon_s^{(4)}|A_j| \geq \varepsilon_n^{(3)}|V_n|$.
Multiplying (4) by $(1-p)$ and using the fact that $R_n = (R_n - D_n) \cup (D_n \cap R_n)$, we obtain
(6) $|(1-p)\delta_{R_n-D_n}(U_i,U_j) + (1-p)\delta_{D_n\cap R_n}(U_i,U_j) - (1-p)\frac{1}{2}| < (1-p)\varepsilon_n^{(1)}$ whenever $U_i \subset A_i$, $|U_i| > \varepsilon_s^{(4)}|A_i|$, $U_j \subset A_j$, $|U_j| > \varepsilon_s^{(4)}|A_j|$.
Multiplying (3) by p and using the fact that $R_n \div D_n = (R_n - D_n) \cup (D_n - R_n)$, we obtain
(7) $\frac{p}{2}(1+u_{i,j}) - p\varepsilon_s^{(4)} < p\delta_{R_n-D_n}(U_i,U_j) + p\delta_{D_n-R_n}(U_i,U_j) < \frac{p}{2}(1+u_{i,j}) + p\varepsilon_s^{(4)}$ whenever $U_i \subset A_i$, $|U_i| > \varepsilon_s^{(4)}|A_i|$, $U_j \subset A_j$, $|U_j| > \varepsilon_s^{(4)}|A_j|$, for all but $t_s^2\varepsilon_s^{(4)}$ pairs A_i,A_j.
Adding (6) and (7) we get
(8) $\frac{1}{2}(1+pu_{i,j}) - p\varepsilon_s^{(4)} - (1-p)\varepsilon_n^{(1)} < \delta_{R_n-D_n}(U_i,U_j) + (1-p)\delta_{R_n\cap D_n}(U_i,U_j) + p\delta_{D_n-R_n}(U_i,U_j) < \frac{1}{2}(1+pu_{i,j}) + p\varepsilon_s^{(4)} + (1-p)\varepsilon_n^{(1)}$ whenever $U_i \subset A_i$, $|U_i| > \varepsilon_s^{(4)}|A_i|$, $U_j \subset A_j$, $|U_j| > \varepsilon_s^{(4)}|A_j|$, for all but $t_s^2\varepsilon_s^{(4)}$ pairs A_i,A_j.
Similarly, adding (5) and (8) we get
(9) $\frac{1}{2}(1+pu_{i,j}) - p\varepsilon_s^{(4)} - (1-p)\varepsilon_n^{(1)} - \varepsilon_n^{(3)} < \delta_{p(R_n,D_n)} < \frac{1}{2}(1+pu_{i,j}) + p\varepsilon_s^{(4)} + (1-p)\varepsilon_n^{(1)} + \varepsilon_n^{(3)}$, whenever $U_i \subset A_i$, $|U_i| > \varepsilon_s^{(4)}|A_i|$, $U_j \subset A_j$, $|U_j| > \varepsilon_s^{(4)}|A_j|$, for all but $t_s^2\varepsilon_s^{(4)}$ pairs A_i,A_j.
Let $p\vec{u}_s$ be the t_s-vector with entries $pu_{i,j}$, and set $\varepsilon_s^{(5)} = p\varepsilon_s^{(4)} + (1-p)\varepsilon_n^{(1)} + \varepsilon_n^{(3)}$. Then $\varepsilon_s^{(5)} \to 0$ as $s \to \infty$, and thus for each s, the t_s-vector $p\vec{u}_s$ $p\varepsilon_s^{(5)}$-

represents the graph $p(R_{n_s}, D_{n_s})$. Since $l_s \to \infty$, also $t_s \to \infty$.

Let $p_\lambda = \mu_\lambda$ from Lemma 15. Fix a p such that $0 < p \le p_\lambda$. If $c_4(p(\mathcal{R},\mathcal{D}))$ exists, then $c_4(p(\mathcal{R},\mathcal{D})) = \lim_{n\to\infty} c_4(p(R_n, D_n)) = \lim_{s\to\infty} c_4(p(R_{n_s}, D_{n_s}))$. By Lemma 8 (a), $\lim_{s\to\infty} c_4(p(R_{n_s}, D_{n_s})) =$ $\lim_{s\to\infty} C_4(p\vec{u}_s)$. By the assumption of the theorem, $d(\mathcal{R} \div \mathcal{D}) \ge \lambda$, and so (as each $\vec{u}_s$ $\varepsilon_s^{(4)}$-represents the graph $R_{n_s} \div D_{n_s}$), for some s_0 big enough, $D(\vec{u}_s) \ge \lambda$ for every $s \ge s_0$. $C_4(p\vec{u}_s) = \frac{1}{32} + a(\vec{u}_s)p^6 + b(\vec{u}_s)p^4 + c(\vec{u}_s)p^2 \ge \frac{1}{32} + \frac{1}{8}(\lambda - \frac{3}{8})p^4$ by Lemma 15. It follows that $\lim_{s\to\infty} C_4(p\vec{u}_s) \ge \frac{1}{32} + \frac{1}{8}(\lambda - \frac{3}{8})p^4$, and so $c_4(p(\mathcal{R},\mathcal{D})) \ge \frac{1}{32} + \frac{1}{8}(\lambda - \frac{3}{8})p^4$. $\square$

Acknowledgement. The authors would like to express their thanks to Peter Frankl for his helpful ideas and discussions concerning the topic of this paper.

References

[1] F.R.K. Chung, R.L. Graham, R.M. Wilson, *Quasi-random Graphs*, Combinatorica 9 (1989), no.4, 345-362.

[2] P. Erdös, *On the number of complete subgraphs contained in certain graphs*, Publ. Math. Inst. Hung. Acad. Sci., VII, ser. A 3 (1962), 459-464.

[3] P. Erdös and J.W. Moon, *On subgraphs on the complete bipartite graph*, Canad.Math.Bull. 7 (1964), 35-39.

[4] F. Franek and V. Rödl, *2-colorings of complete graphs with small number of monochromatic K_4 subgraphs*, to appear in Discr. Math.

[5] R.L. Graham, J.H. Spencer, *A constructive solution to a tournament problem*, Canad.Math.Bull. 14 (1971), 45-48.

[6] A.W. Goodman, *On sets of acquaintances and strangers at any party*, Amer.Math.Monthly, 66 (1959), 778-783.

[7] G. Giraud, *Sur le probleme de Goodman pour les quadrangles et la majoration des nombres de Ramsey*, J.Combin.Theory Ser. B 30 (1979), 237-253.

[8] P. Frankl, V. Rödl, R.M. Wilson, *The number of submatrices of given type in a Hadamard matrix and related results*, J.Comb.Theory, 44 (1988), 317-328.

[9] V. Rödl, *On universality of graphs with uniformly distributed edges*, Discr. Math. 59 (1986), no. 1-2, 125-134.

[10] E. Szemerédi, *Regular partitions of graphs*, in *Proc. Colloque Internat. CNRS (J.-C. Bermond et. al., eds.)*, Paris, 1978, 399-401.

[11] A.F. Sidorenko, *Tsikly v grafakh i funktsional'nye neravenstva*, Matematicheskie Zametki, 46 (1989), no. 5, 72-79 (in Russian).

[12] A. Thomason, *A disproof of a conjecture of Erdös in Ramsey theory*, J. London Math. Soc. (2), 39 (1898), no. 2, 246-255.

[13] A. Thomason, *Pseudo-random graphs*, in "Proceedings of Random Graphs, Poznan, '85", (M. Karonski, ed.), *North-Holland Math. Stud.*, 144, *North-Holland*, *Amsterdam*, 1987.

Received: September 13, 1989
Revised: October 1, 1991