Erdös' conjecture on multiplicities of complete subgraphs for nearly quasirandom graphs

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Outline

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$k_t(G)$ the number of cliques of order t in a graph G

$$c_t(G) = \frac{k_t(G) + k_t(\overline{G})}{\binom{|G|}{t}}$$

$$c_t(n) = \min\{c_t(G) : |G| = n\}$$

$$c_t = \lim_{n\to\infty} c_t(n)$$

A 1962 conjecture of Erdös related to Ramsey's theorem states that

$$c_t = 2^{1-\binom{t}{2}}$$

The motivation for the conjecture:

- trivially true for $t = 2$ (edges)
- from *Goodman*'s (1957) work follows for $t = 3$ (triangles)
- true for random graphs

(1987) Shown false by *A. Thomason* for all $t \geq 4$ by providing upper bounds:

- $c_4 < 0.976 \cdot 2^{-5}$
- $c_5 < 0.906 \cdot 2^{-9}$
- $c_t < 0.936 \cdot 2^{1-\binom{t}{2}}$, for $t > 5$

- (1993) *F.* and *Rödl* using a computer search provided a simpler counterexample for $t = 4$ with the same bound
- (1996) *Jagger*, *Šťovíček*, *Thomason*: $c_5 \leq 0.8801 \cdot 2^{-9}$
- (2002) *F.*: $c_6 \leq 0.744514 \cdot 2^{-14}$
- (1968) The only known lower bound is due to *Giraud*: $c_4 > \frac{1}{46}$

Outline

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   - Background
   - Preliminaries
2. The contribution
   - Main Result
   - Basic Ideas for the Proofs

Quasirandom and nearly quasirandom graphs

It was known that $c_t(G) \sim 2^{1-\binom{t}{2}}$ whenever G is a quasirandom graph.

Quasirandom graphs - the graphs “that behave like random graphs" - were introduced and studied by *F.R.K. Chung*, *R.L. Graham*, *R.M. Wilson*, and *A. Thomason*.

The aim of this presentation is to show that for $t = 4$, $c_t(G) \geq 2^{1-\binom{t}{2}}$, if G is a nearly quasirandom graph, i.e. a graph arising from quasirandom by a small perturbation.

Quasirandom and nearly quasirandom graphs

Quasirandom graphs are defined as graphs with the property that

- $|N(v)| \sim \frac{1}{2}|V|$, and
- $|N(u) \cap N(v)| \sim \frac{1}{4}|V|$ for almost all $v \in V$ and almost all pairs $u, v \in V$.

where $N(v)$ denotes the neighbourhood of vertex v.

For any fixed t, $k_t(R) + k_t(\overline{R}) \sim 2^{1-\binom{t}{2}}\binom{|V|}{t}$ for any sufficiently large quasirandom graph R with vertex set V.

Quasirandom and nearly quasirandom graphs

A **quasirandom sequence** of graphs $\mathcal{R} = \{R_n\}_{n=0}^{\infty}$

- for all but $o(|V(R_n)|)$ vertices $u \in V(R_n)$, $d(u) = |N(u)|$ satisfies $\left| d(u) - \frac{|V(R_n)|}{2} \right| < o(|V(R_n)|)$, and
- for all but $o\left(\binom{|V(R_n)|}{2}\right)$ pairs of vertices $u, v \in V(R_n)$, the size $d(u, v)$ of their common neighbourhood $N(u) \cap N(v)$ satisfies $\left| d(u, v) - \frac{|V(R_n)|}{4} \right| < o(|V(R_n)|)$.

Quasirandom and nearly quasirandom graphs

Theorem (Chung,Graham,Wilson,Thomason)

Let $\mathcal{R} = \{R_n\}$ be a quasirandom sequence of graphs, then there exists a sequence of positive reals $\{\varepsilon_n\}$ so that $\varepsilon_n \to 0$ as $n \to \infty$ and so that for every $V \subset V(R_n)$, $|V| \geq \varepsilon_n |V(R_n)|$, $\left(\frac{1}{2} - \varepsilon_n\right)\binom{|V|}{2} < e < \left(\frac{1}{2} + \varepsilon_n\right)\binom{|V|}{2}$, where the e is the number of edges of R_n induced on a set V.

Quasirandom and nearly quasirandom graphs

For a graph $D = (V, E)$ and $U \subset V$ let $\delta_D(U) = \frac{E \cap [U]^2}{\binom{|U|}{2}}$ denote the edge density of the subgraph induced on U.

For a sequence $\mathcal{D} = \{D_n\}$ and $0 < p \leq 1$ let

$p\mathcal{D} = \{pD_n\}$ be any sequence with the following property:

$V_n = V(pD_n) = V(D_n)$, and there exists $\varepsilon_n \to 0$ such that

$\left|\delta_{pD_n}(U) - p\delta_{D_n}(U)\right| < \varepsilon_n$ as $n \to \infty$ for any $U \subset V_n$,

$|U| > \varepsilon_n |V_n|$.

Quasirandom and nearly quasirandom graphs

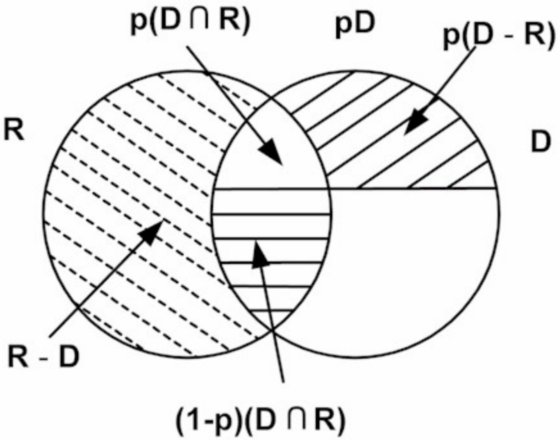
We can think of pD as a graph obtained from the graph D by flipping a p-biased coin for each edge of D to decide to remove it or to leave it. (p remove it, $(1-p)$ leave it)

$\mathcal{D} = \{D_n\}$ an arbitrary sequence of graphs
$\mathcal{R} = \{R_n\}$ a quasirandom sequence

$p(\mathcal{R}, \mathcal{D}) = \{p(R_n, D_n)\} = \{R_n \triangle pD_n\}$

$\triangle$ *denotes symmetric difference*

Quasirandom and nearly quasirandom graphs



Quasirandom and nearly quasirandom graphs

$p(\mathcal{R}, \mathcal{D}) = \{p(R_n, D_n)\}$ has the following property:

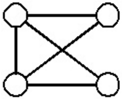
there exists a sequence $\{\varepsilon_n\}$ of positive reals such that $\varepsilon_n \to 0$ and for every $U \subset V_n$, $|U| > \varepsilon_n |V_n|$,
$|\delta_{p(R_n,D_n)}(U) - \delta_{R_n - D_n}(U) - (1-p)\delta_{R_n \cap D_n}(U) - p\delta_{D_n - R_n}(U)| < \varepsilon_n$.

So the farther we go in the sequence, the more it looks like the diagram

Quasirandom and nearly quasirandom graphs

$d_H(G) = \frac{i_H(G)+i_H(\overline{G})}{2}$, where $i_H(G)$ is the number of isomorphic copies (not necessarily induced) of H in G.

$Z = K_4$ less one edge



$d(G) = d_Z(G)$.

For $\mathcal{G} = \{G_n\}$, $d(\mathcal{G}) = \liminf d(G_n)$.

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Theorem 1

Theorem

Let $\mathcal{G}$ be a sequence of graphs. Then $d(\mathcal{G}) \geq \frac{3}{8}$ and equality holds if and only if $\mathcal{G}$ is a quasirandom sequence.

This answered a question of Erdös

Theorem 2

Theorem

For every $\lambda > \frac{3}{8}$ there exists p_λ, $0 < p_\lambda \leq 1$, such that for every quasirandom sequence of graphs $\mathcal{R} = \{R_n\}$, and for every sequence of graphs $\mathcal{D} = \{D_n\}$ with $d(\mathcal{R} \triangle \mathcal{D}) \geq \lambda$, if $c_4(p(\mathcal{R}, \mathcal{D}))$ exists, then $c_4(p(\mathcal{R}, \mathcal{D})) \geq \frac{1}{32} + \frac{1}{8}(\lambda - \frac{3}{8})p^4$ whenever $0 < p \leq p_\lambda$.

Loosely speaking: counterexamples to Erdös' conjecture have to differ essentially from quasirandom graphs.

We call $p(\mathcal{R}, \mathcal{D})$ a nearly quasirandom sequence.

Outline

Basic Ideas for the Proofs

We use t-vectors to represent sequences of graphs.

$\vec{x}$ is a t-vector with t^2 real valued entries $x_{i,j}$, $1 \leq i, j \leq t$ and so that $x_{i,j} = x_{j,i}$.

$B_t = \{\vec{x} \in R^{t^2} : \vec{x}$ is a t-vector & $|x_{i,j}| \leq 1$ *for all* $1 \leq i, j \leq t\}$. unit ball

V, W disjoint sets of vertices of a graph G are ε-uniform if $|\delta(V, W) - \delta(V', W')| < \varepsilon$ whenever $V' \subset V$ and $|V'| \geq \varepsilon \cdot |V|$, and $W' \subset W$ and $|W'| \geq \varepsilon \cdot |W|$.

Basic Ideas for the Proofs

t-vector $\vec{x}$ ε-represents a graph G

- the vertex set of G can be partitioned into t disjoint classes $A_1, \ldots, A_t$
- $\big||A_i| - |A_j|\big| \leq 1$ for all $1 \leq i, j \leq t$, and
- all but $t^2\varepsilon$ pairs $\{A_i, A_j\}$, are ε-uniform, and
- $\delta(A_i, A_j) = \frac{1}{2}(1+x_{i,j})$ for all $1 \leq i, j \leq t$, $i \neq j$, and
- $\delta(A_i, A_i) = \delta(A_i)$ for all $1 \leq i \leq t$.

Basic Ideas for the Proofs

t-vector $\vec{x}$ represents a sequence of graphs $\mathcal{G}$ iff there is a sequence of positive reals $\{\varepsilon_n\}$ so that $\varepsilon_n \to 0$ and $\vec{x}$ ε_n-represents G_n, for every n.

Theorem 1 can be reformulated as: *$\vec{x}$ represents a quasirandom sequence iff $\vec{x} = \vec{o}$.*

Basic Ideas for the Proofs

- $C_4(\vec{x}) = \frac{1}{2^6 \cdot t^4} \sum_{1 \le i,j,k,l \le t} [(1+x_{i,j})(1+x_{i,k})(1+x_{i,l})(1+x_{j,k})(1+x_{j,l})(1+x_{k,l}) + (1-x_{i,j})(1-x_{i,k})(1-x_{i,l})(1-x_{j,k})(1-x_{j,l})(1-x_{k,l})]$
- $D(\vec{x}) = \frac{6}{2^5 \cdot t^4} \sum_{1 \le i,j,k,l \le t} [(1+x_{i,j})(1+x_{i,k})(1+x_{i,l})(1+x_{j,k})(1+x_{j,l}) + (1-x_{i,j})(1-x_{i,k})(1-x_{i,l})(1-x_{j,k})(1-x_{j,l})]$

Basic Ideas for the Proofs

- $c(\vec{x}) = \frac{3}{2^5 \cdot t^4} \left(4t \sum_{1 \le i,j,k \le t} x_{i,j} x_{j,k} + \sum_{1 \le i,j,k,l \le t} x_{i,j} x_{k,l} \right)$
- $b(\vec{x}) = \frac{3}{2^5 \cdot t^4} \left(\sum_{1 \le i,j,k,l \le t} x_{i,j} x_{i,l} x_{j,k} x_{k,l} + 4 \sum_{1 \le i,j,k,l \le t} x_{i,j} x_{i,l} x_{j,l} x_{k,l} \right)$
- $a(\vec{x}) = \frac{1}{2^5 \cdot t^4} \sum_{1 \le i,j,k,l \le t} x_{i,j} x_{i,k} x_{i,l} x_{j,k} x_{j,l} x_{k,l}$

Basic Ideas for the Proofs

- If $\varepsilon_n \to 0$, $t_n \to \infty$, each t_n-vector $\vec{x}_n$ ε-represents G_n, then $\lim_{n\to\infty} c_4(G_n) = \lim_{n\to\infty} C_4(\vec{x}_n)$
- If t-vector $\vec{x}$ represents a graph sequence $\mathcal{G}$, then $d(\mathcal{G}) = D(\vec{x})$
- For any t-vector $\vec{x}$, $C_4(\vec{x}) = \frac{1}{32} + c(\vec{x}) + b(\vec{x}) + a(\vec{x})$
- For any t-vector $\vec{x}$, $D(\vec{x}) = \frac{3}{8} + 4\Big(2c(\vec{x}) + b(\vec{x})\Big)$
- For any t-vector $\vec{x} \in B_t$, $|a(\vec{x})| \leq \frac{1}{32}$
- For any t-vector $\vec{x}$, $c(\vec{x}) \geq 0$

Basic Ideas for the Proofs

The facts established up to here are sufficient to prove Theorem 1. More facts needed to prove Theorem 2.

- $D(\vec{x})$ is strictly minimal for $\vec{x} = \vec{o}$
- For any t-vector $\vec{x}$, $2c(\vec{x}) + b(\vec{x}) \geq 0$ The equality is attained iff $\vec{x} = \vec{o}$
- For any $\lambda > \frac{3}{8}$ there is μ_λ, $0 < \mu_\lambda \leq 1$, so that for any positive integer t and for any $\vec{u} \in B_t$ with $D(\vec{u}) \geq \lambda$, $f_{\vec{u}}(\mu) = a(\vec{u})\mu^6 + b(\vec{u})\mu^4 + c(\vec{u})\mu^2 \geq \frac{1}{8}(\lambda - \frac{3}{8})\mu^4$ for any $\mu \in [0, \mu_\lambda]$

Basic Ideas for the Proofs

- *Szemerédi's Uniformity Lemma*
  Given $\varepsilon > 0$, and a positive integer l. Then there exist positive integers $m = m(\varepsilon, l)$ and $n = n(\varepsilon, l)$ with the property that the vertex set of every graph G of order $\geq n$ can be partitioned into t disjoint classes $A_1, ..., A_t$ such that
  (a) $l \leq t \leq m$,
  (b) $\big||A_i| - |A_j|\big| \leq 1$ for all $1 \leq i, j \leq t$,
  (c) All but at most $t^2\varepsilon$ pairs A_i, A_j, $1 \leq i, j \leq t$, are ε-uniform.

The facts established up to here are sufficient to prove Theorem 2.

Summary

- When counting monochromatic copies of Z, the quasirandom graph attains the minimum $\geq \frac{3}{8}$ *answering a question of Erdös*
- For counting monochromatic copies of K_4, Erdös' conjecture holds true for nearly quasirandom graphs *though in general the conjecture is not true*
- Further research will concentrate on pushing down the upper bounds (cf. presentation by A. Baker).

THANK YOU