# TWO-FACTORIZATIONS OF SMALL COMPLETE GRAPHS

F.Franek and A.Rosa, McMaster University

Abstract. We enumerate 2-factorizations of $K_9$ of all types, as well as those of several types for $K_{11}$.

## 1. Introduction

A *2-factor* of a graph $G$ is a spanning subgraph of $G$ which is regular of degree 2. Each component of a 2-factor is a cycle.

A *2-factorization* of $G$ is an (edge-disjoint) decomposition of $G$ into 2-factors. A graph $G$ which has a 2-factorization is necessarily regular of even degree. Petersen's theorem (see, e.g., König (1936)) states that the converse is also true.

**Petersen's Theorem.** *Every regular graph of even degree has a 2-factor (and hence, a 2-factorization).*

The *type* of a 2-factor $F$ in an $n$-vertex graph $G$ is a partition $\pi$ of $n$ whose parts are the lengths of the components of $F$. Clearly, there is a bijection between the set of all types of 2-factors of $K_n$ and the set of all partitions of $n$ into parts not less than 3.

Let $\Omega$ be the set of all types of 2-factors of $K_{2m+1}$ and let $<$ be an order (say, lexicographic) on $\Omega$. The *type of a 2-factorization* $\mathcal{F}$ of $K_{2m+1}$ is a nondecreasing sequence $T = (t_1, t_2, \ldots, t_m)$ where $t_i \in \Omega$ for $i \in \{1, \ldots, m\}$. A 2-factorization $\mathcal{F}$ whose all 2-factors are isomorphic (i.e. of the same type $t$) is of *uniform type* (or just *uniform*).

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Clearly, if the number of types of 2-factors of $K_{2m+1}$ is $x$ then the number of distinct types of 2-factorizations of $K_{2m+1}$ is $\binom{m+x-1}{m}$.

The main existence problem for 2-factorizations of $K_{2m+1}$ is to characterize those types $T$ for which a 2-factorization of type $T$ exists.

For uniform 2-factorizations this problem is the well known Oberwolfach problem due to Ringel (see Alspach (1996)). Special cases of the Oberwolfach problem date back to the last century: the Hamiltonian cycle decompositions (each 2-factor is connected) (Lucas (1883), cf. also Alspach, Bermond and Sotteau (1990)) and the Kirkman schoolgirl problem (each 2-factor consists of triangles only) solved only in 1970 by Ray-Chaudhuri and Wilson (1971). It is believed that apart from the type $T = (4,5)$ when $m = 4$ and the type $T = (3,3,5)$ when $m = 5$ for which no 2-factorizations exist, a uniform 2-factorization of $K_{2m+1}$ exists of every other type. [See Alspach (1996) for a brief survey of known results on the Oberwolfach problem.]

As for the existence of 2-factorizations of other than uniform types, general existence results are lacking. Concerning the enumeration, with the exception of Colbourn's results on Hamiltonian decompositions (Colbourn (1982)) and Stinson and Vanstone's lower bound on the number of Kirkman triple systems (Stinson and Vanstone (1984)), no results appear to be known.

The purpose of this paper is to enumerate all 2-factorizations of $K_9$ (by types), as well as those of several types for $K_{11}$.

## 2. Two-factorizations of $K_n$, $n \le 11$

Trivially, there is only one type of 2-factors when $n$ =3 or 5, and in both cases the 2-factorizations are also unique.

The first interesting case is $n = 7$ where there are two types of 2-factors: $a = (3,4), b = (7)$, and four possible types of 2-factorizations. There are two nonisomorphic Hamiltonian 2-factorizations of $K_7$ (Colbourn (1982)), in two other cases the 2-factorization is unique, and in one case it does not exist (see Table 1).

| Type | Solutions | Group orders | Reference |
|---|---|---|---|
| aaa | 1 | 24 | |
| aab | 0 | - | |
| abb | 1 | 4 | |
| bbb | 2 | 42,6 | Colbourn (1982) |

Table 1 2-factorizations of $K_7$

When $n = 9$, there are 4 distinct types of 2-factors:
$A = (3, 3, 3), B = (3, 6), C = (4, 5), D = (9)$,
and 35 possible types of 2-factorizations.

Previously, it was known that there exist exactly 122 nonisomorphic Hamiltonian decompositions (Colbourn (1982)) further that there is a unique solution of type AAAA (a Kirkman triple system of order 9, a.k.a. affine plane of order 3), and that no solution of type CCCC exists (see Alspach (1996)).

We enumerated all nonisomorphic 2-factorizations of $K_9$ on a Sun Sparcstation using backtracking and a simple isomorph rejection algorithm. In the process, the results mentioned above were confirmed. Solutions turn out to exist for 26 of the 35 possible types. The results are summarized in Table 2.

| Type | # | $\|G\|$ | Type | # | $\|G\|$ |
|---|---|---|---|---|---|
| AAAA | 1 | 432 | ACDD | 5 | 2×2,3×1 |
| AAAB | 0 | - | ADDD | 10 | 54,6,3,2×2,5×1 |
| AAAC | 0 | - | BBBB | 5 | 48,2×12,2×2 |
| AAAD | 0 | - | BBBC | 2 | 2,1 |
| AABB | 1 | 8 | BBBD | 9 | 18,3,3×2,4×1 |
| AABC | 0 | - | BBCC | 4 | 4,2,2×1 |
| AABD | 0 | - | BBCD | 19 | 4×2,15×1 |
| AACC | 0 | - | BBDD | 50 | 2×4,9×2,39×1 |
| AACD | 0 | - | BCCC | 5 | 2×12,4,3,2 |
| AADD | 2 | 12,4 | BCCD | 16 | 2×2,14×1 |
| ABBB | 2 | 36,12 | BCDD | 81 | 6×2,75×1 |

| | | | | | |
|---|---|---|---|---|---|
| ABBC | 0 | - | BDDD | 122 | 6×6,2×3,16×2,98×1 |
| ABBD | 2 | 2×1 | CCCC | 0 | - |
| ABCC | 1 | 4 | CCCD | 10 | 2×6,4×2,4×1 |
| ABCD | 2 | 2×1 | CCDD | 36 | 6×2,30×1 |
| ABDD | 7 | 6×2,1 | CDDD | 110 | 16×2,94×1 |
| ACCC | 1 | 12 | DDDD | 122 | 2×8,5×4,3,11×2,103×1 |
| ACCD | 1 | 1 | | | |

Table 2 2-factorizations of $K_9$

Summary of group orders

| 432 | 54 | 48 | 36 | 18 | 12 | 8 | 6 | 4 | 3 | 2 | 1 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 7 | 3 | 9 | 11 | 6 | 92 | 493 | 626 |

A listing of one representative solution for each of the 26 types of 2-factorizations of $K_9$ for which solutions exist is given in Appendix 1. A complete list of all 626 nonisomorphic 2-factorizations of $K_9$ can be obtained from *http://www.cas.mcmaster.ca/∼franya*.

When $n = 11$, there are 6 distinct types of 2-factors:
$E = (3,3,5), F = (3,4,4), G = (3,8), H = (4,7), I = (5,6), J = (11)$,
and 252 possible types of 2-factorizations. It was known that for at least one of these types (EEEEE, i.e. for the Oberwolfach problem OP(3,3,5)) there exists no solution (Piotrowski (1979); cf. also Alspach (1996)). In his paper devoted to the enumeration of Hamiltonian decompositions, Colbourn (1982) noted the combinatorial explosion that occurs for the Hamiltonian decompositions of $K_{11}$: in addition to obtaining all 3140 nonisomorphic Hamiltonian decompositions of $K_{11}$ with nontrivial automorphism group, he generated more than 45 thousand automorphism-free ones before abandoning the task.

As an exceedingly large number of nonisomorphic solutions may be expected whenever the type of a 2-factorization includes at least one type of 2-factors with "large" cycles, we concentrated on establishing the existence of (at least one) 2-factorization of given type, and on the enumeration of those types of 2-factorizations that include only 2-factors of types including only ”small” cycles, i.e. those of types E and F.

Somewhat surprisingly, it turns out that 2-factorizations of $K_{11}$ exist of all but one of the 252 possible types. The only type not admitting a 2-factorization of $K_{11}$ is the already mentioned type EEEEE (thus, in the process, we verified the result of Piotrowski (1979) that there exists no solution to the Oberwolfach problem OP(3,3,5)).

A listing of one representative solution for each of the 251 other types of 2-factorizations of $K_{11}$ can be obtained from
*http://www.cas.mcmaster.ca/~franya.*

There are altogether 15 nonisomorphic 2-factorizations of $K_{11}$ containing 2-factors of types E and F only. A brief summary is given in Table 3. The 15 solutions are listed in Appendix 2.

| Types | Solutions | Group orders |
|---|---|---|
| $E^5$ | 0 | - |
| $E^4F$ | 4 | $4, 2\times 2, 1$ |
| $E^3F^2$ | 1 | 1 |
| $E^2F^3$ | 6 | $6\times 1$ |
| $EF^4$ | 3 | 4,2,1 |
| $F^5$ | 1 | 4 |

Table 3 2-factorizations of $K_{11}$ with 2-factors of type E, F

We note that when $n = 13$, there are 10 types of 2-factors and 5005 possible types of 2-factorizations.

## 3. Conclusion

Let $s = s_n(s \leq (n-1)/2)$ be the largest integer such that given any list $(t_1, \dots, t_s)$ of $s$ (not necessarily distinct) types of 2-factors of $K_n$, there exists a 2-factorization of $K_n$ containing $s$ two-factors of the types $(t_1, \dots, t_s)$.

The results of the previous section show that $s_7 = 2$, and $s_9 = 2$ (when $n = 9$, of the 20 distinct lists with 3 types, one cannot be completed to a 2-factorization: AAC), and $s_{11} = 4$.

Note that in each of these cases, $s_n < (n-1)/2$. However, we believe that for $n \geq 13, s_n = (n-1)/2$.

The situation is different, however, if one asks what is the largest $s = s_n$ such that given any set of $s$ (edge-disjoint) 2-factors of $K_n$, it can be completed to a 2-factorization by using additional 2-factors of prescribed types. It follows easily from Hoffman, Rodger and Rosa (1993) and Rees, Rosa and Wallis (1994) that for any $s \geq (n+1)/4 \quad (s \leq (n-1)/2$, of course), there exists a set $\mathcal{F}$ of $s$ 2-factors such that not a *single* 2-factor of type $t$ (for some type $t$) may be added disjointly to $\mathcal{F}$.

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## Appendix 1

**Sample 2-factorizations of $K_9$ by type.**

Below is a list of 26 two-factorizations of $K_9$, one of each type for which a solution exists. A 2-factorization from among those having an automorphism group of the largest order is given for each type.

| Type AAAA | Type AABB | Type AADD |
|---|---|---|
| (012)(345)(678) | (012)(345)(678) | (012)(345)(678) |
| (036)(147)(258) | (036)(147)(258) | (036)(147)(258) |
| (048)(156)(237) | (051837)(246) | (042375618) |
| (057)(138)(246) | (132756)(048) | (051384627) |
| $\vert G\vert = 432$ | $\vert G\vert = 8$ | $\vert G\vert = 12$ |

| Type ABBB | Type ABBD | Type ABCC |
|---|---|---|
| (012)(345)(678) | (012)(345)(678) | (012)(345)(678) |
| (046238)(157) | (138256)(047) | (147258)(036) |
| (056137)(248) | (142758)(036) | (13756)(0428) |
| (147258)(036) | (051732648) | (23846)(0517) |
| $\vert G\vert = 36$ | $\vert G\vert = 1$ | $\vert G\vert = 4$ |

| Type ABCD | Type ABDD | Type ACCC |
|---|---|---|
| (012)(345)(678) | (012)(345)(678) | (012)(345)(678) |
| (142758)(036) | (142758)(036) | (23846)(0517) |
| (04738)(1526) | (046152837) | (24758)(0316) |
| (056482317) | (056231748) | (25637)(0418) |
| $\vert G\vert = 1$ | $\vert G\vert = 2$ | $\vert G\vert = 12$ |

| Type ACCD | Type ACDD | Type ADDD |
|---|---|---|
| (012)(345)(678) | (012)(345)(678) | (012)(345)(678) |
| (17238)(0465) | (24758)(0316) | (036147258) |
| (24758)(0316) | (046381527) | (048156237) |
| (073625148) | (056237148) | (057138246) |
| $\vert G\vert = 1$ | $\vert G\vert = 2$ | $\vert G\vert = 54$ |

| Type BBBB | Type BBBC | Type BBBD |
|---|---|---|
| (012345)(678) | (135724)(068) | (012345)(678) |
| (026357)(148) | (154628)(037) | (147265)(038) |
| (047316)(258) | (163847)(025) | (163758)(024) |
| (156427)(038) | (01234)(5678) | (064825317) |

| | | |
|---|---|---|
| $\lvert G\rvert = 48$ | $\lvert G\rvert = 2$ | $\lvert G\rvert = 18$ |

| Type BBCC | Type BBCD | Type BBDD |
|---|---|---|
| (036418)(257) | (136847)(025) | (012345)(678) |
| (062847)(135) | (157246)(038) | (026357)(148) |
| (01234)(5678) | (01234)(5678) | (031728564) |
| (16837)(0245) | (062814537) | (061524738) |
| $\lvert G\rvert = 4$ | $\lvert G\rvert = 2$ | $\lvert G\rvert = 4$ |

| Type BCCC | Type BCCD | Type BCDD |
|---|---|---|
| (037415)(268) | (031846)(257) | (024157)(368) |
| (01234)(5678) | (01234)(5678) | (01234)(5678) |
| (02457)(1638) | (16837)(0245) | (035271648) |
| (13527)(0648) | (074153628) | (054731826) |
| $\lvert G\rvert = 12$ | $\lvert G\rvert = 2$ | $\lvert G\rvert = 2$ |

| Type BDDD | Type CCCD | Type CCDD |
|---|---|---|
| (012345)(678) | (01234)(5678) | (01234)(5678) |
| (025163748) | (16837)(0245) | (13684)(0257) |
| (035718264) | (25746)(0318) | (035461728) |
| (065831427) | (063514827) | (051837426) |
| $\lvert G\rvert = 6$ | $\lvert G\rvert = 6$ | $\lvert G\rvert = 2$ |

| Type CDDD | Type DDDD |
|---|---|
| (01234)(5678) | (012345678) |
| (024516837) | (024163857) |
| (035714628) | (031527486) |
| (052748136) | (046281735) |
| $\lvert G\rvert = 2$ | $\lvert G\rvert = 8$ |

## Appendix 2

**2-factorizations of $K_{11}$ with 2-factors of type E and F only.**

Each of the 15 two-factorizations below contains the 2-factor (0123)(4567)(89A) which is not listed.

Type $E^4F$

1. (02849)(157)(36A)
(04A25)(169)(378)
(0685A)(134)(279)
(07A18)(246)(359)
$|G| = 1$

2. (04185)(269)(37A)
(08759)(136)(24A)
(25348)(06A)(179)
(38649)(027)(15A)
$|G| = 2$

3. (04368)(159)(27A)
(05387)(16A)(249)
(0964A)(137)(258)
(3975A)(026)(148)
$|G| = 2$

4. (04819)(257)(36A)
(05968)(137)(24A)
(28539)(07A)(146)
(34978)(026)(15A)
$|G| = 4$

Type $E^3F^2$

5. (04178)(25A)(369)
(05729)(16A)(348)
(0794A)(135)(268)
(0246)(1859)(37A)
$|G| = 1$

Type $E^2F^3$

6. (05A17)(268)(349)
(0614A)(259)(378)
(0248)(1579)(36A)
(0469)(1358)(27A)
$|G| = 1$

7. (04169)(27A)(358)
(05A17)(268)(349)
(0248)(1579)(36A)
(064A)(1378)(259)
$|G| = 1$

8. (04359)(168)(27A)
(06937)(14A)(258)

9. (05817)(26A)(349)
(0914A)(257)(368)

(0248)(1579)(36A)
(1387)(2649)(05A)
$|G| = 1$

(0248)(1596)(37A)
(135A)(2879)(046)
$|G| = 1$

10. (0634A)(158)(279)
(07839)(146)(25A)
(0248)(196A)(357)
(0495)(13A7)(268)
$|G| = 1$

11. (06A59)(134)(278)
(24936)(07A)(158)
(0468)(192A)(357)
(1697)(384A)(025)
$|G| = 1$

Type $EF^4$

12. (26859)(07A)(134)
(0248)(1579)(36A)
(0496)(1738)(25A)
(0539)(164A)(278)
$|G| = 1$

13. (34978)(025)(16A)
(0426)(1859)(37A)
(0728)(14A5)(369)
(092A)(1357)(468)
$|G| = 2$

14. (35869)(027)(14A)
(0428)(1597)(36A)
(052A)(1649)(378)
(0629)(1348)(57A)
$|G| = 4$

Type $F^5$

15. (0248)(1597)(36A)
(0496)(1358)(27A)
(0529)(164A)(378)
(075A)(1439)(268)
$|G| = 4$