Isomorphisms of Infinite Steiner Triple Systems

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Abstract.

A generalization and extension of the method used in [GGP] is presented to prove that for any infinite cardinal κ there are 2^λ mutually non-isomorphic Steiner triple systems of size κ that admit 2^κ automorphisms, where $\lambda = min\{\kappa, 2^{\aleph_0}\}$. If $\kappa \leq 2^{\aleph_0}$, then there are 2^κ mutually non-isomorphic rigid Steiner triple systems of size κ [rigid = admits no non-trivial automorphism], and also there are 2^κ mutually non-isomorphic Steiner triple systems of size κ that admit exactly one non-trivial automorphism.

Introduction.

Mathematicians have been mostly interested in finite Steiner systems, and so the published literature dealing with finite Steiner systems is quite extensive (see e.g. [DR]). There has been very little published on infinite Steiner systems (see e.g. [So], [Si], [V], [GGP], [N]).

We are going to present a general method to generate Steiner triple systems of any desirable (infinite) size κ with features controlling the number of quadrilaterals and anti-quadrilaterals in them. Such systems will admit 2^κ automorphisms (see (3.1)). Similarly as in [GGP], the "structure" of the quadrilateral graph (which is an invariant under isomorphism) will be used to generate the desired number of non-isomorphic systems with many automorphisms (see (3.1)), and to "eliminate" possible automorphisms as well (for rigid systems see (3.2), for systems with exactly one non-trivial automorphism see (3.3)). Since by definition quadrilateral graphs are countable in their nature and hence there are at most $2^{\aleph_0}$ "different" ones, this method does work only for numbers up to $2^{\aleph_0}$. For higher cardinalities a different invariant has to be used.

(1) Notation and definitions.

The standard set-theoretical notation is used. Lower case Greek letters will denote ordinal numbers. Z denotes the set of all integers, N denotes the set of all non-negative integers, Q is the set of all rational numbers.

Let $\langle G, +\rangle$ be an additive Abelian group of size κ (κ infinite) with O its identity element.

(1.1) As usual, define $n{\cdot}x$ for any $n{\in}Z$ and any $x{\in}G$ by

$0{\cdot}x = O$,

for $n > 0$ by induction : $(n{+}1){\cdot}x = n{\cdot}x{+}x$,

for $n < 0$ by $n{\cdot}x = (-n){\cdot}(-x)$.

Assume that the group G satisfies

(1.2) $(\forall x{\in}G{-}\{O\})(\exists y{\in}G{-}\{O\})(x = 2{\cdot}y)$.

(1.3) $(\forall x{\in}G)(\forall n{\in}Z)(n{\cdot}x = O \Rightarrow (n = 0 \vee x = O))$.

It is easy to prove that the y such that $x = 2{\cdot}y$ from (1.2) is unique and so we shall denote it as $\frac{x}{2}$ or $2^{-1}{\cdot}x$.

Also, the following properties hold for any $n, m{\in}Z$, and $x{\in}G$:

$n{\cdot}x{+}n{\cdot}y = n{\cdot}(x{+}y)$,

$n{\cdot}x{+}m{\cdot}x = (n{+}m){\cdot}x$,

$-(n{\cdot}x) = (-n){\cdot}x = n{\cdot}(-x)$,

$n(m{\cdot}x) = (nm){\cdot}x$,

$n{\cdot}x = m{\cdot}x \Rightarrow x = O \vee n = m$.

Define a relation $\sim$ on $G{-}\{O\}$ by

(1.4) $x{\sim}y$ iff $(\exists n{\in}Z)(x = (-2)^n{\cdot}y)$.

It is easy to show that $\sim$ is an equivalence relation on $G{-}\{O\}$.

(1.5) For $x{\in}G{-}\{O\}$ define $[x] = \{y{\in}G{-}\{O\}{:}y{\sim}x\}$.

$|[x]| = \aleph_0$ (define $f{:}[x]{\to}Z$ by $f((-2)^n{\cdot}x) = n$; then f is a bijection).

(1.6) Let V be a set. Then

$\langle V, B\rangle$ is a **Steiner Triple System** iff $B{\subset}[V]^3$ and for any $\{x, y\}{\in}[V]^2$ there is a unique $b{\in}B$ so that $\{x, y\}{\subset}b$. The elements of B are usually called **blocks** or **triples**, set V is usually called **index set** of the system.

(1.7) The **size** of the system is the size of its set of blocks, B.

(1.8) Let $f{:}V_1 \to V_2$, then f **induces** a mapping $f^*{:}[V_1]^3 \to [V_2]^3$ defined by $f^*(\{x,y,z\}) = \{f(x), f(y), f(z)\}$ and a mapping $f^{**}{:}[[V_1]^3]^4] \to [[V_2]^3]^4$ defined by $f^{**}(\{b_1,b_2,b_3,b_4\}) = \{f^*(b_1), f^*(b_2), f^*(b_3), f^*(b_4)\}$.

If f is a bijection, so are f^* and f^{**}. Moreover,

$f^*(b_1 \cup b_2) = \{f(x){:}x \in b_1 \cup b_2\}$; $f^*(b_1 \cap b_2) = \{f(x){:}x \in b_1 \cap b_2\}$; $f^*(b_1 - b_2) = \{f(x){:}x \in b_1 - b_2\}$;

$f^{**}(q_1 \cup q_2) = \{f^*(x){:}x \in q_1 \cup q_2\}$; $f^{**}(q_1 \cap q_2) = \{f^*(x){:}x \in q_1 \cap q_2\}$; $f^{**}(q_1 - q_2) = \{f^*(x){:}x \in q_1 - q_2\}$.

(1.9) Let $\langle V_1, B_1 \rangle$ and $\langle V_2, B_2 \rangle$ be Steiner triple systems. A mapping $f{:}V_1 \to V_2$ is an **isomorphism** iff $f{:}V_1 \to V_2$ is a bijection, and also the induced mapping f^* restricted to B_1 is a bijection onto B_2.

(1.10) An **automorphism** of a Steiner triple system S is an isomorphism from S onto S.

(1.11) A Steiner triple system is **rigid** iff its group of automorphisms has size 1 (i.e. it admits no non-trivial automorphism).

(1.12) A group of 4 blocks is called a **quadrilateral** iff their (set) union has size 6 and every pair-wise (set) intersection has size 1.

Note that a quadrilateral is determined by 6 elements such that each occurs exactly in two blocks; e.g. $\{a,b,c\}, \{a,y,z\}, \{x,b,z\}$, and $\{x,y,c\}$.

All 6 elements of a quadrilateral are completely determined by any 3 blocks of the quadrilateral.

(1.13) A **quadrilateral graph** of a Steiner triple system S is a graph whose vertices are quadrilaterals of S and two quadrilaterals form an edge if their (set) intersection is non-empty (i.e. they have at least one block in common).

(1.14) If b_1, b_2, b_3, and b_4 form a quadrilateral, the **complementary quadrilateral** is formed by c_1, c_2, c_3, and c_4 where $c_i = \bigcup\{b_j{:}1 \le j \le 4\} - b_i$, $1 \le i \le 4$. In other words, each c_i is formed by the complement (in the union of the quadrilateral) of b_i. If q is a quadrilateral $\bar{q}$ will denote the complementary quadrilateral. Note that $\overline{(\bar{q})} = q$.

(2) Construction and auxiliary results.

(2.1) Let $S_1 = \langle V_1, B_1 \rangle$ and $S_2 = \langle V_2, B_2 \rangle$ be Steiner triple systems. Let $f{:}V_1 \to V_2$ be an isomorphism of S_1 onto S_2. Then

(i) $q \in [[V_1]^3]^4$ is a quadrilateral iff $f^{**}(q) \in [[V_2]^3]^4$ is a quadrilateral;

(ii) q is a quadrilateral of S_1 (i.e. $q \in [B_1]^4$ and it is a quadrilateral) iff $f^{**}(q)$ is a quadrilateral of S_2;

(iii) if $q \in [[V_1]^3]^4$ is a quadrilateral, then $f^{**}(\bar{q}) = \overline{f^{**}(q)}$, and $\bar{q} \in [B_1]^4$ iff $\overline{f^{**}(q)} \in [B_2]^4$.

(2.2) Let κ be an infinite cardinal. Let $x \in {}^{\kappa}Q$ (i.e. $x: \kappa \to Q$). Then define $supp(x) = \{\alpha \in \kappa : x(\alpha) \neq 0\}$. Let $G = \{x \in {}^{\kappa}Q : |supp(x)| < \aleph_0\}$. Then $|G| = \kappa$. For $x, y \in G$ define $x+y$ by $(x+y)(\alpha) = x(\alpha)+y(\alpha)$ for all $\alpha \in \kappa$. Clearly, $(x+y) \in G$ (since $supp(x+y) \subset supp(x) \cup supp(y)$) and it is an associative and commutative binary operation on G. Define $O(\alpha) = 0$ for all $\alpha \in \kappa$. Then $O \in G$ and it is the identity element of G with respect to $+$. Define $-x$ by $(-x)(\alpha) = -x(\alpha)$ for all $\alpha \in \kappa$. Then $-x \in G$, in fact $-x$ is the inverse element of x.

Thus, G together with $+$ form an additive Abelian group of size κ. Since Q satisfies (1.2) and (1.3), so does G.

(2.3) Let G be as in (2.2). Then $supp(x) \neq supp(y) \Rightarrow x \notin [y] \cup [-y]$.

By the way of contradiction assume that $x \in [y] \cup [-y]$. $x = (-2)^n \cdot z$ for some $n \in Z$, $z = y$ or $-y$. By the definition of G, $x(\alpha) = ((-2)^n \cdot z)(\alpha)$ for every $\alpha \in \kappa$. Then $x(\alpha) = (-2)^n \cdot (z(\alpha))$ for every $\alpha \in \kappa$ and so by (1.3) $x(\alpha) = 0$ iff $z(\alpha) = 0$ for every $\alpha \in \kappa$. It follows that $supp(x) = supp(z) = supp(y)$, a contradiction.

(2.4) Let G be as in (2.2). Then $(\forall x, y \in G - \{O\})(\exists z \in G - \{O\})(z \notin [y] \cup [-y]$ & $(-x-z) \notin [y] \cup [-y]$ & $(-x-y-z) \notin [y] \cup [-y])$.

For given $x, y \in G - \{O\}$ choose any $z \in G - \{O\}$ so that $supp(z) \cap (supp(x) \cup supp(y)) = \emptyset$. Then $supp(z) \neq supp(y)$, and so by (2.3) $z \notin [y] \cup [-y]$.

$supp(-x-z) = supp(x) \cup supp(z) \neq supp(y)$, and so by (2.3) $(-x-z) \notin [y] \cup [-y]$.

$supp(-x-y) \subset supp(x) \cup supp(y)$. Since $supp(z) \cap supp(-x-y) = \emptyset$, $supp(-x-y-z) = supp(-x-y) \cup supp(z) \neq supp(y)$. So $(-x-y-z) \notin [y] \cup [-y]$ by (2.3).

(2.5) Let G be as in (2.2). Then there are $\{f_\varphi : \varphi \in {}^{\kappa}2\}$, and $G_1 \subset G - \{O\}$ so that

(i) each f_φ is a (group) automorphism of G;

(ii) $f_{\varphi_1} \neq f_{\varphi_2}$ whenever $\varphi_1 \neq \varphi_2$;

(iii) $f_\varphi(f_\varphi(x)) = x$ for any $x{\in}G$;

(iv) $[x]{\cup}[-x]{\subset}G_1$ whenever $x{\in}G_1$, $|G_1| = |G{-}G_1| = \kappa$ and for every $\varphi{\in}{}^\kappa 2$ and every $x{\in}G_1$, $f_\varphi(x) = x$.

Let $\kappa_1{\subset}\kappa$ so that $|\kappa_1| = |\kappa{-}\kappa_1| = \kappa$. For every $\varphi{:}\kappa{\to}2$ so that $\varphi(\alpha) = 0$ for all $\alpha{\in}\kappa_1$ define f_φ by

$$f_\varphi(x)(\alpha) = \begin{cases} x(\alpha), & \text{if } \varphi(\alpha) = 0; \\ -x(\alpha), & \text{otherwise.} \end{cases}$$

Clearly, $supp(f_\varphi(x)) = supp(x)$, and so $f_\varphi(x){\in}G$. If $x \neq y$, then if $supp(x) \neq supp(y)$, it follows that $supp(f_\varphi(x)) \neq supp(f_\varphi(y))$ and so $f_\varphi(x) \neq f_\varphi(y)$. If $supp(x) = supp(y)$, it follows that $x(\alpha) \neq y(\alpha)$ for some $\alpha{\in}supp(x)$. But then also $-x(\alpha) \neq -y(\alpha)$ and so $f_\varphi(x)(\alpha) \neq f_\varphi(y)(\alpha)$ and so $f_\varphi(x) \neq f_\varphi(y)$. Thus f_φ is injective.

It follows directly from the definition of f_φ that $f_\varphi(f_\varphi(x)) = x$ (so (iii) is proven), and thus f_φ is surjective, and hence f_φ is a bijection from G onto G.

Let $\varphi(\alpha) = 0$, then $f_\varphi(x{+}y)(\alpha) = (x{+}y)(\alpha) = x(\alpha){+}y(\alpha) = f_\varphi(x)(\alpha){+}f_\varphi(y)(\alpha)$. Let $\varphi(\alpha) = 1$, then $f_\varphi(x{+}y)(\alpha) = -(x{+}y)(\alpha) = -x(\alpha){+}(-y(\alpha)) = f_\varphi(x)(\alpha){+}f_\varphi(y)(\alpha)$.

henceforth $f_\varphi(x{+}y) = f_\varphi(x){+}f_\varphi(y)$ and so f_φ is a (group) automorphism of G ((i) is proven).

Let $\varphi_1 \neq \varphi_2$. WLOG asssume that for some α, $\varphi_1(\alpha) = 0$ and $\varphi_2(\alpha) = 1$. Let $x{\in}G{-}\{O\}$ so that $\alpha{\in}supp(x)$. Then $f_{\varphi_1}(x)(\alpha) = x(\alpha) \neq -x(\alpha) = f_{\varphi_2}(x)(\alpha)$. Hence $f_{\varphi_1}(x) \neq f_{\varphi_2}(x)$, and so $f_{\varphi_1} \neq f_{\varphi_2}$ ((ii) is proven).

Define $G_1 = \{x{\in}G{:}supp(x){\subset}\kappa_1\}$. If $x{\in}G_1$ and $x{\sim}y$, then $y{\in}G_1$ as by (2.3) $supp(x) = supp(y)$. Hence $[x]{\subset}G_1$. Since $supp(x){=}supp(-x)$, $-x{\in}G_1$ and so by the previous argument $[-x]{\subset}G_1$. $|G_1| = |G{-}G_1| = \kappa$ and if $x{\in}G_1$, then $f_\varphi(x) = x$ as $\varphi(\alpha) = 0$ for any $\alpha{\in}supp(x)$, for any φ ((iv) is proven).

(2.6) Let G be as in (2.2). Let $\{f_\varphi{:}\varphi{\in}{}^\kappa 2\}$ be as in (2.5). Then there is $\{R_\alpha{:}\alpha{\in}\kappa\}$ so that $G{-}\{O\} = \bigcup\{[R_\alpha]{:}\alpha{\in}\kappa\}$, $R_\alpha{\notin}[R_\beta]$ whenever $\alpha \neq \beta$, and for any $\alpha{\in}\kappa$ and any $\varphi{\in}{}^\kappa 2$, there is $\beta{\in}\kappa$ so that $f_\varphi(R_\alpha) = R_\beta$.

Since for any $n{\in}N{-}\{0\}$, $\{n\}{\times}\kappa^n{\times}\{0\}{\times}Z^{n-1}{\times}N^n{\subset}\kappa^{2n+1}$, then $|\{n\}{\times}\kappa^n{\times}\{0\}{\times}Z^{n-1}{\times}N^n| = \kappa$. Hence $\bigcup\{\{n\}{\times}\kappa^n{\times}\{0\}{\times}Z^{n-1}{\times}N^n{:}n{\in}N{-}\{0\}\}$ has size κ. Enumerate $\bigcup\{\{n\}{\times}\kappa^n{\times}\{0\}{\times}Z^{n-1}{\times}N^n\} = \{t_\alpha{:}\alpha{\in}\kappa\}$. It is easy to see that there is $\{r_n{:}n{\in}N\}$ so that $Z{-}\{0\} = \bigcup\{[r_n]{:}n{\in}N\}$, and $r_n{\notin}[r_m]$ whenever $n \neq m$. Define R_α by :

consider $t_\alpha = \langle n, \alpha_1 ... \alpha_n, k_1 ... k_n, m_1 ... m_n\rangle$, where $k_1 = 0$. Define $supp(R_\alpha) = \{\alpha_1, ..., \alpha_n\}$, and $R_\alpha(\alpha_i) = (-2)^{k_i} r_{m_i}$ for $i \leq n$. Clearly, $R_\alpha \in G - \{O\}$.

Consider $R_\alpha \sim R_\beta$. Then $R_\alpha = (-2)^k \cdot R_\beta$ for some $k \in Z$. Hence $supp(R_\alpha) = supp(R_\beta)$. Assume that $t_\alpha = \langle n, \alpha_1 ... \alpha_n, k_1 ... k_n, m_1 ... m_n\rangle$, and $t_\beta = \langle n, \alpha_1 ... \alpha_n, s_1 ... s_n, p_1 ... p_n\rangle$, where $k_1 = s_1 = 0$. Then $R_\alpha(\alpha_i) = (-2)^{k_i} r_{m_i}$, and $R_\beta(\alpha_i) = (-2)^{s_i} r_{p_i}$ for all $i \leq n$. henceforth $(-2)^{k_i} r_{m_i} = (-2)^{k+s_i} r_{p_i}$, so $r_{m_i} \sim r_{p_i}$, and so $r_{m_i} = r_{p_i}$ for all $i \leq n$. Therefore $r_{m_1} = (-2)^{k_1} r_{m_1} = (-2)^{k+s_1} r_{p_1} = (-2)^k r_{m_1}$, thus $k = 0$. So $R_\alpha = R_\beta$.

Let $x \in G - \{O\}$. Let $supp(x) = \{\alpha_1, ..., \alpha_n\}$. Since $x(\alpha_i) \in Z$ for every $i \leq n$, there are unique $k_i \in Z$ and $m_i \in N$ so that $x(\alpha_i) = (-2)^{k_i} r_{m_i}$. There is $\alpha \in \kappa$ so that $t_\alpha = \langle n, \alpha_1 ... \alpha_n, l_1 ... l_n, m_1 ... m_n\rangle$, where $l_i = k_i - k_1$. Then $R_\alpha(\alpha_i) = (-2)^{l_i} r_{m_i}$, and so $(-2)^{k_1} R_\alpha(\alpha_i) = (-2)^{k_i} r_{m_i} = x(\alpha_i)$. Hence $x = (-2)^{k_1} \cdot R_\alpha$ and so $x \sim R_\alpha$. Thus $G - \{O\} = \bigcup\{[R_\alpha] : \alpha \in \kappa\}$ and $R_\alpha \notin [R_\beta]$ whenever $\alpha \neq \beta \in \kappa$.

Let $\varphi \in {}^\kappa 2$, and let $\alpha \in \kappa$. $f_\varphi(R_\alpha) = ?$. Let $t_\alpha = \langle n, \alpha_1 ... \alpha_n, k_1 ... k_n, m_1 ... m_n\rangle$, where $k_1 = 0$. Then $R_\alpha(\alpha_i) = (-2)^{k_i} r_{m_i}$ for all $i \leq n$. $f_\varphi(R_\alpha)(\alpha_i) = \pm(-2)^{l_i} r_{m_i}$. If $\varphi(\alpha_i) = 0$, then put $h_i = m_i$, otherwise h_i is the (unique) $m \in N$ so that $r_{m_i} = -r_m$. Then $f_\varphi(R_\alpha)(\alpha_i) = (-2)^{k_i} r_{h_i}$ for all $i \leq n$. There is $\beta \in \kappa$ so that $t_\beta = \langle n, \alpha_1 ... \alpha_n, l_1 ... l_n, h_1 ... h_n\rangle$, then $f_\varphi(R_\alpha)(\alpha_i) = R_\beta(\alpha_i)$ for all $i \leq n$, and so $f_\varphi(R_\alpha) = R_\beta$.

(2.7) Thus we have proven that for every infinite cardinal κ there is an additive Abelian group G of size κ satisfying (1.2) and (1.3) and such that

- (i) it has κ classes of the equivalence $\sim$ (as defined in (1.4)),
- (ii) $(\forall x, y \in G - \{O\})(\exists z \in G - \{O\})(z \notin [y] \cup [-y]$ & $(-x-z) \notin [y] \cup [-y]$ & $(-x-y-z) \notin [y] \cup [-y])$.
- (iii) there are $\{f_\varphi : \varphi \in {}^\kappa 2\}$, $\{R_\alpha : \alpha \in \kappa\}$, and $G_1 \subset G - \{O\}$ so that $[x] \cup [-x] \subset G_1$ whenever $x \in G_1$, $|G_1| = |G - G_1| = \kappa$; $G - \{O\} = \bigcup\{[R_\alpha] : \alpha \in \kappa\}$; $R_\alpha \notin [R_\beta]$ whenever $\alpha \neq \beta$; each f_φ is a (group) automorphism of G; $f_{\varphi_1} \neq f_{\varphi_2}$ whenever $\varphi_1 \neq \varphi_2$; for any φ and any $x \in G_1$, $f_\varphi(x) = x$; for any φ and any $x \in G$, $f_\varphi(f_\varphi(x)) = x$; and for any φ and any $\alpha \in \kappa$ there is $\beta \in \kappa$ so that $f_\varphi(R_\alpha) = R_\beta$.

(2.8) Let κ be an infinite cardinal and let $\langle G, +\rangle$ with the identity element O be a group satisfying (2.7). First we shall define a few functions which will be parameters of our construction.

For each class of $\sim$ pick a representative, i.e. choose a function $R : \kappa \to G - \{O\}$ so that $[R(\alpha)] \neq [R(\beta)]$ whenever $\alpha \neq \beta$, and $G - \{O\} = \bigcup\{[R(\alpha)] : \alpha \in \kappa\}$. We shall call R the **rep function**.

(2.9) For each rep function R define a function $C_R{:}G-\{O\}\to\kappa$ by $C_R(x)=\alpha$ iff $x\in[R(\alpha)]$. We shall call C_R the **R-class function**, and $C_R(x)$ will be called **R-class of** x (we shall use simple "class function" and "class of x" if no confusion arises).

(2.10) For each rep function R define a function $T_R{:}G-\{O\}\to\{-1,+1\}$ by :

if $x\in G-\{O\}$, then $x=(-2)^n{\cdot}R(\alpha)$ for some $n\in Z$. Define

$$T_R(x)=\begin{cases}-1, & \text{if } n \text{ is odd;}\\ +1, & \text{if } n=0, \text{ or } n \text{ is even.}\end{cases}$$

We shall call T_R the **R-type function**, and $T_R(x)$ will be called **R-type of** x (we shall use simple "type function" and "type of x" if no confusion arises).

(2.11) Choose a function $CT{:}\kappa\to\{-1,+1\}$. We shall call CT the **class-type function**.

It is easy to prove that for every $x\in G-\{O\}$

(2.12) $C_R(x)=C_R(-2{\cdot}x)$;

(2.13) $T_R(x)\neq T_R(-2{\cdot}x)$.

(2.14) Let G, $\{R_\alpha{:}\alpha\in\kappa\}$, and $\{f_\varphi{:}\varphi\in{}^\kappa 2\}$ be as in (2.7). Then there are a rep function $R{:}\kappa\to G-\{O\}$ and a class-type function $CT{:}\kappa\to\{-1,+1\}$ so that $T_R(x)CT(C_R(x))=T_R(f_\varphi(x))CT(C_R(f_\varphi(x)))$ for any $x\in G-\{O\}$ and any $\varphi\in{}^\kappa 2$.

Define $R(\alpha)=R_\alpha$ for all $\alpha\in\kappa$. Obviously, R is a rep function. Define $CT(\alpha)=+1$ for every $\alpha\in\kappa$. Now let $x\in G-\{O\}$. There are (unique) $m\in Z$ and $\alpha\in\kappa$ so that $x=(-2)^m{\cdot}R(\alpha)$. Then $f_\varphi(x)=(-2)^m{\cdot}f_\varphi(R(\alpha))=(-2)^m{\cdot}R(\beta)$ for some $\beta\in\kappa$. Hence $T_R(x)=T_R(f_\varphi(x))$. Since $CT(\alpha)=CT(\beta)=+1$, $T_R(x)CT(C_R(x))=T_R(x)CT(\alpha)=T_R(f_\varphi(x))CT(\beta)=T_R(f_\varphi(x))CT(C_R(f_\varphi(x)))$.

(2.15) Now we shall describe the construction of a Steiner triple system of size κ from a given group G satisfying (2.7), a given rep-function R, and a given class-type function CT:

Choose any two distinct elements $e_0,e_1\notin G$. The index set of our system will be $V=G\cup\{e_0,e_1\}$. Fix a rep function R and a class-type function CT. Following [GGP] we shall build three types of blocks, A, B, and C :

A-blocks : $\{x,y,z\}$ where $x,y,z\in G, x\neq y, x\neq z, y\neq z$, and $x{+}y{+}z=O$.

B-blocks : $\{x,-2{\cdot}x,e\}$ where $x\in G-\{O\}$, and where

$$e=\begin{cases}e_0, & \text{if } T_R(x)CT(C_R(x))=-1;\\ e_1, & \text{otherwise (i.e. if } T_R(x)CT(C_R(x))=+1).\end{cases}$$

the C-block : $\{O, e_0, e_1\}$.

(2.16) **Lemma :** All A-blocks, B-blocks and the C-block as described in (2.15) form a Steiner triple system of size κ.

Proof.

$V = G \cup \{e_0, e_1\}$. Let $\{x, y\} \in [V]^2$. Let us discuss possible cases.

(i) $x, y \in G$ and $x \neq -2{\cdot}y, y \neq -2{\cdot}x$.

Clearly, $\{x, y\}$ is covered by the (unique) A-block $\{x, y, -x-y\}$. Since neither $x = -2{\cdot}y$ nor $y = -2{\cdot}x$, no B-block can cover it. Since $x, y \in G$, the C-block does not cover it either. Thus no other block covers it.

(ii) $x \in G - \{O\}$ and $y = -2{\cdot}x$.

Then no A-block can cover it, the C-block does not cover it either, and $\{x, -2{\cdot}x, e\}$ (whatever e is, e_0 or e_1) is the only B-block covering it.

(iii) $x \in G - \{O\}$ and $y = e_0$.

Then no A-block and not the C-block can cover it. There are two possibilities :

(α) $T_R(x) CT(C_R(x)) = -1$. Then $\{x, -2{\cdot}x, e_0\}$ is in our system and covers $\{x, e_0\}$. Is it the only B-block covering it? By the way of contradiction assume that $\{a, -2{\cdot}a, e_0\}$ is a different B-block covering it. Then $x = -2{\cdot}a$ and by (2.12) and (2.13) $T_R(a) CT(C_R(a)) \neq T_R(-2{\cdot}a) CT(C_R(-2{\cdot}a)) = T_R(x) CT(C_R(x))$. So $T_R(a) CT(C_R(a)) = +1$ and thus $\{a, -2{\cdot}, e_1\}$ is in our system and $\{a, -2{\cdot}, e_0\}$ is not, a contradiction with our assumption.

(β) $T_R(x) CT(C_R(x)) = +1$. Then by (2.12) and (2.13) $T_R(\frac{x}{2}) CT(C_R(\frac{x}{2})) = -1$. Therefore $\{\frac{x}{2}, -x, e_0\}$ is in our system and covers $\{x, e_0\}$. To show that it is the only B-block covering it is almost identical to the discussion in (α), so we omit it here.

(iv) $x \in G - \{O\}$ and $y = e_1$.

Almost identical to (iii), so we omit it here.

(v) $x = 0, y = e_0$ or e_1.

The only block covering $\{x, y\}$ is the C-block.

Hence all blocks together form a Steiner triple system.

Let B be the set of all A-blocks, B-blocks and the C-block. Since $|G| = \kappa$ and $V = G \cup \{e_0, e_1\}$, $|V| = \kappa$ and also $|[V]^3| = \kappa$. Now, $|G| \leq |B| \leq |[V]^3|$. Hence the size of B is κ. $\square$

(2.17) **Lemma :** In a Steiner triple system constructed as in (2.15) every quadrilateral consists of 2 B-blocks and 2 A-blocks and has the form $\{x, -2\cdot x, e\}, \{-x, 2\cdot x, e\}, \{x, -x, O\}$, and $\{2\cdot x, -2\cdot x, O\}$ for some $x \in G - \{O\}$ and some $e \in \{e_0, e_1\}$.

Proof.

(i) No quadrilateral contains the C-block.

By the way of contradiction assume that there is a quadrilateral with the C-block $\{O, e_0, e_1\}$. Call it block I. In the quadrilateral there must a block (distinct from I) which contains O, it must be an A-block $\{x, -x, O\}$; call it block II. Also there must be a block (distinct from I, and II) which contains e_0, it must be a B-block $\{y, -2\cdot y, e_0\}$; call it block III. And finally, there must be a block (distinct from I,II, and III) which contains e_1, it must be a B-block $\{z, -2\cdot z, e_1\}$; call it block IV. Clearly, $O, e_0, e_1, x, -x$ are all distinct elements. There are four possibilities for the sixth element :

(α) y is the sixth element. Then either $-2\cdot y = x$ (which forces z to be either $-x$ and that makes $-2\cdot z$ seventh distinct element, or y and that makes blocks III and IV identical), or $-2\cdot y = -x$ (which forces z to be either x and that makes $-2\cdot z$ seventh distinct element, or y and that makes blocks III and IV identical). Both cases lead to contradictions.

Almost identical discussions (for the situations are symmetric) show that (β) $-2\cdot y$ is the sixth element, (γ) z is the sixth element, and (δ) $-2\cdot z$ is the sixth element, also lead to contradictions.

(ii) No quadrilateral contains 3 A-blocks.

By the way of contradiction assume that there is such a quadrilateral. Clearly, all 4 blocks must be A-blocks. Let $\{x, y, -x-y\}$ be one of the A-blocks ; call it block I. Call block II the other block (distinct from I) containing x. Call block III the other block (distinct from I, and II) containing y. Finally, call block IV the other block (distinct from I,II and III) containing $-x-y$. Hence I $= \{x, y, -x-y\}$, II $= \{x, a, -x-a\}$, III $= \{y, b, -b-y\}$, IV $= \{-x-y, \bullet, \bullet\}$. Clearly, $x, y, -x-y, a, -x-a$ are distinct. Now, we shall discuss possibilities for b :

$b = x$ or $b = -x-y$ implies that I and III are identical, a contradiction;

$b = a$ makes III $= \{a, y, -y-a\}$ and IV $= \{-x-y, -x-a, -y-a\}$. So $-x-y-x-a-y-a = O$, hence $a = -x-y$, which is a contradiction;

$b = -x-a$ makes III $= \{-x-a, y, -x-a-y\}$ and IV $= \{-x-y, a, -y+x+a\}$. So $-x-y+a-y+ x+a = O$, hence $a = y$, a contradiction.

Thus $x, y, -x-y, a, -x-a, b$ are all distinct. Let us discuss possibilities for $-b-y$:

$-b-y = x$ implies that $b = -x-y$, a contradiction;

$-b-y = y$ implies that $b = -2{\cdot}y$ and that makes III $= \{y, -2{\cdot}y, y\}$, a contradiction;

$-b-y = -x-y$ implies $b = x$, a contradiction;

$-b-y = a$ makes III $= \{y, -y-a, a\}$, and IV $= \{-x-y, -x-a, -y-a\}$. It follows that $a = -x-y$, a contradiction;

$-b-y = -x-a$ makes III $= \{y, x+a-y, -x-a\}$, and IV $= \{-x-y, a, x+a-y\}$. It follows that $a = y$, a contradiction;

$-b-y = b$ makes III $= \{-2{\cdot}b, b, b\}$, a contradiction.

Thus all possibilities lead to contradictions, claim (ii) is proven.

(iii) No quadrilateral contains 3 B-blocks.

(α) Assume all 3 B-blocks contain the same e.

Let I $= \{x, -2{\cdot}x, e\}$, II $= \{y, -2{\cdot}y, e\}$, and III $= \{z, -2{\cdot}z, e\}$. $x, -2{\cdot}x, y, -2{\cdot}y, z$, and $-2{\cdot}z$ must be 5 distinct elements. By symmetry, either $x = -2{\cdot}y$, or $y = -2{\cdot}z$, or $z = -2{\cdot}x$. WLOG we can discuss just the case $x = -2{\cdot}y$.

Then I $= \{-2{\cdot}y, 4{\cdot}y, e\}$ and II $= \{y, -2{\cdot}b, e\}$, which contradicts (2.12) and (2.13).

(β) Assume that only two of the 3 B-blocks contain the same e.

WLOG let I $= \{x, -2{\cdot}x, e_0\}$, II $= \{y, -2{\cdot}y, e_0\}$, and III $= \{z, -2{\cdot}z, e_1\}$. x, $-2{\cdot}x$, y, $-2{\cdot}y$, z, and $-2{\cdot}z$ must be 4 distinct elements. Since $x, -2{\cdot}x, y$, and $-2{\cdot}y$ must be distinct, let us discuss what z can be.

$z = x$ implies III $= \{x, -2{\cdot}x, e_1\}$, a contradiction;

$z = -2{\cdot}x$ makes $-2{\cdot}z = 4{\cdot}x$ which is the 5th distinct element, a contradiction.

Situations $z = y$ and $z = -2{\cdot}y$ are similar and lead to contradictions as well.

(iv) A quadrilateral has the form $\{x, -2{\cdot}x, e\}, \{-x, 2{\cdot}x, e\}, \{x, -x, O\}$, and $\{2{\cdot}x, -2{\cdot}x, O\}$ for some $x \in G - \{O\}$ and some $e \in \{e_0, e_1\}$.

I $= \{x, -2{\cdot}x, e\}$, II $= \{y, -2{\cdot}y, e\}$, III $= \{x, \bullet, \bullet\}$, IV $= \{-2{\cdot}x, \bullet, \bullet\}$. There are two possibilities :

(α) III $= \{x, y, -x-y\}$ and IV $= \{-2{\cdot}x, -2{\cdot}y, 2{\cdot}x+2{\cdot}y\}$. It follows that $-x-y = 2{\cdot}x+2{\cdot}y$ and so $x = -y$. henceforth the quadrilateral has the desired form.

(β) III $= \{x, -2{\cdot}y, 2{\cdot}y-x\}$ and IV $= \{-2{\cdot}x, y, 2{\cdot}x-y\}$. It follows that $2{\cdot}y-x = 2{\cdot}x-y$ and so $x = y$, a contradiction.

Thus claim (2.17) is proven. $\square$

The combinatorics of R, C_R, T_R, and CT determines the number of quadrilaterals in a Steiner triple system constructed as in (2.15).

(2.18) From (2.17) follows that in a Steiner triple system S constructed from a group G satisfying (2.7) as in (2.15) a quadrilateral graph consists of countably infinite **quadrilateral chains**; each quadrilateral chain has "ordertype" Z and is formed by quadrilaterals with blocks containing only elements from $[x] \cup [-x]$ (not considering e_0, e_1, and O), e.g. :

[each column represents a quadrilateral] :

$$\begin{array}{lllllll}
\ldots\ldots & \{\frac{x}{2}, -x, e_1\} & — & \{x, -2{\cdot}x, e_0\} & — & \{2{\cdot}x, -4{\cdot}x, e_1\} & \ldots. \\
\ldots\ldots & \{-\frac{x}{2}, x, e_1\} & — & \{-x, 2{\cdot}x, e_0\} & — & \{-2{\cdot}x, 4{\cdot}x, e_1\} & \ldots. \\
\ldots\ldots & \{\frac{x}{2}, -\frac{x}{2}, O\} & — & \{x, -x, O\} & — & \{2{\cdot}x, -2{\cdot}x, O\} & \ldots. \\
\ldots\ldots & \{x, -x, O\} & — & \{2{\cdot}x, -2{\cdot}x, O\} & — & \{4{\cdot}x, -4{\cdot}x, O\} & \ldots.
\end{array}$$

The quadrilateral chains of the quadrilateral graph are mutually disjoint and each chain is uniquely determined by any element (distinct from e_0, e_1, O) from any block of any quadrilateral of the chain.

(2.19) **Lemma :** Let κ be an infinite cardinal. Let λ be a (finite or infinite) cardinal $\leq \kappa$. Then there is a Steiner triple system of size κ with exactly λ quadrilateral chains.

Proof.

Fix a λ. Fix a group G as in (2.7). Choose $\{r_\alpha : \alpha \in \kappa\} \subset G - \{O\}$ so that $r_\alpha \notin [r_\beta]$ and $r_\alpha \notin [-r_\beta]$ for every $\beta \neq \alpha$ and so that $G - \{O\} = \bigcup\{[r_\alpha] : \alpha \in \kappa\} \cup \bigcup\{[-r_\alpha] : \alpha \in \kappa\}$. Partition κ into two disjoint sets

A and B of size κ. Let $A = \{\gamma_\alpha{:}\alpha{\in}\kappa\}$ and $B = \{\delta_\alpha{:}\alpha{\in}\kappa\}$. Define a rep function R by $R(\gamma_\alpha) = r_{\gamma_\alpha}$ and $R(\delta_\alpha) = r_{\delta_\alpha}$ for all $\alpha{\in}\kappa$. Select a class-type function CT so that $CT(\gamma_\alpha) = CT(\delta_\alpha)$ if $\alpha < \lambda$, and $CT(\gamma_\alpha) \neq CT(\delta_\alpha)$ if $\lambda{\leq}\alpha < \kappa$.

Construct a Steiner triple system S as in (2.15), using the group G.

Let $x{\in}G{-}\{O\}$. If $CT(x) = \gamma_\alpha$, then $CT(-x) = \delta_\alpha$, and vice versa. Thus $T_R(x) = T_R(-x)$. Clearly, $T_R(x)CT(T_R(x)) = T_R(-x)CT(T_R(-x))$ iff $\alpha < \lambda$. Hence from (2.17) follows that the B-blocks $\{x, -2{\cdot}x, \bullet\}$ and $\{-x, 2{\cdot}x, \bullet\}$ have the same $\bullet$ ($\bullet = e_0, e_1$) iff $\alpha < \lambda$, and so they form (a part of) a quadrilateral iff $\alpha < \lambda$. It is obvious from the construction (2.15) that such a quadrilateral is in a quadrilateral chain. Hence there are exactly λ quadrilateral chains in S. $\square$

Note. If $\lambda = 0$, then S does not have any quadrilaterals, i.e. it is a so-called **anti-Pasch** Steiner triple system.

For $\lambda{\leq}\aleph_0$, S has exactly $\aleph_0$ quadrilaterals.

For an infinite λ, S has exactly $\aleph_0{\cdot}\lambda = \lambda$ quadrilaterals.

(2.20) From (2.17) follows that in a Steiner triple system S constructed from a group G satisfying (2.7) as in (2.15) it is possible for an element $x{\in}G{-}\{O\}$ to be a part of what we shall call an **anti-quadrilateral chain**; each anti-quadrilateral chain has "ordertype" Z and is formed by **anti-quadrilaterals** with blocks containing only elements from $[x]{\cup}[-x]$ (not considering e_0, e_1, and O), e.g. :

[each column represents an anti-quadrilateral] :

$$
\begin{array}{llllllll}
\ldots\ldots & \{\frac{x}{2}, -x, e_1\} & — & \{x, -2{\cdot}x, e_0\} & — & \{2{\cdot}x, -4{\cdot}x, e_1\} & \ldots. \\
\ldots\ldots & \{-\frac{x}{2}, x, e_0\} & — & \{-x, 2{\cdot}x, e_1\} & — & \{-2{\cdot}x, 4{\cdot}x, e_0\} & \ldots. \\
\ldots\ldots & \{\frac{x}{2}, -\frac{x}{2}, O\} & — & \{x, -x, O\} & — & \{2{\cdot}x, -2{\cdot}x, O\} & \ldots. \\
\ldots\ldots & \{x, -x, O\} & — & \{2{\cdot}x, -2{\cdot}x, O\} & — & \{4{\cdot}x, -4{\cdot}x, O\} & \ldots.
\end{array}
$$

The anti-quadrilateral chains are mutually disjoint and each chain is uniquely determined by any element (distinct from e_0, e_1, O) from any block of any anti-quadrilateral of the chain.

(2.21) **Lemma :** Let κ be an infinite cardinal. Let λ be a (finite or infinite) cardinal $\leq\kappa$. Then there is a Steiner triple system of size κ with exactly λ anti-quadrilateral chains.

Proof.

Fix a λ. Fix a group G as in (2.7). Choose $\{r_\alpha:\alpha\in\kappa\}\subset G-\{O\}$ so that $r_\alpha\notin[r_\beta]$ and $r_\alpha\notin[-r_\beta]$ for every $\beta\neq\alpha$ and so that $G-\{O\}=\bigcup\{[r_\alpha]:\alpha\in\kappa\}\cup\bigcup\{[-r_\alpha]:\alpha\in\kappa\}$. Partition κ into two disjoint sets A and B of size κ. Let $A=\{\gamma_\alpha:\alpha\in\kappa\}$ and $B=\{\delta_\alpha:\alpha\in\kappa\}$. Define a rep function R by $R(\gamma_\alpha)=r_{\gamma_\alpha}$ and $R(\delta_\alpha)=r_{\delta_\alpha}$ for all $\alpha\in\kappa$. Select a class-type function CT so that $CT(\gamma_\alpha)\neq CT(\delta_\alpha)$ if $\alpha<\lambda$, and $CT(\gamma_\alpha)=CT(\delta_\alpha)$ if $\lambda\leq\alpha<\kappa$.

Construct a Steiner triple system S as in (2.15), using the group G.

Let $x\in G-\{O\}$. If $CT(x)=\gamma_\alpha$, then $CT(-x)=\delta_\alpha$, and vice versa. Thus $T_R(x)=T_R(-x)$. Clearly, $T_R(x)CT(T_R(x))\neq T_R(-x)CT(T_R(-x))$ iff $\alpha<\lambda$. Hence from (2.17) follows that the B-blocks $\{x,-2{\cdot}x,\bullet\}$ and $\{-x,2{\cdot}x,\bullet\}$ have the same $\bullet$ ($\bullet=e_0,e_1$) iff $\alpha\geq\lambda$, and so they form (a part of) a quadrilateral iff $\alpha\geq\lambda$. It is obvious from the construction (2.15) that such a quadrilateral is in a quadrilateral chain. Hence there are exactly $\kappa-\lambda$ quadrilateral chains and exactly λ anti-quadrilateral chains in S. $\square$

(2.22) Consider a quadrilateral chain containing the quadrilateral $\{x,-2{\cdot}x,e\}$, $\{-x,2{\cdot}x,e\}$, $\{x,-x,O\}$, and $\{2{\cdot}x,-2{\cdot}x,O\}$. Replace this quadrilateral in the system S by its complementary quadrilateral $\{-x,2{\cdot}x,O\}$, $\{x,-2{\cdot}x,O\}$, $\{2{\cdot}x,-2{\cdot}x,e\}$, and $\{x,-x,e\}$. It is easy to check that the new system still is a Steiner triple system, that it has the same index set and the same number of blocks as the original system. Since none of the blocks of the complementary quadrilateral is of type A, B or C, this new quadrilateral does not have any common block with any other quadrilateral of the original system. The "neighbouring" quadrilaterals $\{\frac{x}{2},-x,\bar{e}\}$, $\{-\frac{x}{2},x,\bar{e}\}$, $\{-\frac{x}{2},\frac{x}{2},O\}$, $\{-x,x,O\}$, and $\{2{\cdot}x,-4{\cdot}x,\bar{e}\}$, $\{-2{\cdot}x,4{\cdot}x,\bar{e}\}$, $\{-2{\cdot}x,2{\cdot}x,O\}$, $\{-4{\cdot}x,4{\cdot}x,O\}$ (where $\bar{e}$ denotes e_0 if e is e_1, and it denotes e_1 if e is e_0) will disappear, for the blocks $\{-x,x,O\}$ and $\{-2{\cdot}x,2{\cdot}x,O\}$ were replaced by blocks $\{-x,x,\bar{e}\}$ and $\{-2{\cdot}x,2{\cdot}x,\bar{e}\}$. In fact, this "breaks" the quadrilateral chain into two unconnected parts (it sort of punches a "double-hole" in the quadrilateral chain) :

the quadrilateral chain before :

=O===O===O===O===O=

q_2 $\quad q_1$ $\quad q_0$ $\quad q_3$ $\quad q_4$

the quadrilateral chain after "punching out" quadrilateral q_0 :

=O X □ X O=

q_2 $\hat{q}_1$ $\bar{q}_0$ $\hat{q}_3$ q_4

We shall define a **quadrilateral fragment** of a Steiner triple system S to be a group of 3 blocks of S so that a replacement of a quadrilateral of S by its complementary quadrilateral completes this group of 3 blocks to a quadrilateral. By this definition, after q_0 was "punched out", we ware left with two quadrilateral fragments denoted $\hat{q}_1$ and $\hat{q}_3$.

We say that two quadrilateral chains are **brothers** iff a replacement of a quadrilateral by its complementary quadrilateral joins them to a single quadrilateral chain. A transitive closure of the relation "being brothers" will be called a **quadrilateral family**.

(2.23) Let f be an isomorphism of a Steiner triple system $S_1 = \langle V_1, B_1\rangle$ onto a Steiner triple system $S_2 = \langle V_2, B_2\rangle$.

(i) $\{b_1, b_2, b_3\}\subset B_1$ is a quadrilateral fragment of S_1 iff $\{f^*(b_1), f^*(b_2), f^*(b_3)\}\subset B_2$ is a quadrilateral fragment of S_2;

(follows from (2.1))

(ii) let $\{..., q_1\}$ and $\{q_2, ...\}$ be two quadrilateral chains in S_1 that are brothers. Then $\{..., f^{**}(q_1)\}$ and $\{f^{**}(q_2), ...\}$ are two quadrilateral chains in S_2 that are are also brothers;

(follows from (2.1))

(iii) f^{**} maps a quadrilateral family onto a quadrilateral family.

(follows from (ii))

(2.24) Consider the quadrilateral chain containing the quadrilateral $\{x, -2{\cdot}x, e\}$, $\{-x, 2{\cdot}x, e\}$, $\{x, -x, O\}$, and $\{2{\cdot}x, -2{\cdot}x, O\}$. First, enumerate the quadrilaterals in the quadrilateral chain starting with the above quadrilateral (or any other as a matter of fact) as $qad(0)$, the quadrilateral with $2{\cdot}x$ as $qad(1)$, the quadrilateral with $\frac{x}{2}$ as $qad(-1)$ and so forth. Let $a\in{}^{\omega}\omega$. Let $p(0) = 5, p(1) = 7, p(2) = 11$ and so forth, enumerating all primes ≥ 5. By induction we shall "punch out" some quadrilaterals (i.e. replace them by their complementary quadrilaterals) in the quadrilateral chain according to a : first punch out $qad(0)$. Then punch out $qad(p(0)^{a(0)})$. Then punched out $qad(-p(1)^{a(1)})$. Then punch out $qad(p(2)^{a(2)})$, then $qad(-p(3)^{a(3)})$. And so forth. This will break the quadrilateral chain into finite length segments (no two segments have the same length) forming a quadrilateral family. The lengths of the segments are determined by a : the nth segment has length $p(n)^{a(n)} - 3$. We shall call a the **type** of the family. Thus, by this procedure $2^{\aleph_0}$ quadrilateral families of different types can be produced.

Note. It is important to punch the holes at least five quadrilaterals apart so there is something left connected in the original quadrilateral chain. That is why $p(0)$ starts with 5, rather than with 1.

(2.25) Let f be an isomorphism of S_1 onto S_2 as in (2.23).

(i) f^{**} maps a finite length quadrilateral segment onto a quadrilateral segment of the same length. By the way of contradiction and WLOG assume that f^{**} maps a shorter segment onto a longer one. Proceed by induction over the length of the shorter segment :

(a) length=2.

The shorter segment :

□ X O===O X □

$\bar{q}_4$ $\hat{q}_3$ q_1 q_2

Assume that $f^{**}(q_2) = q$ where q is not an end-quadrilateral of the longer segment, i.e. :

□ X O= $\cdots$ =O===O===O= $\cdots$ =O X □

q_5 q q_6

Consider q_3 before q_4 was punched out. Since $|q_3 \cap q_1| = 1$, by (2.1) $|f^{**}(q_3) \cap f^{**}(q_1)| = |f^{**}(q_3) \cap q| = 1$. Hence $f^{**}(q_3) = q_5$ or q_6. Since both, $q_5, q_6 \in [B_2]^4$, by (2.1) $q_3 \in [B_1]^4$, but it is not. Hence f^{**} must map both, q_1, q_2 on end-quadrilaterals of the longer segment. Consider a "second" quadrilateral of the longer segment (i.e. a "neighbour" of one of the end-quadrilaterals). It must be an f^{**}-image of a quadrilateral which has an empty intersection with both, q_1, q_2. Hence it itself must have an empty intersection with both end-quadrilaterals, a contradiction.

(b) length $= n+1$, and assume the assertion holds true for n.

The discussion is very much like in (a), so we omit it here.

(ii) f^{**} maps a quadrilateral family (obtained as in (2.24)) onto a quadrilateral family of the same type.

Assume that F_1 is a family of type a_1 in S_1, and F_2 is a family of type a_2 in S_2, and that f^{**} maps F_1 onto F_2. By (i) f^{**} preserves the length of quadrilateral segments, and so f^{**} maps the mth segment of F_1 (its length is $p(m)^{a_1(m)}-3$) onto an nth segment of F_2 (its length is $p(n)^{a_2(n)}-3$). Since $p(m)^{a_1(m)}-3 = p(n)^{a_2(n)}-3$, it follows that $n = m$ and also $a_1(m) = a_2(n)$. Hence $a_1 = a_2$.

(iii) Let f be an automorphism of S_1. Let F_1 be a quadrilateral family of S_1 obtained from a quadrilateral chain determined by $[x]\cup[-x]$ for some $x\in G-\{O\}$. Let f^{**} map F_1 onto itself. Then f^{**} is identity on quadrilaterals of the type $\{y, -2{\cdot}y, e\}$, $\{-y, 2{\cdot}y, e\}$, $\{-y, y, O\}$, $\{-2{\cdot}y, 2{\cdot}y, O\}$, and their complementary quadrilaterals, for $y\in[x]\cup[-x]$.

Similar discussion as in (ii) shows that f^{**} maps every segment of F_1 onto itself. Since being the end-quadrilateral of a segment must be preserved by f^{**}, f^{**} is either identity on a segment, or it reverses the segment. If f^{**} reverses one segment, then it has to switch over the two neighbouring segments, a contradiction. Thus f^{**} is identity on segments, and thus identity on the quadrilaterals of segments. By the same token the "punched out" quadrilaterals must be mapped onto themselves, as they are in fact "segments" of length 1. When f^{**} preserves a quadrilateral, then it also preserves its complementary quadrilateral (easy to prove). Thus we are almost done, the only unresolved cases are the quadrilaterals whose quadrilateral fragments were left when the family F_1 was created. By the way of contradiction assume that there is such a quadrilateral q which is not preserved by f^{**}. There must be a quadrilateral q_1 which was "punched out" and that destroyed q. Hence $|q\cap\bar{q}_1| = 1$. $f^{**}(\bar{q}_1) = \bar{q}_1$ and so $|f^{**}(q)\cap\bar{q}_1|$ $= 1$. There are only two quadrilaterals, call them q_2, q_3, with non-empty intersections with q_1, the ones which get destroyed by punching q_1 out. Hence q is either q_2 or q_3. WLOG assume that $q = q_2$. Then $f^{**}(q)$ must be q_3 to preserve a non-empty intersection with q_1. Then f^{**} must switch over the neighbouring segments, and that is a contradiction.

(3) Main results

Now we are ready to prove the main theorems.

(3.1) **Theorem :** Let κ be an infinite cardinal, and let $\lambda = min\{\kappa, 2^{\aleph_0}\}$. Then there are 2^λ mutually non-isomorphic Steiner triple systems of size κ that admit 2^κ automorphisms.

Proof.

Let G, G_1, $\{f_\varphi{:}\varphi{\in}{}^\kappa 2\}$ be as in (2.7), and let $R{:}\kappa{\to}G-\{O\}$, and $CT{:}\kappa{\to}\{-1,+1\}$ be as in (2.14). Build a Steiner triple system S of size κ with the index set $V = G{\cup}\{e_0, e_1\}$ as in (2.15) with no anti-quadrilateral chains (see (2.21)). Let $A{\subset}\{\alpha{\in}\kappa{:}R(\alpha){\in}G_1\}$ so that $|A| = \lambda$. Let $\{a_\alpha{:}\alpha{\in}A\}{\subset}{}^\omega\omega$. Fix $\chi{\in}{}^A 2$. For each $\alpha{\in}A$ so that $\chi(\alpha) = 1$ change the quadrilateral chain determined by $R(\alpha)$ to a quadrilateral family of type a_α (according to (2.24)). The resulting system will be called S_χ, and it is a Steiner triple system of size κ. Let $\varphi{\in}{}^\kappa 2$. Extend f_φ to map V onto V by defining $f_\varphi(e_0) = e_0$ and $f_\varphi(e_1) = e_1$. Then f_φ is an automorphism of S_χ : there are four possibilities for a block $b{\in}S_\chi$;

(i) $b = \{x, y, -x{-}y\}$ is an A-block. Then $f^*_\varphi(b) = \{f_\varphi(x), f_\varphi(y), -f_\varphi(x){-}f_\varphi(y)\}$ since f_φ is a (group) automorphism of G. Hence $f^*_\varphi(b)$ is an A-block. Is this A-block in S_χ? If not, then it was a block in a quadrilateral which was "punched out" and it means that $f_\varphi(x){\in}[R(\alpha)]{\cup}[-R(\alpha)]$ for some $\alpha{\in}A$. Since f_φ is an identity on $[R(\alpha)]{\cup}[-R(\alpha)]$ (as $[R(\alpha)]{\cup}[-R(\alpha)]{\subset}G_1$), we have $f_\varphi(x) = f_\varphi(f_\varphi(x)) = x$. Similarly $f_\varphi(y) = y$ and $-x{-}y = f_\varphi(-x{-}y) = -f_\varphi(x){-}f_\varphi(y)$. Hence $f^*_\varphi(b) = b$ and so $b{\in}S_\chi$, a contradiction.

(ii) $b = \{x, -2{\cdot}x, e\}$ is a B-block. Then $f^*_\varphi(b) = \{f_\varphi(x), -2{\cdot}f_\varphi(x), e\}$ since f_φ is a (group) automorphism of G and also $f_\varphi(e) = e$. Hence $f^*_\varphi(b)$ is a B-block. Is this B-block in S? Yes, for $T_R(x)CT(C_R(x)) = T_R(f_\varphi(x))CT(C_R(f_\varphi(x)))$ (see (2.14)). Is this B-block in S_χ? Similar discussion as above for A-blocks shows that if assumed not, then $f^*_\varphi(b) = b$ and so $f^*_\varphi(b)$ is in S_χ, a contradiction.

(iii) $b = \{e_0, e_1, O\}$ is the C-block. Then $f^*_\varphi(b) = b$ and so $f^*_\varphi(b)$ is in S_χ.

(iv) b is not an A-block, a B-block, not the C-block. Then b was in a quadrilateral which was "punched out" and the same discussion as above for the A-block will show that $f^*_\varphi(b) = b$ and so $f^*_\varphi(b)$ is in S_χ.

We have shown that, indeed, f_φ is an automorphism of S_χ, and so S_χ admits 2^κ automorphisms.

Now consider $\chi_1 \neq \chi_2{\in}{}^A 2$. WLOG assume that $\chi_1(\alpha) = 1$ and $\chi_2(\alpha) = 0$ for some α. Then S_{χ_1} and S_{χ_2} are not isomorphic, for S_{χ_1} contains a quadrilateral family of type a_α while S_{χ_2} does not. $\square$

(3.2) **Theorem :** Let κ be an infinite cardinal $\leq 2^{\aleph_0}$. Then there are 2^κ mutually non-isomorphic rigid Steiner triple systems of size κ.

Proof.

Using a group G as in (2.7) construct Steiner triple systems S of size κ, with exactly 1 anti-quadrilateral chain (see (2.21)). Let $\{a_\alpha : \alpha\in\kappa\}\subset{}^\omega\omega$. Fix a $g\in[\kappa]^\kappa$. For every $\alpha\in g$ change exactly one of quadrilateral chains (by "punching" holes in them according to (2.24)) to a quadrilateral family of type a_α. Do it is so that every of κ quadrilateral chains is changed to a (unique) quadrilateral family. The result is a Steiner triple system S_g of size κ with the same underlying index set as S. Now, if $g\neq h$, then S_g and S_h are not isomorphic : let $\alpha\in\kappa$ be so that $\alpha\notin g\cap h$. WLOG assume that $\alpha\in g$ and $\alpha\notin h$. Then S_g contains exactly one quadrilateral family of type a_α, while S_h does not. Thus any isomorphism would have to map this quadrilateral family of type a_α in S_g to a quadrilateral family of a different type in S_h, a contradiction.

Let us consider an automorphism f of S_g. Let $x\in(G-\{O\})-([a]\cup[-a])$, where a is an element determining the only anti-quadrilateral chain. By (2.25)(iii) f^{**} must preserve quadrilateral $\{x,-2\cdot x,e\}$, $\{-x,2\cdot x,e\}$, $\{-x,x,O\}$, $\{-2\cdot x,2\cdot x,O\}$, and so f may map x to x, or to $-x$, or to $2\cdot x$, or to $-2\cdot x$, or to O, or to e. By the same token, f^{**} must preserve quadrilateral $\{\frac{x}{2},-x,\bar{e}\}$, $\{-\frac{x}{2},x,\bar{e}\}$, $\{\frac{x}{2},-\frac{x}{2},O\}$, $\{x,-x,O\}$. So f may map x to x, or to $-x$, or to $\frac{x}{2},$ or to $-\frac{x}{2}$, or to O, or to $\bar{e}$. The conditions from these two quadrilaterals force f to map x to x, or to $-x$, or to O. If $f(x)=O$, then $\{x,-2\cdot x,e\}$ will be mapped onto $\{O,f(-2\cdot x),f(e)\}$ and so $f(e)$ must be either x, or $-x$, or $2\cdot x$, or $-2\cdot x$. Now consider quadrilateral q : $\{2\cdot x,-4\cdot x,e\}$, $\{-2\cdot x,4\cdot x,e\}$, $\{2\cdot x,-2\cdot x,O\}$, $\{4\cdot x,-4\cdot x,O\}$. $f^{**}(q)\neq q$ for any of the possibilities for $f(e)$, thus f cannot map x onto O and so f can map x either to x, or to $-x$.

We have established that for any $x\in(G-\{O\})-([a]\cup[-a])$, $f(x)=x$ or $f(x)=-x$.

Now, assume that there is at least one $x\in(G-\{O\})-([a]\cup[-a])$ so that $f(x)=-x$. It is easy to see, that this "propagates" through $[x]\cup[-x]$, i.e. $f(y)=-y$ for every $y\in[x]\cup[-x]$ (it follows from (2.25)(iii)).

Let $y\in(G-\{O\})-([a]\cup[-a])$ so that $y\notin[x]\cup[-x]$ and $(-x-y)\notin[a]\cup[-a]$. Then $f(y)=y$ or $-y$. Assume that $f(y)=y$. Then f^{**} maps the A-block $\{x,y,-x-y\}$ either onto the A-block $\{-x,y,-x-y\}$ (which implies that $x = O$), a contradiction, or onto the A-block $\{-x, y, x+y\}$ (which implies that $y = O$), also a contradiction. Thus $f(y) = -y$.

We have established that $f(y) = -y$ for every $y \in G - \{O\}$ so that $y, (-x-y) \notin [a] \cup [-a]$.

Let $b \in [a] \cup [-a]$. Since f^{**} cannot map the anti-quadrilateral $\{b, -2{\cdot}b, e\}$, $\{-b, 2{\cdot}b, \bar{e}\}$, $\{-b, b, O\}$, and $\{-2{\cdot}b, 2{\cdot}b, O\}$ onto a quadrilateral, $f(b) = c$ for some $c \in [a] \cup [-a]$. By (2.7)(ii), for x, b there is some $z \in G - \{O\}$ so that $z \notin [b] \cup [-b]$, $(-x-z) \notin [b] \cup [-b]$, and $(-x-b-z) \notin [b] \cup [-b]$. Since $[a] \cup [-a] = [b] \cup [-b]$, $f(z) = -z$ and also $f(-b-z) = b+z$. Thus f^{**} maps the A-block $\{b, z, -b-z\}$ onto the A-block $\{c, -z, b+z\}$, and so $c = -b$.

Thus, we have established that $f(b) = -b$ for any $b \in [a] \cup [-a]$.

Finally, let $y \in (G - \{O\}) - ([a] \cup [-a])$ so that $y \notin [x] \cup [-x]$ and $(-x-y) \in [a] \cup [-a]$. Then $f(-x-y) = x+y$ and so f^{**} maps the A-block $\{x, y, -x-y\}$ onto the A-block $\{-x, f(y), x+y\}$, and so $f(y) = -y$.

So, we have proven that if $f(x) = -x$ for some $x \in (G - \{O\}) - ([a] \cup [-a])$, then $f(x) = -x$ for every $x \in G - \{O\}$.

Now consider the anti-quadrilateral $\{a, -2{\cdot}a, e\}$, $\{-a, 2{\cdot}a, \bar{e}\}$, $\{-a, a, O\}$, and $\{-2{\cdot}a, 2{\cdot}a, O\}$, which is being mapped by f^{**} onto $\{-a, 2{\cdot}a, e\}$, $\{a, -2{\cdot}a, \bar{e}\}$, $\{a, -a, O\}$, and $\{2{\cdot}a, -2{\cdot}a, O\}$. But the block $\{-a, 2{\cdot}a, e\}$ is not in the system S_g. Hence there is no $x \in (G - \{O\}) - ([a] \cup [-a])$ so that $f(x) = -x$ and so $f(x) = x$ for all $x \in (G - \{O\}) - ([a] \cup [-a])$.

Let $b \in [a] \cup [-a]$. As discussed above, $f(b) = c$ for some $c \in [a] \cup [-a]$. By (2.7)(ii), for x, b there is some $z \in G - \{O\}$ so that $z \notin [b] \cup [-b]$, $(-x-z) \notin [b] \cup [-b]$, and $(-x-b-z) \notin [b] \cup [-b]$. Since $[a] \cup [-a] = [b] \cup [-b]$, $f(z) = z$ and also $f(-b-z) = -b-z$. Thus f^{**} maps the A-block $\{b, z, -b-z\}$ onto the A-block $\{c, z, -b-z\}$, and so $c = b$.

We have just proven that $f(x) = x$ for all $x \in G - \{O\}$. Since, clearly, f cannot move O, e_0, and e_1, f must be the identity. $\square$

(3.3) **Theorem :** Let κ be an infinite cardinal $\leq 2^{\aleph_0}$. Then there are 2^κ mutually non-isomorphic Steiner triple systems of size κ that admit exactly one non-trivial automorphism.

Proof.

Using a group G as in (2.7) construct Steiner triple systems S of size κ with no anti-quadrilateral chain (see (2.21)). Let $\{a_\alpha{:}\alpha{\in}\kappa\}{\subset}^\omega\omega$. Fix a $g{\in}[\kappa]^\kappa$. For every $\alpha{\in}g$ change exactly one of quadrilateral chains (by "punching" holes in them according to (2.24)) to a quadrilateral family of type a_α. Do it is so that every of κ quadrilateral chains is changed to a (unique) quadrilateral family. The result is a Steiner triple system S_g of size κ with the same underlying index set as S. Now, if $g \neq h$, then S_g and S_h are not isomorphic : let $\alpha{\in}\kappa$ be so that $\alpha{\notin}g{\cap}h$. WLOG assume that $\alpha{\in}g$ and $\alpha{\notin}h$. Then S_g contains exactly one quadrilateral family of type a_α, while S_h does not. Thus any isomorphism would have to map this quadrilateral family of type a_α in S_g to a quadrilateral family of a different type in S_h, a contradiction.

Let us consider an automorphism f of S_g. Let $x{\in}G{-}\{O\}$. By (2.25)(iii) f^{**} must preserve quadrilateral $\{x,-2{\cdot}x,e\}$, $\{-x,2{\cdot}x,e\}$, $\{-x,x,O\}$, $\{-2{\cdot}x,2{\cdot}x,O\}$, and so f may map x to x, or to $-x$, or to $2{\cdot}x$, or to $-2{\cdot}x$, or to O, or to e. By the same token, f^{**} must preserve quadrilateral $\{\frac{x}{2},-x,\bar{e}\}$, $\{-\frac{x}{2},x,\bar{e}\}$, $\{\frac{x}{2},-\frac{x}{2},O\}$, $\{x,-x,O\}$. So f may map x to x, or to $-x$, or to $\frac{x}{2,}$ or to $-\frac{x}{2}$, or to O, or to $\bar{e}$. The conditions from these two quadrilaterals force f to map x to x, or to $-x$, or to O. If $f(x) = O$, then $\{x,-2{\cdot}x,e\}$ will be mapped onto $\{O,f(-2{\cdot}x),f(e)\}$ and so $f(e)$ must be either x, or $-x$, or $2{\cdot}x$, or $-2{\cdot}x$. Now consider quadrilateral q : $\{2{\cdot}x,-4{\cdot}x,e\}$, $\{-2{\cdot}x,4{\cdot}x,e\}$, $\{2{\cdot}x,-2{\cdot}x,O\}$, $\{4{\cdot}x,-4{\cdot}x,O\}$. $f^{**}(q) \neq q$ for any of the possibilities for $f(e)$, thus f cannot map x onto O and so f can map x either to x, or to $-x$.

We have established that for any $x{\in}G{-}\{O\}$, $f(x) = x$ or $f(x) = -x$.

Now, assume that there is at least one $x{\in}G{-}\{O\}$ so that $f(x) = -x$. It is easy to see, that this "propagates" through $[x]{\cup}[-x]$, i.e. $f(y) = -y$ for every $y{\in}[x]{\cup}[-x]$ (it follows from (2.25)(iii)).

Let $y{\in}G{-}\{O\}$ so that $y{\notin}[x]{\cup}[-x]$. Since $f(y) = y$ or $-y$, and also $f(-x-y) = (-x-y)$ or $(x+y)$, f^{**} maps the A-block $\{x,y,-x-y\}$ either onto the A-block $\{-x,f(y),-x-y\}$ or onto the A-block $\{-x,f(y),x+y\}$. Assuming $f(y) = y$, it follows in the first case that $x = 0$, which is a contradiction, and in the latter case that $y = 0$, also a contradiction. Hence $f(y) = -y$.

We have established that if $f(x) = -x$ for some $x{\in}G{-}\{O\}$, then $f(x) = -x$ for all $x{\in}G{-}\{O\}$. Since f cannot move O, e_0, and e_1, there are only two automorphisms of S_g : the identity and the one moving x onto $-x$ for all $x{\in}G{-}\{O\}$. $\square$

References

[DR] J.Doyen, A.Rosa, *An updated bibliography of Steiner systems* in *Topics on Steiner systems* (Ed. C.C.Lindner and A. Rosa), Ann. Discrete Math. 7(1980), 317-349.

[N] J.Nesetril, *On infinite precise objects*, Matematica Slovaca 28(1978),253-260.

[Si] W.Sierpinski,*Sur un probleme de triades*, C.R.Soc.Sci.Varsovie 33-38(1946),13-16.

[So] B.Sobocinski, *A theorem of Sierpinski on triads and the axiom of choice*, Notre Dame J. Formal Logic 5(1964),51-58.

[Vu] V.Vuckovic, *Note on a theorem of W.Sierpinski*, Notre Dame J. Formal Logic 6(19865), 180-182.

[GGP] M.J.Grannell, T.S.Griggs, J.S.Phelan, *Countably infinite Steiner triple systems*, to appear in Ars Combinatoria.

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