Isomorphisms of Infinite Steiner Triple Systems II

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Abstract.

A combinatorial method in conjuction with the results presented in [F] is introduced to prove that for any infinite cardinal κ, and every cardinal λ, $0 \leq \lambda \leq \kappa$, there are 2^κ mutually non-isomorphic Steiner triple systems of size κ that admit exactly 2^λ automorphisms. In particular, there are 2^κ mutually non-isomorphic rigid Steiner triple systems of size κ.

Introduction.

Mathematicians have been mostly interested in finite Steiner systems, and so the published literature dealing with finite Steiner systems is quite extensive (see e.g. [DR]). There has been very little published on infinite Steiner systems (see e.g. [So], [Si], [V], [GGP], [N], [F]).

We are going to present a combinatorial method to generate a family of mutually non-isomorphic "nice" Steiner triple systems of any desirable (infinite) size κ with features controlling the number of their automorphisms. The method utilizes results presented in [F], and so it extends these to all cardinalities.

(1) Notation and definitions.

The standard set-theoretical notation is used. (x, y) denotes an ordered pair, $< x, ..., y >$ a sequence. Lower case Greek letters denote ordinal numbers. Many terms used here are defined in [F] (namely: quadrilateral, complementary quadrilateral, quadrilateral chain, quadrilateral family, anti-quadrilateral chain, a quadrilateral graph).

Let $\underline{S}=\langle V, S\rangle$ be a Steiner triple system (from now on **STS**).

(1.1) We shall call $\underline{S}$ **quadrilateral family complete** iff for any $x \in V$ there are $y, z \in V$ and quadrilateral families F_1, F_2 of $\underline{S}$ so that $\{x, y, z\} \in S$ and $y \in \bigcup\bigcup F_1$ and $z \in \bigcup\bigcup F_2$.

(1.2) We shall call $\underline{S}$ **quadrilateral complete** iff for every $x{\in}V$ there is a quadrilateral q of $\underline{S}$ such that $x{\in}\bigcup q$.

(1.3) The **weak quadrilateral graph** of $\underline{S}$ is an undirected graph whose vertices are the quadrilaterals of $\underline{S}$, and quadrilaterals q_1, q_2 are connected by an edge iff $\bigcup q_1{\cap}\bigcup q_2{\neq}\emptyset$.

(1.4) $\underline{S}$ is **quadrilateral connected** iff its weak quadrilateral graph is connected.

(1.5) $\underline{S}$ is **rich** iff for every $x, y, z{\in}V$ there is a quadrilateral q of $\underline{S}$ such that $x, y, z{\in}\bigcup q$.

(1.6) A STS $\underline{S}$ is **nice** iff it is quadrilateral complete, quadrilateral connected and quadrilateral family complete.

(1.7) Let $\Im = \{\underline{S}_\alpha = \langle V, S_\alpha\rangle : \alpha{\in}\kappa\}$ be a family of STS's, κ an infinite cardinal. A **product of** $\Im$ is a STS $\underline{T}{=}\langle W, T\rangle$ obtained as follows:

Let $\underline{R}{=}\langle\kappa, R\rangle$ be a rich STS. Let $\underline{A}{=}\langle V, A\rangle$ be an anti-Pasch STS. Then $W{=}\kappa{\times}V$ and T is defined by $\{(\alpha, x), (\beta, y), (\gamma, z)\}{\in}T$ iff

(i) $\alpha{=}\beta{=}\gamma$ and $\{x, y, z\}{\in}S_\alpha$, or

(ii) $\{\alpha, \beta, \gamma\}{\in}R$ and $x{=}y{=}z$, or

(iii) $\{\alpha, \beta, \gamma\}{\in}R$ and $\{x, y, z\}{\in}A$.

(1.8) Definition (1.7) is correct and the product of $\Im$ is, indeed, a STS of size $\kappa{\times}|V|$.

The existence of a rich STS $\underline{R}$ of size κ is assured by Lemma (2.1). The existence of an anti-Pasch STS $\underline{A}$ of size $|V|$ is assured by Lemma (2.19) in [F]. It si easy to verify that the definition of blocks in the product assures that every pair is covered by a unique block.

(1.9) Consider the weak quadrilateral graph of $\underline{S}$. $\langle q_i : i{\leq}n\rangle$ is a **path** connecting q_0 and q_n iff $\{q_i, q_j\}$ is an edge if and only if $j{=}i{+}1$.

Let q_0, q_1 be quadrilaterals in $\underline{S}$. Let $\langle q_i : i{\leq}n\rangle$ be a path connecting q_0 and q_n. q_m $(1{\leq}m{<}n)$ is a κ**-oscilation point** (κ a cardinal) of the path iff there are exactly κ quadrilaterals q so that $\langle q_0, ..., q_{m-1}, q, q_{m+1}, ..., q_n\rangle$ is a path connecting q_0 and q_n.

q_m $(1{\leq}m{<}n)$ is an **oscilation point** iff it is a κ**-oscilation point** for some κ, κ is then called the **magnitude** of the oscilation.

(1.10) Let $\underline{T}{=}\langle W, T\rangle$ be a STS. $\underline{S}$ is **nicely included** in $\underline{T}$ (we shall denote it $\underline{S}{\prec}\underline{T}$) iff

(1) $V{\subset}W$, and $S{\subset}T$;

(2) $|S| < |T|$;

(3) for all quadrilaterals q_0, q_1 in $\underline{S}$, every $\underline{T}$-path connecting q_0 and q_1 which has an $|T|$-oscilation point has at least two distinct $|T|$-oscilation points;

(4) for every quadrilateral q_0 in $\underline{S}$, and every quadrilateral q_1 in $\underline{T}{-}\underline{S}$, there is an $\underline{T}$-path connecting q_0 and q_1 which has at most one $|T|$-oscilation point and all other oscilation points are of magnitude $<|T|$.

Note: It is easy to prove that "nice inclusion" is transitive. In particular, if $\underline{S} \prec \underline{T}$, then any oscilation point on an $\underline{S}$-path remains with the same magnitude in $\underline{T}$.

(1.11) $\{\underline{S}_{\alpha\beta} = \langle V_\beta, S_{\alpha\beta}\rangle : \alpha{<}\aleph_{\delta+1},\ \beta{\le}\delta\}$ is an $\aleph_\delta$**-telescope system** iff

(1) for every α, for every β, $\underline{S}_{\alpha\beta}$ is a nice STS of size $\aleph_\beta$;

(2) for every β, for every $\alpha_0, \alpha_1{<}\aleph_{\beta+1}$, $\underline{S}_{\alpha_0\beta}$ and $\underline{S}_{\alpha_1\beta}$ are not isomorphic;

(3) for every α, for every β_0, β_1, if $\beta_0{<}\beta_1$, then $\underline{S}_{\alpha\beta_0} \prec \underline{S}_{\alpha\beta_1}$;

(4) for every α, for every β limit, $\underline{S}_{\alpha\beta}$ is a product of some subfamily of size $\aleph_\beta$ of the family $\{\underline{T}_\alpha : \alpha{<}\aleph_\beta\}$, where $\underline{T}_\alpha = \bigcup\{\underline{S}_{\alpha\gamma} : \gamma{<}\beta\}$;

(5) for every α, for every $\beta{+}1{<}\delta$, $\underline{S}_{\alpha(\beta+1)}$ is a product of some subfamily of size $\aleph_{\beta+1}$ of the family $\{\underline{S}_{\alpha\beta} : \alpha{<}\aleph_{\beta+1}\}$.

(1.12) Let $T(\aleph_\gamma) = \{\underline{S}^0_{\alpha\beta} : \alpha{<}\aleph_{\gamma+1},\ \beta{\le}\gamma\}$ be an $\aleph_\gamma$-telescope system. Let $T(\aleph_\delta) = \{\underline{S}^1_{\alpha\beta} : \alpha{<}\aleph_{\delta+1},\ \beta{\le}\delta\}$ be an $\aleph_\delta$-telescope system. Let $\gamma{\le}\delta$. We say that $T(\aleph_\delta)$ **extends** $T(\aleph_\gamma)$ (we shall denote it by $T(\aleph_\gamma){\subset}T(\aleph_\delta)$) iff and $\underline{S}^0_{\alpha\beta} = \underline{S}^1_{\alpha\beta}$ for all $\alpha{<}\aleph_{\gamma+1}$ and all $\beta{\le}\gamma$.

(2) Auxiliary results.

(2.1) **Lemma:** For every infinite cardinal κ there is a rich STS of size κ.

Proof:

Let B be a Boolean algebra of size κ, let 0_B be its zero. Consider the binary operation "symmetric difference" on B defined by: $a{\triangle}b{=}(a{-}b){\vee}(b{-}a){=}(a{\vee}b){-}(a{\wedge}b)$. This operation satisfies the following: $a{\triangle}b{=}b{\triangle}a$, $a{\triangle}(a{\triangle}b){=}b$, and $(a{\triangle}b){\triangle}(a{\triangle}c){=}b{\triangle}c$. Let us define a STS $\underline{R}{=}\langle V, R\rangle$ by: $V{=}B{-}0_B$ and $R{=}\{\{a, b, a{\triangle}b\} : a, b{\in}V\}$. Given the properties of $\triangle$, it is easy to check that $\underline{R}$ is a STS of size κ.

Prove that $\underline{R}$ is rich: given $x,y,z{\in}V$. If $\{x,y,z\}{\in}R$, then for any $t{\in}V$ $q{=}\{\{x,y,z\},\{x,t,x\triangle t\}$, $\{y,t,y\triangle t\}$, $\{z,t,z\triangle t\}\}$ is a quadrilateral of $\underline{R}$ and $x,y,z{\in}\bigcup q$. If, on the other hand $\{x,y,z\}{\notin}R$, then for any $t{\in}V$, $q{=}\{\{x,y,x\triangle y\}$, $\{x,t,x\triangle t\}$, $\{y,t,y\triangle t\}$, $\{x\triangle y, x\triangle t, y\triangle t\}\}$ is a quadrilateral of $\underline{R}$ and $x,y,z{\in}\bigcup q$. $\square$

(2.2) **Lemma:** Let a STS $\underline{T}$ be a product of a family of STS's $\Im$ as in (1.7). Then $\underline{T}$ is quadrilateral complete and every quadrilateral has either "horizontal" form, i.e. $\{\{(\alpha,x),(\beta,x),(\gamma,x)\}$, $\{(\alpha,x),(\delta,x),(\varepsilon,x)\}$, $\{(\beta,x),(\delta,x),(\vartheta,x)\}$, $\{(\gamma,x),(\varepsilon,x),(\vartheta,x)\}\}$ for some $x{\in}V$ where $\{\{\alpha,\beta,\gamma\}$, $\{\alpha,\delta,\varepsilon\}$, $\{\beta,\delta,\vartheta\}$, $\{\gamma,\varepsilon,\vartheta\}\}$ is a quadrilateral in $\underline{R}$, or "vertical" form, i.e. $\{\{(\alpha,x),(\alpha,y),(\alpha,z)\}$, $\{(\alpha,x),(\alpha,x_1),(\alpha,y_1)\}$, $\{(\alpha,y),(\alpha,x_1),(\alpha,z_1)\}$, $\{(\alpha,z),(\alpha,y_1),(\alpha,z_1)\}\}$ for some $\alpha{\in}\kappa$ where $\{\{x,y,z\}$, $\{x,x_1,y_1\}$, $\{y,x_1,z_1\}$, $\{z,y_1,z_1\}\}$ is a quadrilateral in $\underline{S}_\alpha$.

Proof:

First consider an arbitrary element (α,x). Since $\underline{R}$ is rich, there is a quadrilateral $q{=}\{\{\alpha,\beta,\gamma\}$, $\{\alpha,\delta,\varepsilon\}$, $\{\beta,\delta,\vartheta\}$, $\{\gamma,\varepsilon,\vartheta\}\}$ in $\underline{R}$ for any β,δ. Then $(\alpha,x){\in}\bigcup q$.

Now, for the other part. Let $q{=}\{\{(\alpha,x),(\beta,y),(\gamma,z)\}$, $\{(\alpha,x),(\delta,x_1),(\varepsilon,y_1)\}$, $\{(\beta,y),(\delta,x_1),(\vartheta,z_1)\}$, $\{(\gamma,z),(\varepsilon,y_1),(\vartheta,z_1)\}\}$ be a quadrilateral in $\underline{T}$. If every block of q was of type (1.7)(iii), then $\{\{x,y,z\}$, $\{x,x_1,y_1\}$, $\{y,x_1,z_1\}$, $\{z,y_1,z_1\}\}$ would be a quadrilateral in $\underline{A}$, a contradiction. Hence at least one of the blocks of q must be of type (1.7)(i) or (1.7)(ii). WLOG assume it is the first block.

First consider the case that $\{x,y,z\}$ is of type (1.7)(i), i.e. $\alpha{=}\beta{=}\gamma$. Thus $q{=}\{\{(\alpha,x),(\alpha,y),(\alpha,z)\}$, $\{(\alpha,x),(\delta,x_1),(\varepsilon,y_1)\}$, $\{(\alpha,y),(\delta,x_1),(\vartheta,z_1)\}$, $\{(\alpha,z),(\varepsilon,y_1),(\vartheta,z_1)\}\}$. From the form of the second and third blocks of q follows that $\varepsilon{=}\vartheta$. Hence $q{=}\{\{(\alpha,x),(\alpha,y),(\alpha,z)\}$, $\{(\alpha,x),(\delta,x_1),(\varepsilon,y_1)\}$, $\{(\alpha,y),(\delta,x_1),(\varepsilon,z_1)\}$, $\{(\alpha,z),(\varepsilon,y_1),(\varepsilon,z_1)\}\}$. Now from the form of the fourth block follows that $\alpha{=}\varepsilon$, and thus $q{=}\{\{(\alpha,x),(\alpha,y),(\alpha,z)\}$, $\{(\alpha,x),(\delta,x_1),(\alpha,y_1)\}$, $\{(\alpha,y),(\delta,x_1),(\alpha,z_1)\}$, $\{(\alpha,z),(\alpha,y_1),(\alpha,z_1)\}\}$. From the form of the second and third blocks follows that $\delta{=}\alpha$ and so q has "vertical" form. In order q was a quadrilateral, $\{\{x,y,z\}$, $\{x,x_1,y_1\}$, $\{y,x_1,z_1\}$, $\{z,y_1,z_1\}\}$ must be a quadrilateral in $\underline{S}_\alpha$.

Second consider the case that $\{x,y,z\}$ is of type (1.7)(ii), i.e. $x{=}y{=}z$. Thus $q{=}\{\{(\alpha,x),(\beta,x),(\gamma,x)\}$, $\{(\alpha,x),(\delta,x_1),(\varepsilon,y_1)\}$, $\{(\beta,x),(\delta,x_1),(\vartheta,z_1)\}$, $\{(\gamma,x),(\varepsilon,y_1),(\vartheta,z_1)\}\}$. From the form of the second and third blocks follows that $y_1{=}z_1$. Hence $q{=}\{\{(\alpha,x),(\beta,x),(\gamma,x)\}$, $\{(\alpha,x),(\delta,x_1),(\varepsilon,y_1)\}$,

$\{(\beta,x),(\delta,x_1),(\vartheta,y_1)\},\ \{(\gamma,x),(\varepsilon,y_1),(\vartheta,y_1)\}\}$. From the form of the third block follows that $x{=}y_1$. Thus $q{=}\{\{(\alpha,x),(\beta,x),(\gamma,x)\},\ \{(\alpha,x),(\delta,x_1),(\varepsilon,x)\},\ \{(\beta,x),(\delta,x_1),(\vartheta,x)\},\ \{(\gamma,x),(\varepsilon,x),(\vartheta,x)\}$. Now, from the form of the second and third blocks follows that $x{=}x_1$ and henceforth $q =$ $\{\{(\alpha,x),(\beta,x),(\gamma,x)\},\ \{(\alpha,x),(\delta,x),(\varepsilon,x)\},\ \{(\beta,x),(\delta,x),(\vartheta,x)\},\ \{(\gamma,x),(\varepsilon,x),(\vartheta,x)\}\}$. So, q has "horizontal" form, and in order q was a quadrilateral in $\underline{T}$, $\{\{\alpha,\beta,\gamma\},\ \{\alpha,\delta,\varepsilon\},\ \{\beta,\delta,\vartheta\},\ \{\gamma,\varepsilon,\vartheta\}\}$ must be a quadrilateral in $\underline{R}$. $\square$

Note: since every quadrilateral family started in fact as "a straight chain" of quadrilaterals being connected by having a block in common (see [F]), every quadrilateral family (i.e. its quadrilaterals) in a product is of "vertical" type.

Note: if $\alpha{\in}\kappa$, q is a quadrilateral of $\underline{S}_\alpha$, then αq will be used to denote the corresponding "vertical" quadrilateral of the product; if $x{\in}V$, q is a quadrilateral of $\underline{R}$, then qx will be used to denote the corresponding "horizontal" quadrilateral of the product. Similarly for quadrilateral families.

(2.3) **Lemma:** Let $\Im$ be a family of nice STS's of size less than κ, while $|\Im| = \kappa$. Let a STS $\underline{T}$ be a product of the family $\Im$ as defined in (1.7). Then for every $\alpha{\in}\kappa$, $\underline{S}_\alpha{\prec}\underline{T}$.

Proof:

Let $\underline{R}$ be the rich STS used in the product $\underline{T}$.

($\star$) Consider a $\underline{T}$-path $\{q_i : i{\leq}n\}$. Let for some m, $0{\leq}m{\leq}n{-}2$, q_m be "vertical" and q_{m+1} be "horizontal". Then q_{m+1} is a $|T|$-oscilation point.

Let $q_m{=}\alpha q$ for some $\alpha{\in}\kappa$. Let $q_{m+1}{=}q'x$ for some $x{\in}V$. Let (α,x) be a common point between αq and $q'x$. Let (β,x) be a common point between $q'x$ and q_{m+2}. Pick any $\gamma{\in}\kappa$. Since $\underline{R}$ is rich, there is a quadrilateral q^γ in $\underline{R}$ so that $\alpha,\beta,\gamma{\in}\bigcup q^\gamma$. Then quadrilateral $q^\gamma x$ has (α,x) in common with q_m and (β,x) in common with q_{m+2}. Hence $\langle q_0,...,q_m,q^\gamma x,q_{m+2},...,q_n\rangle$ is a $\underline{T}$-path connecting q_0 and q_n.

($\star\star$) Consider a $\underline{T}$-path $\{q_i : i{\leq}n\}$. Let for some m, $1{\leq}m{\leq}n{-}1$, q_m be "horizontal" and q_{m+1} be "vertical". Then q_m is a $|T|$-oscilation point.

Proof is practically identical to the proof of $\star$, and so omitted here.

Fix $\alpha{\in}\kappa$.

Note: for simplicity we shall treat the canonic isomorphic embedding of $\underline{S}_\alpha$ into $\underline{T}$ (defined by $\phi(x)=(\alpha,x)$) as an inclusion.

Then $V\subset W$, $S_\alpha\subset T$, and $|S_\alpha|<|T|$.

Consider quadrilaterals q_0, q_1 from $\underline{S}_\alpha$. Consider a $\underline{T}$-path $\wp$ connecting αq_0 and αq_1 and having a $|T|$-oscilation point. Since every $\underline{S}_\alpha$-path has all oscilation points of magnitude $\leq|\underline{S}_\alpha|$, $\wp$ is not an $\underline{S}_\alpha$-path. Thus $\wp$ has to include a "horizontal" quadrilateral. The next quadrilateral in the path must also be "horizontal" (since path $\cdots-\beta q-q'x-\beta q''-\cdots$ implies that $\{\beta q,\beta q'\}$ is an edge of the weak quadrilateral graph of $\underline{T}$, and hence the above would not be a path, a contradiction). To get back to the α's ("vertical") component on a different ("horizontal") level, there must be at least two edges of type $\{q,q'\}$ where one of the quadrilateral is "vertical" and the other is "horizontal". So by $\star$ and $\star\star$ the path has to have at least two $|T|$-oscilation points.

Consider a quadrilateral αq_0 (q_0 from $\underline{S}_\alpha$) and a quadrilateral q_1 from $\underline{T}-\underline{S}_\alpha$.

<u>*Case I*</u>: $q_1=qx$ for some $x\in V$, and some quadrilateral q of $\underline{R}$.

<u>*Subcase Ia*</u>: $(\alpha,x)\in\bigcup qx$.

Since $\underline{S}_\alpha$ is quadrilateral complete, there is a quadrilateral q' in $\underline{S}_\alpha$ so that $x\in\bigcup q'$. Since $\underline{S}_\alpha$ is quadrilateral complete there is an $\underline{S}_\alpha$-path $\langle q'_i : i\leq k\rangle$ connecting q_0 and q'. Every oscilation on this path is of magnitude $\leq|S_\alpha|$. Then $\langle \alpha q'_i : i\leq k\rangle$ is a $\underline{T}$-path connecting αq_0 and $\alpha q'$. One more edge connects $\alpha q'$ and qx. This path has all oscilations of magnitude $\leq|S_\alpha|$.

<u>*Subcase Ib*</u>: $(\alpha,x)\notin\bigcup qx$.

There is some $\beta\in\kappa$ so that $(\beta,x)\in\bigcup qx$. Since $\underline{R}$ is rich, for any $\gamma\in\kappa$, $\gamma\neq\alpha$, $\gamma\neq\beta$, there is a quadrilateral q^γ so that $\alpha,\beta\in\bigcup q^\gamma$. By Subcase Ia there is a $\underline{T}$-path connecting αq_0 and $q^\gamma x$ with all oscilations of magnitude $\leq|S_\alpha|$. Since $q^\gamma x$ and qx form an edge, we have a $\underline{T}$-path which has one $|T|$-oscilation point (q^γ), all other oscilations are of magnitude $\leq|S_\alpha|$. $\square$

<u>*Case II*</u>: $q_1=\beta q$ for some $\beta\in\kappa$, and some quadrilateral q of $\underline{S}_\beta$.

Let $x\in V$ be so that $(\beta,x)\in\bigcup q_1$. Since $\underline{S}_\alpha$ is quadrilateral complete, there is a quadrilateral q' in $\underline{S}_\alpha$ so that $x\in\bigcup q'$. Since $\underline{S}_\alpha$ is quadrilateral connected, there is an $\underline{S}_\alpha$-path $\wp$ connecting q_0 and q'. For any $\gamma\in\kappa$, $\gamma\neq\alpha$, $\gamma\neq\beta$ there is a quadrilateral q^γ in $\underline{R}$ so that $\alpha,\beta,\gamma\in\bigcup q^\gamma$ as $\underline{R}$ is rich. Hence $\{\alpha q', q^\gamma x\}$ is an edge of the weak quadrilateral graph of $\underline{T}$, as well as $\{q^\gamma x, \beta q\}$. These two edges extend the path $\alpha\wp$ connecting αq_0 and $\alpha q'$ to a path connecting αq_0 and $\beta q{=}q_1$. This path has one $|T|$-oscilation point $(q^\gamma x)$, all the other oscilation points are of magnitude $\leq |\underline{S}_\alpha|$. $\square$

(2.4) **Lemma:** Let $\Im^0 = \{\underline{S}^0_\alpha = \langle V^0, S^0_\alpha\rangle : \alpha{\in}\kappa\}$, and let $\Im^1 = \{\underline{S}^1_\alpha = \langle V^1, S^1_\alpha\rangle : \alpha{\in}\kappa\}$, be families of nice STS's, κ an infinite cardinal, $|V^0| < \kappa$, $|V^1| < \kappa$. Let $\underline{T}^0$ be a product of $\Im^0$, and let $\underline{T}^1$ be a product of $\Im^1$. Let $\underline{T}^0$ and $\underline{T}^1$ be isomorphic. Then for every $\alpha{\in}\kappa$ there is a unique $\beta{\in}\kappa$ so that $\underline{S}^0_\alpha$ is isomorphic to $\underline{S}^1_\beta$, and vice versa.

Proof:

Consider an isomorphism $\phi{:}\underline{T}^0{\to}\underline{T}^1$.

Fix $\alpha{\in}\kappa$. Consider some quadrilateral families F_0, F_1 in $\underline{S}^0_\alpha$. Consider a quadrilateral q_0 from F_0 and a quadrilateral q_1 from F_1. By the note after Lemma (2.2) both q_0 and q_1 must be mapped on "vertical" quadrilaterals. Assume that ϕ maps αq_0 onto βq_2 in $\underline{S}^1_\beta$ and that it maps αq_1 onto γq_3 in $\underline{S}^1_\gamma$. If $\beta{\neq}\gamma$, then by Lemma (2.3) there is an $\underline{T}^1$-path connecting βq_2 and γq_3 which has exactly one κ-oscilation points, and all the other oscilations are of magnitude $<\kappa$. Thus (as ϕ is an isomorphism) there must be such a path in $\underline{T}^0$ connecting αq_0 and αq_1. By Lemma (2.3) such path has to have at least two κ-oscilation points, a contradiction. Hence $\beta{=}\gamma$, and so all quadrilateral families of $\underline{S}^0_\alpha$ are mapped by ϕ onto quadrilateral families of $\underline{S}^1_\beta$. Consider an element (α, x). Since $\underline{S}^0_\alpha$ is nice, and hence quadrilateral family complete, there are $(\alpha, y), (\alpha, z)$, and quadrilateral families F_0, F_1, and quadrilaterals αq_0 from F_0 and αq_1 from F_1 so that $\{(\alpha, x), (\alpha, y), (\alpha, z)\}$ is a block in $\underline{S}^0_\alpha$ and $(\alpha, y){\in}\bigcup\alpha q_0$ and $(\alpha, z){\in}\bigcup\alpha q_1$. So ϕ maps both (α, y) and (α, z) into $\underline{S}^1_\beta$. Since ϕ maps a block onto a block, it must map (α, x) into $\underline{S}^1_\beta$. Thus ϕ maps everything from $\underline{S}^0_\alpha$ into $\underline{S}^1_\beta$. By the same token ϕ^{-1} maps everything from $\underline{S}^1_\beta$ into $\underline{S}^0_\alpha$. Thus, ϕ (restricted to $\underline{S}^0_\alpha$) is an isomorphism of $\underline{S}^0_\alpha$ onto $\underline{S}^1_\beta$.
$\square$

(2.5) **Lemma:** There are $2^{\aleph_0}$ non-isomorphic nice rigid STS of size $\aleph_0$.

Proof:

Let $\underline{A}_\alpha{=}\langle V, A_\alpha\rangle$ be a system of $2^{\aleph_0}$ non-isomorphic rigid quadrilateral family complete STS's of size $\aleph_0$ constructed as in the proof of Theorem (3.2) in [F] from a STS $\underline{A}{=}\langle V, A\rangle$ with exactly one anti-quadrilateral chain determined by $[a]{\cup}[-a]$ using a group G, where $V{=}G{\cup}\{e_0, e_1\}$. Let $\underline{B}_\alpha{=}\langle V, B_\alpha\rangle$

be a system of $2^{\aleph_0}$ non-isomorphic rigid quadrilateral family complete STS's of size $\aleph_0$ constructed as in the proof of Theorem (3.2) in [F] from a STS $\underline{B}{=}\langle V, B\rangle$ with exactly one anti-quadrilateral chain determined by $[b]{\cup}[-b]$ using the same group G, $V{=}G{\cup}\{e_0, e_1\}$, and so that $[a]{\cup}[-a]{\neq}[b]{\cup}[-b]$.

Define $\underline{S}_0^0 = \underline{A}_0$, $\underline{S}_0^1 = \underline{B}_0$ and

$$\underline{S}_{n+1}^i = \begin{cases} \underline{A}_{n+1}, & \text{if } \underline{S}_n^i = \underline{B}_n; \\ \underline{B}_{n+1}, & \text{if } \underline{S}_n^i = \underline{A}_n. \end{cases}$$

for all $n{\in}\omega$; $i{=}0, 1$.

Let $\chi{:}\omega{\to}2$ be a non-oscilating function, i.e. $\chi(n){=}\chi(n{+}1)$ for some $n{\in}\omega$. Let $\underline{T}_\chi$ be a product of the family $\{\underline{S}_n^{\chi(n)} : n{\in}\omega\}$. By Lemma (2.2) $\underline{T}_\chi$ is quadrilateral complete, and since every $\underline{S}_n^{\chi(n)}$ is quadrilateral family complete, so is $\underline{T}_\chi$.

Let us show that $\underline{T}_\chi$ is quadrilateral connected.

Consider two distinct quadrilaterals of $\underline{T}_\chi$:

(i) nq_0, nq_1. Since $O{\in}\bigcup q_0{\cap}\bigcup q_1$ (see [F]), then $(n, O){\in}\bigcup nq_0{\cap}\bigcup nq_1$, so nq_0 and nq_1 are connected.

(ii) nq_0, mq_1, $n{\neq}m$. Since $\underline{R}$ is rich, there is a quadrilateral (in fact a lot) q so that $n, m{\in}\bigcup q$. Then "horizontal" quadrilateral qO is connected to nq_0 as well as to mq_1. Thus nq_0 and mq_1 are connected.

($\star$) For every $x{\in}V$ and for every $n{\in}\omega$ so that $\chi(n){=}\chi(n{+}1)$ either there is a quadrilateral q in $\underline{S}_n^{\chi(n)}$ so that $x{\in}\bigcup q$, or there is a quadrilateral q in $\underline{S}_{n+1}^{\chi(n+1)}$ so that $x{\in}\bigcup q$.

If $x{=}O, e_0$, or e_1, then clearly true (see [F] about the form of all quadrilaterals). Let us assume that $x{\in}G{-}\{O\}$. Now assume that $x{\notin}[a]{\cup}[-a]$ (where $[a]{\cup}[-a]$ determines the only anti-quadrilateral chain in $\underline{A}_\alpha$'s). Then there is such a quadrilateral q in every $\underline{A}_m$ for every m. Since $\chi(n{+}1){=}\chi(n)$, either $\underline{S}_{n+1}^{\chi(n+1)} = \underline{A}_{n+1}$, or $\underline{S}_n^{\chi(n)} = \underline{A}_n$, so we are done. On the other hand, if $x{\in}[a]{\cup}[-a]$, then $x{\notin}[b]{\cup}[-b]$ (where $[b]{\cup}[-b]$ determines the only anti-quadrilateral chain in $\underline{B}_\alpha$'s). Now proceed as in the previous case, but with $\underline{B}_n$'s.

(iii) nq_0, q_1x. By ($\star$) there are $m{\in}\omega$ and a quadrilateral q_2 in $\underline{S}_m^{\chi(m)}$ so that $x{\in}q_2$. Hence $(m, x){\in}\bigcup mq_2$. Let $(k, x){\in}q_1x$. Since $\underline{R}$ is rich, there is a quadrilateral q_3 in $\underline{R}$ so that $m, k{\in}\bigcup q_3$. Hence $(m, x), (k, x){\in}\bigcup q_3x$. Thus q_1x is connected to q_3x, which is connected to mq_2, which si connected to nq_0 (by (i) or (ii)).

(iv) q_0x, q_1y. As in (iii), it can be reduced to (i) or (ii).

Hence $\underline{T}_\chi$ is quadrilateral connected, and therefore nice.

Consider $\chi,\theta{:}\omega\to 2$ non-oscilating. Consider an isomorphisms $\phi{:}\underline{T}_\chi\to\underline{T}_\theta$. Let $n\in\omega$. Consider quadrilateral families F_0, F_1 in $\underline{S}_n^{\chi(n)}$. Consider a quadrilateral q_0 from F_0 and a quadrilateral q_1 from F_1. Assume that ϕ maps nq_0 to $\underline{S}_m^{\theta(m)}$ and nq_1 to $\underline{S}_k^{\theta(k)}$. Since $\bigcup nq_0\cap\bigcup nq_1\neq\emptyset$, $m{=}k$. Hence all quadrilateral families from $\underline{S}_n^{\chi(n)}$ must be mapped to $\underline{S}_m^{\theta(m)}$. Since $\underline{S}_n^{\chi(n)}$ is quadrilateral family complete, all elements of $\underline{S}_n^{\chi(n)}$ must be mapped to $\underline{S}_m^{\theta(m)}$ (see e.g. proof of Lemma (2.4)). Since some of quadrilateral families of $\underline{S}_m^{\theta(m)}$ are mapped by ϕ^{-1} to $\underline{S}_n^{\chi(n)}$, consequently all elements of $\underline{S}_m^{\theta(m)}$ are mapped by ϕ^{-1} to $\underline{S}_n^{\chi(n)}$. Therefore $\underline{S}_n^{\chi(n)}$ and $\underline{S}_m^{\theta(m)}$ are isomorphic, and so $n{=}m$ and so $\chi(n){=}\theta(n)$, and thus ϕ must be the identity on $\underline{S}_n^{\chi(n)}$. This is true for every n and so ϕ must be the identity. If $\chi\neq\theta$, then this is impossible, and so $\underline{T}_\chi$ and $\underline{T}_\theta$ are not isomorphic. If $\chi{=}\theta$, then there is only one automorphism, the trivial one. Hence $\underline{T}_\chi$ is rigid. $\square$

(2.6) **Lemma:** There are $2^{\aleph_0}$ non-isomorphic nice STS of size $\aleph_0$ with the same index set and a bijection of this set which is the only non-trivial automorphism of all of them.

Proof:

Let $\underline{A}_\alpha{=}\langle V, A_\alpha\rangle$ be a system of $2^{\aleph_0}$ non-isomorphic quadrilateral family complete STS's of size $\aleph_0$ constructed as in the proof of Theorem (3.3) in [F] from a STS $\underline{A}{=}\langle V, A\rangle$ with no anti-quadrilateral chain, using a group G, where $V{=}G\cup\{e_0, e_1\}$. They all admit only one non-trivial automorphism f defined by $f(x){=}{-x}$ for $x\in G{-}\{O\}$, $f(O){=}O$, $f(e_0){=}e_0$, and $f(e_1){=}e_1$.

Define $\underline{S}_n^0{=}\underline{A}_n$ and $\underline{S}_n^1{=}\underline{A}_{\omega+n}$ for all $n\in\omega$.

Let $\chi{:}\omega\to 2$. Let $\underline{T}_\chi$ be a product of the family $\{\underline{S}_n^{\chi(n)} : n\in\omega\}$. By Lemma (2.2) $\underline{T}_\chi$ is quadrilateral complete, and since every $\underline{S}_n^{\chi(n)}$ is quadrilateral family complete, so is $\underline{T}_\chi$.

Let us show that $\underline{T}_\chi$ is quadrilateral connected.

Consider two distinct quadrilaterals of $\underline{T}_\chi$:

(i) nq_0, nq_1. Since $O\in\bigcup q_0\cap\bigcup q_1$ (see [F]), then $(n,O)\in\bigcup nq_0\cap\bigcup nq_1$, so nq_0 and nq_1 are connected.

(ii) nq_0, mq_1, $n\neq m$. Since $\underline{R}$ is rich, there is a quadrilateral (in fact a lot) q so that $n,m\in\bigcup q$. Then "horizontal" quadrilateral qO is connected to nq_0 as well as to mq_1. Thus nq_0 and mq_1 are connected.

($\star$) For every $x\in V$ and for every $n\in\omega$ there is a quadrilateral q in $\underline{S}_n^{\chi(n)}$ so that $x\in\bigcup q$.

Check the proof of Theorem (3.3) in [F] that ($\star$) holds for every $\underline{A}_\alpha$.

(iii) nq_0, q_1x. By ($\star$) there are $m\in\omega$ and a quadrilateral q_2 in $\underline{S}_m^{\chi(m)}$ so that $x\in\bigcup q_2$. Hence $(m,x)\in\bigcup mq_2$. Let $(k,x)\in q_1x$. Since $\underline{R}$ is rich, there is a quadrilateral q_3 in $\underline{R}$ so that $m,k\in\bigcup q_3$. Hence $(m,x),(k,x)\in\bigcup q_3x$. Thus q_1x is connected to q_3x, which is connected to mq_2, which si connected to nq_0 (by (i) or (ii)).

(iv) q_0x, q_1y. As in (iii), it can be reduced to (i) or (ii).

Hence $\underline{T}_\chi$ is quadrilateral connected, and therefore nice.

Consider $\chi,\theta{:}\omega\to 2$. Consider an isomorphism $\phi{:}\underline{T}_\chi\to\underline{T}_\theta$. Let $n\in\omega$. Consider quadrilateral families F_0,F_1 in $\underline{S}_n^{\chi(n)}$. Consider a quadrilateral q_0 from F_0 and a quadrilateral q_1 from F_1. Assume that ϕ maps nq_0 to $\underline{S}_m^{\theta(m)}$ and nq_1 to $\underline{S}_k^{\theta(k)}$. Since $\bigcup nq_0\cap\bigcup nq_1\neq\emptyset$, $m{=}k$. Hence all quadrilateral families from $\underline{S}_n^{\chi(n)}$ must be mapped to $\underline{S}_m^{\theta(m)}$. Since $\underline{S}_n^{\chi(n)}$ is quadrilateral family complete, all elements of $\underline{S}_n^{\chi(n)}$ must be mapped to $\underline{S}_m^{\theta(m)}$ (see a.g. proof of Lemma (2.4)). Since some of quadrilateral families of $\underline{S}_m^{\theta(m)}$ are mapped by ϕ^{-1} to $\underline{S}_n^{\chi(n)}$, consequently all elements of $\underline{S}_m^{\theta(m)}$ are mapped by ϕ^{-1} to $\underline{S}_n^{\chi(n)}$. Therefore $\underline{S}_n^{\chi(n)}$ and $\underline{S}_m^{\theta(m)}$ are isomorphic, and so $n{=}m$ and $\chi(n){=}\theta(m)$. Since this must be true for all n, it is impossible if $\chi\neq\theta$. Henceforth $\underline{T}_\chi$ and $\underline{T}_\theta$ are not isomorphic if $\chi\neq\theta$.

In case $\chi{=}\theta$, ϕ must be either the identity on $\underline{S}_n^{\chi(n)}$, or it must be equal to f.

Pick any $m\neq n$, and any $x\in V$. There is a quadrilateral q in $\underline{R}$ so that $n,m\in\bigcup q$. The isomorphism ϕ must map qx onto some $q'y$ (for if it mapped qx onto a "vertical" $\beta q'$ than ϕ^{-1} would map $\beta q'$ onto a "horizontal" quadrilateral. But $\beta q'$ is in some quadrilateral family, and hence must be mapped onto a "vertical" quadrilateral, a contradiction). Since $\phi(n,x){=}(n,f(x))$, $y{=}f(x)$ and thus $\phi(m,x){=}(m,f(x))$. Thus ϕ is either the identity on all $\underline{S}_n^{\chi(n)}$'s, and hence the identity on $\underline{T}_\chi$, or it is equal to f on all $\underline{S}_n^{\chi(n)}$'s. Thus $\underline{T}_\chi$ has exactly one non-trivial automorphism. $\square$

(2.7) **Lemma:** Let $T(\aleph_\delta)=\{\underline{S}_{\alpha\beta}=\langle V_\beta,S_{\alpha\beta}\rangle : \alpha{<}\aleph_{\delta+1},\ \beta{\leq}\delta\}$ be an $\aleph_\delta$-telescope system such that all $\underline{S}_{\alpha\delta}$'s for all $\alpha{<}\aleph_{\delta+1}$ are rigid. Then

(1) there are $2^{\aleph_{\delta+1}}$ non-isomorphic nice rigid STS's of size $\aleph_{\delta+1}$;

(2) there is an $\aleph_{\delta+1}$-telescope system extending $T(\aleph_\delta)$.

Proof:

Split the sequence $\{\underline{S}_{\alpha\delta} : \alpha{<}\aleph_{\delta+1}\}$ into two disjoint sequences $\{\underline{S}^0_\alpha : \alpha{<}\aleph_{\delta+1}\}$, and $\{\underline{S}^1_\alpha : \alpha{<}\aleph_{\delta+1}\}$.

Let $\chi{:}\aleph_{\delta+1}{\to}2$. Define $\underline{T}_\chi$ to be a product (as defined in (1.7)) of the family $\{\underline{S}^{\chi(\alpha)}_\alpha : \alpha{<}\aleph_{\delta+1}\}$. By Lemmas (2.2) and (2.3), $\underline{T}_\chi$ is a nice STS of size $\aleph_{\delta+1}$ and every $\underline{S}^{\chi(\alpha)}_\alpha$ is nicely included in $\underline{T}_\chi$.

Consider $\chi, \theta{:}\aleph_{\delta+1}{\to}2$. Let $\phi{:}\underline{T}_\chi{\to}\underline{T}_\theta$. By Lemma (2.4) for every $\alpha{<}\aleph_{\delta+1}$ there is $\beta{<}\aleph_{\delta+1}$ so that ϕ maps isomorphicaly $\underline{S}^{\chi(\alpha)}_\alpha$ onto $\underline{S}^{\theta(\beta)}_\beta$, and vice versa. So $\alpha{=}\beta$ and $\chi(\alpha){=}\theta(\beta)$.

If $\chi{\neq}\theta$, then this is impossible, and so there is no isomoprhism ϕ.

If $\chi{=}\theta$, then ϕ maps $\underline{S}^{\chi(\alpha)}_\alpha$ onto itself and hence (as it is rigid) ϕ must be the identity on $\underline{S}^{\chi(\alpha)}_\alpha$. Since this is true for any α, ϕ must be the identity. Hence $\underline{T}_\chi$ is rigid. (1) has been proven.

To prove (2), choose a sequence $\{\underline{T}_\alpha : \alpha{<}\aleph_{\delta+2}\}$ from the $2^{\aleph_{\delta+1}}$ non-isomorphic nice rigid STS's of size $\aleph_{\delta+1}$ obtained in (1). Let us define a system $T(\aleph_{\delta+1}) = \{\underline{T}_{\alpha\beta} : \alpha{<}\aleph_{\delta+2},\ \beta{\leq}\delta{+}1\}$ by:

$\underline{T}_{\alpha\beta}{=}\underline{S}_{\alpha\beta}$ for $\alpha{<}\aleph_{\delta+1}$, $\beta{\leq}\delta$;

$\underline{T}_{\alpha\beta}{=}\underline{S}_{0\beta}$ for $\aleph_{\delta+1}{\leq}\alpha{<}\aleph_{\delta+2}$, $\beta{\leq}\delta$;

$\underline{T}_{\alpha\delta}{=}\underline{T}_\alpha$ for $\alpha{<}\aleph_{\delta+2}$. It is now straightforward to check that $T(\aleph_{\delta+1})$ is an $\aleph_{\delta+1}$-telescop extending $T(\aleph_\delta)$. $\square$

(2.8) **Lemma:** Let $T(\aleph_\delta) = \{\underline{S}_{\alpha\beta} = \langle V_\beta, S_{\alpha\beta}\rangle : \alpha{<}\aleph_{\delta+1},\ \beta{\leq}\delta\}$ be an $\aleph_\delta$-telescope system such that there is a bijection f of V_δ which is the only non-trivial automorphism of all $\underline{S}_{\alpha\delta}$'s for all $\alpha{<}\aleph_{\delta+1}$. Then

(1) there are a bijection ϕ of a set W and $2^{\aleph_{\delta+1}}$ non-isomorphic nice STS's of size $\aleph_{\delta+1}$ with the index set W, so that ϕ is their only non-trivial automorphism, and ϕ extends f;

(2) there is an $\aleph_{\delta+1}$-telescope system extending $T(\aleph_\delta)$.

Proof:

Split the sequence $\{\underline{S}_{\alpha\delta} : \alpha{<}\aleph_{\delta+1}\}$ into two disjoint sequences $\{\underline{S}^0_\alpha : \alpha{<}\aleph_{\delta+1}\}$, and $\{\underline{S}^1_\alpha : \alpha{<}\aleph_{\delta+1}\}$.

Let $\chi{:}\aleph_{\delta+1}{\to}2$. Define $\underline{T}_\chi$ to be a product (as defined in (1.7)) of the family $\{\underline{S}^{\chi(\alpha)}_\alpha : \alpha{<}\aleph_{\delta+1}\}$. By Lemmas (2.2) and (2.3), $\underline{T}_\chi$ is a nice STS of size $\aleph_{\delta+1}$ and every $\underline{S}^{\chi(\alpha)}_\alpha$ is nicely included in $\underline{T}_\chi$.

Consider $\chi, \theta{:}\aleph_{\delta+1}{\to}2$. Let $\phi{:}\underline{T}_\chi{\to}\underline{T}_\theta$. By Lemma (2.4) for every $\alpha{<}\aleph_{\delta+1}$ there is $\beta{<}\aleph_{\delta+1}$ so that ϕ maps isomorphicaly $\underline{S}^{\chi(\alpha)}_\alpha$ onto $\underline{S}^{\theta(\beta)}_\beta$, and vice versa. So $\alpha{=}\beta$ and $\chi(\alpha){=}\theta(\beta)$.

If $\chi{\neq}\theta$, then this is impossible, and so there is no isomoprhism ϕ.

If $\chi=\theta$, then ϕ maps $\underline{S}_\alpha^{\chi(\alpha)}$ onto itself and hence ϕ must be the identity or equal to the only non-trivial automorphism of $\underline{S}_\alpha^{\chi(\alpha)}$, f. If ϕ is not the identity on some $\underline{S}_\alpha^{\chi(\alpha)}$, then it is not the identity on all of them (see e.g. the proof of Lemma (2.6)). Hence $\underline{T}_\chi$ has exactly one non-trivial automorphism. (1) has been proven.

To prove (2) is identical to the proof of (2) in Lemma (2.7) and so omitted here. $\square$

(2.9) **Lemma:** Let $\aleph_\delta$ be a limit cardinal (i.e. δ is a limit ordinal). Let $\{T(\aleph_\beta) : \beta<\delta\}$, be an $\subset$-increasing sequence of $\aleph_\beta$-telescope systems ($\beta<\delta$) with all STS's involved being rigid.

(1) there are $2^{\aleph_\delta}$ non-isomorphic nice rigid STS's of size $\aleph_\delta$;

(2) there is an $\aleph_\delta$-telescope system $T(\aleph_\delta)$ which extends all $T(\aleph_\beta)$'s for all $\beta<\delta$.

Proof:

Let $T(\aleph_\beta) = \{\underline{S}_{\alpha\gamma}^\beta = \langle V_\gamma^\beta, S_{\alpha\gamma}^\beta\rangle : \alpha<\aleph_{\beta+1},\, \gamma\leq\beta\}$, $\beta<\delta$.

For any $\alpha<\aleph_\delta$ define $\underline{T}_\alpha = \bigcup\{\underline{S}_{\alpha\beta}^\beta : \beta<\delta\}$. Since each sequence $\{\underline{S}_{\alpha\beta}^\beta : \beta<\delta\}$ is a $\prec$-increasing sequence of nice STS's, all $\underline{T}_\alpha$'s are nice STS's of size $\aleph_\delta$, all with the same index set $V = \bigcup\{V^\beta : \beta<\delta\}$.

Split the sequence $\{\underline{T}_\alpha : \alpha<\aleph_\delta\}$ into two disjoint sequences $\{\underline{T}_\alpha^0 : \alpha<\aleph_\delta\}$ and $\{\underline{T}_\alpha^1 : \alpha<\aleph_\delta\}$.

Let $\chi{:}\aleph_\delta\to 2$. Define $\underline{S}_\chi$ to be a product of the family $\{\underline{T}_\alpha^{\chi(\alpha)} : \alpha<\aleph_\delta\}$. By Lemma (2.2), $\underline{S}_\chi$ is quadrilateral complete, and since all STS's involved are quadrilateral family complete, it is also quadrilateral family complete. In the same way as in the proof of Lemma (2.3) one can prove that $\underline{S}_\chi$ is quadrilateral connected. Hence $\underline{S}_\chi$ is quadrilateral is a nice STS of size $\aleph_\delta$, with the index set $\aleph_\delta\times V$. Let $\underline{T}_\alpha^{\chi(\alpha)} = \bigcup\{\underline{S}_{\gamma\varepsilon}^\varepsilon : \varepsilon<\delta\}$ for some $\gamma<\delta$. Similarly as in the proof of Lemma (2.3), one can show that each $\underline{S}_{\gamma\varepsilon}^\varepsilon \prec \underline{T}_\alpha^{\chi(\alpha)}$, for all α's.

Let $\chi,\theta{:}\aleph_\delta\to 2$. Let $\phi{:}\underline{S}_\chi\to\underline{S}_\theta$ be an isomorphism.

($\star$) for every $\alpha<\aleph_\delta$ there is $\beta<\aleph_\delta$ such that ϕ (restricted to $\underline{T}_\alpha^{\chi(\alpha)}$) is an isomorphism of $\underline{T}_\alpha^{\chi(\alpha)}$ onto $\underline{T}_\beta^{\theta(\beta)}$.

Let $\alpha\in\aleph_\delta$. $\underline{T}_\alpha^{\chi(\alpha)} = \bigcup\{\underline{S}_{\gamma\varepsilon}^\varepsilon : \varepsilon<\delta\}$ for some $\gamma<\delta$. Consider two quadrilateral families F_0, F_1 in $\underline{T}_\alpha^{\chi(\alpha)}$. Let q_0 be from F_0, let q_1 be from F_1. Assume that ϕ maps αq_0 into $\underline{T}_\beta^{\theta(\beta)}$ and αq_1 into $\underline{T}_{\beta_0}^{\theta(\beta_0)}$. By the way of contradiction assume that $\beta\neq\beta_0$. Similarly as in the proof of

Lemma (2.3) one can show that there is a $\underline{T}_\theta$-path connecting $\phi(\alpha q_0)$ and $\phi(\alpha q_1)$ which has exactly one $\aleph_\delta$-oscilation points and all other oscilation points have magnitude $< \aleph_\delta$. Hence there must be such a $\underline{T}_\chi$-path $\wp$ connecting αq_0 and αq_1. There is $\gamma_0 < \delta$ so that $\wp$ is an $\underline{S}^\varepsilon_{\gamma_0\varepsilon}$-path, and as such it cannot have an ascilation point of magnitude bigger than the size of $\underline{S}^\varepsilon_{\gamma_0\varepsilon}$, a contradiction. Henceforth $\beta{=}\beta_0$. Thus all quadrilateral families of $\underline{T}^{\chi(\alpha)}_\alpha$ are mapped by ϕ onto quadrilateral families of $\underline{T}^{\theta(\beta)}_\beta$, and so by quadrilateral family completeness (as in the proof of Lemma (2.3)) ϕ (restricted to $\underline{T}^{\chi(\alpha)}_\alpha$) is an isomorphism of $\underline{T}^{\chi(\alpha)}_\alpha$ onto $\underline{T}^{\theta(\beta)}_\beta$.

($\star\star$) Let ϕ map $\underline{T}^{\chi(\alpha)}_\alpha$ onto $\underline{T}^{\theta(\beta)}_\beta$. Then $\underline{T}^{\chi(\alpha)}_\alpha = \underline{T}^{\theta(\beta)}_\beta$, and so $\alpha{=}\beta$ and $\chi(\alpha){=}\theta(\beta)$.

Let $\underline{T}^{\chi(\alpha)}_\alpha = \bigcup\{\underline{S}^\varepsilon_{\gamma\varepsilon} : \varepsilon{<}\delta\}$ for some $\gamma{<}\delta$. Let $\underline{T}^{\theta(\beta)}_\beta = \bigcup\{\underline{S}^\varepsilon_{\rho\varepsilon} : \varepsilon{<}\delta\}$ for some $\rho{<}\delta$. Pick a quadrilateral q_0 in $\underline{T}^{\chi(\alpha)}_\alpha$. Since $\alpha,\beta{<}\aleph_\delta$, there is $\delta_0{<}\delta$ so that $\alpha,\beta{<}\aleph_{\delta_0}$. Then there is $\delta_1{\geq}\delta_0$ so that q_0 is a quadrilateral of $\underline{S}^{\delta_1}_{\gamma\delta_1}$ and ϕ maps q_0 into $\underline{S}^{\delta_1}_{\rho\delta_1}$. Pick any quadrilateral q_1 of $\underline{S}^{\delta_1}_{\gamma\delta_1}$ distinct from q_0. If ϕ mapped q_1 outside of $\underline{S}^{\delta_1}_{\rho\delta_1}$, say into $\underline{S}^{\delta_2}_{\rho\delta_2}$ (for some $\delta_1{<}\delta_2{<}\delta$), there would be a $\underline{T}^{\theta(\beta)}_\beta$-path connecting $\phi(q_0)$ and $\phi(q_1)$ with exactly one "big" oscilation point and all other oscilation points of "small" magnitude ("big" means bigger than size of $\underline{S}^{\delta_1}_{\gamma\delta_1}$ which is $\aleph_{\delta_1}$, "small" means smaller or equal to $\aleph_{\delta_1}$), as $\underline{S}^{\delta_1}_{\rho\delta_1}$ is nicely included $\underline{S}^{\delta_2}_{\rho\delta_2}$. On the other hand every $\underline{T}^{\chi(\alpha)}_\alpha$-path connecting q_0, q_1 having a "big" oscilation points has to have at least two "big" oscilation points, a contradiction. Thus ϕ has to map q_1 into $\underline{S}^{\delta_1}_{\rho\delta_1}$, and so by quadrilateral family completeness, $\underline{S}^{\delta_1}_{\gamma\delta_1}$ must be isomorphic to $\underline{S}^{\delta_1}_{\rho\delta_1}$. Thus $\underline{S}^{\delta_1}_{\gamma\delta_1} = \underline{S}^{\delta_1}_{\rho\delta_1}$. Similar argument shows that in fact $\underline{S}^{\delta_3}_{\gamma\delta_3} = \underline{S}^{\delta_3}_{\rho\delta_3}$ for any $\delta_1{\leq}\delta_3{<}\delta$. Hence $\underline{T}^{\chi(\alpha)}_\alpha = \underline{T}^{\theta(\beta)}_\beta$.

By ($\star$) and ($\star\star$) $\chi(\alpha){=}\theta(\alpha)$ for all α's, which is clearly impossible if $\chi{\neq}\theta$, and so $\underline{S}_\chi$ and $\underline{S}_\theta$ are not isomorphic.

If $\chi{=}\theta$ then by ($\star$) and ($\star\star$) for every $\alpha{<}\delta$, ϕ must be an automorphism of all STS's in the sequence $\{\underline{S}^\varepsilon_{\gamma\varepsilon} : \varepsilon{<}\delta\}$ (where $\underline{T}^{\chi(\alpha)}_\alpha = \bigcup\{\underline{S}^\varepsilon_{\gamma\varepsilon} : \varepsilon{<}\delta\}$). But these are all rigid, hence ϕ must be the identity on $\underline{T}^{\chi(\alpha)}_\alpha$, and hence on $\underline{S}_\chi$. Thus $\underline{S}_\chi$ is rigid. Therefore (1) is proven.

To prove (2), first choose a sequence $\{\underline{T}_\alpha : \alpha{<}\aleph_{\delta+1}$ from the STS created in (1). Then define an $\aleph_\delta$-telescope system $T(\aleph_\delta) = \{\underline{S}^\delta_{\alpha\gamma} = \langle V^\delta_\gamma, S^\delta_{\alpha\gamma}\rangle : \alpha{<}\aleph_{\delta+1},\ \gamma{\leq}\delta\}$ by

$\underline{S}^\delta_{\alpha\gamma} = \underline{S}^\gamma_{\alpha\gamma}$ for $\alpha{<}\aleph_\delta$ and $\gamma{<}\delta$,

$\underline{S}^{\delta}_{\alpha\gamma} = \underline{S}^{\gamma}_{0\gamma}$ for $\aleph_\delta \leq \alpha < \aleph_{\delta+1}$, and

$\underline{S}^{\delta}_{\alpha\delta} = \underline{T}_{\alpha}$ for all $\alpha < \aleph_{\delta+1}$.

To verify that it is an $\aleph_\delta$-telescope system extending all $T(\aleph_\beta)$ for all $\beta<\delta$ is left to the reader. $\square$

(2.10) **Lemma:** Let $\aleph_\delta$ be a limit cardinal. Let $\{T(\aleph_\beta) : \beta<\delta\}$, be an $\subset$-increasing sequence of $\aleph_\beta$-telescope systems ($\beta<\delta$). Let $T(\aleph_\beta) = \{\underline{S}^{\beta}_{\alpha\gamma} = \langle V^{\beta}_{\gamma}, S^{\beta}_{\alpha\gamma}\rangle : \alpha<\aleph_{\beta+1},\ \gamma\leq\beta\}$, $\beta<\delta$. Let $\{f_\gamma : \gamma<\delta\}$ be a sequence such that for every $\gamma<\delta$, f_γ is a bijection of V^{γ}_{γ}, and it is the only non-trivial automorphism of all $\underline{S}^{\beta}_{\alpha\gamma}$ (for all $\beta\geq\gamma$), and f_{γ_0} extends f_{γ_1} whenever $\gamma_1<\gamma_0<\delta$.

(1) there are $2^{\aleph_\delta}$ non-isomorphic nice STS's of size $\aleph_\delta$ with the same index set W, and a bijection ϕ of W extedning all f_γ's, and being the only non-trivial automorphism of all of them;

(2) there is an $\aleph_\delta$-telescope system $T(\aleph_\delta)$ which extends all $T(\aleph_\beta)$'s for all $\beta<\delta$.

Proof:

The proof is identical to the proof of Lemma (2.9). Just realize that when ϕ is an automorphism of $\underline{T}^{\chi(\alpha)}_{\alpha}$, it must be an automorphism of all STS's in the sequence $\{\underline{S}^{\varepsilon}_{\gamma\varepsilon} : \varepsilon<\delta\}$ (where $\underline{T}^{\chi(\alpha)}_{\alpha} = \bigcup\{\underline{S}^{\varepsilon}_{\gamma\varepsilon} : \varepsilon<\delta\}$). Clearly, if it is a non-trivial automorphism of one of them, it must be the (only) non-trivial automorphism on all of them, hence there is only one non-trivial automorphism ϕ extending all f_γ's.

$\square$

(3) Main results.

(3.1) **Theorem:** For every infinite cardinal κ there are 2^κ non-isomorphic nice rigid STS's of size κ.

Proof:

We shall proceed by induction over κ.

(1) $\kappa=\aleph_0$.

By Lemma (2.5) there are $2^{\aleph_0}$ non-isomorphic nice rigid STS's of size $\aleph_0$, and one can easily form an $\aleph_0$-telescope system $T(\aleph_0)$ from them.

(2) As the induction hypothesis assume that we have an $\subset$-increasing sequence of $T(\aleph_\alpha)$ of $\aleph_\alpha$-telescope systems, $\alpha\leq\delta$, with all STS's involved being rigid.

Then by Lemma (2.7) there are $2^{\aleph_{\delta+1}}$ non-isomorphic nice rigid STS's of size $\aleph_{\delta+1}$, and also there is an $\aleph_{\delta+1}$-telescope system $T(\aleph_{\delta+1})$ containing some of them and extending all $T(\aleph_\alpha)$'s for all $\alpha\leq\delta$.

(3) As the induction hypothesis assume that we have an $\subset$-increasing sequence of $T(\aleph_\alpha)$ of $\aleph_\alpha$-telescope systems, $\alpha<\delta$, δ a limit ordinal, with all STS's involved being rigid.

By Lemma (2.9) there are $2^{\aleph_\delta}$ non-isomorphic nice rigid STS's of size $\aleph_\delta$, and also there is an $\aleph_\delta$-telescope system $T(\aleph_\delta)$ containing some of them and extending all $T(\aleph_\alpha)$'s for all $\alpha<\delta$. $\square$

(3.2) **Theorem:** For every infinite cardinal κ there are 2^κ non-isomorphic nice STS's of size κ with the same index set and a bijection of the index set which is their only non-trivial automorphism.

Proof:

We shall proceed by induction over κ.

(1) $\kappa=\aleph_0$.

By Lemma (2.6) there are $2^{\aleph_0}$ non-isomorphic nice STS's of size $\aleph_0$ with the same index set, and a bijection of this index set which is their only non-trivial automorphism. One can easily form an $\aleph_0$-telescope system $T(\aleph_0)$ from them.

(2) As the induction hypothesis assume that we have an $\subset$-increasing sequence of $T(\aleph_\alpha)$ of $\aleph_\alpha$-telescope systems, $\alpha\leq\delta$ and an increasing sequence f_α $(\alpha\leq\delta)$ so that all STS's on the level α have the same index set, f_α is a bijection of this index set, and it is the only non-trivial automorphism of all of STS's on level α.

Then by Lemma (2.8) there are $2^{\aleph_{\delta+1}}$ non-isomorphic nice STS's of size $\aleph_{\delta+1}$ with the same index set, and a bijection $f_{\delta+1}$ of this index set, which is the only non-trivial automorphism of all of them, extending all f_α's, $\alpha\leq\delta$. Also, by Lemma (2.8), there is an $\aleph_{\delta+1}$-telescope system $T(\aleph_{\delta+1})$ containig some of them and extending all $T(\aleph_\alpha)$'s for all $\alpha\leq\delta$.

(3) As the induction hypothesis assume that we have an $\subset$-increasing sequence of $T(\aleph_\alpha)$ of $\aleph_\alpha$-telescope systems, $\alpha<\delta$, δ a limit ordinal, and an increasing sequence f_α $(\alpha<\delta)$, such that all STS's on level α have the same index set, and f_α is a bijection of this index set, and it is the only non-trivial automorphism of all of STS's on level α.

By Lemma (2.10) there are $2^{\aleph_\delta}$ non-isomorphic nice STS's of size $\aleph_\delta$ with the same index set, and a bijection f_δ of this index set, extending all f_α's, and being their only non-trivial automorphism. Also there is an $\aleph_\delta$-telescope system $T(\aleph_\delta)$ containing some of them and extending all $T(\aleph_\alpha)$'s for all $\alpha<\delta$.

$\square$

(3.3) **Theorem:** For every infinite cardinal κ and every cardinal λ, $2 \leq \lambda \leq \kappa$, there are 2^{κ} non-isomorphic STS's of size κ admitting exactly 2^{λ} automorphisms.

Proof:

Fix λ. Let $\{\underline{S}^0_\alpha = \langle V, S^0_\alpha\rangle : \alpha \leq \lambda\}$ and $\{\underline{S}^1_\alpha = \langle V, S^1_\alpha\rangle : \alpha \leq \lambda\}$ be disjoint sequences of non-isomorphic nice STS's of size κ with the same index set V, and let f be a bijection of V which is the only non-trivial automorphism of all $\underline{S}^0_\alpha$'s and $\underline{S}^1_\alpha$'s (by Theorem (3.2)).

Let $\chi{:}\kappa \to 2$.

Let $\underline{T}_\chi$ be a product of the family $\{\underline{S}^{\chi(\alpha)}_\alpha : \alpha \leq \lambda\}$ as in (1.7), but with the change that the "underlying" STS $\underline{R}$ be anti-Pasch rather than rich. It is easy to show that the result is an STS of size κ with the index set $\kappa \times V$, and so that all its quadrilaterals are of "horizontal" type.

Consider $\chi, \theta{:}\kappa \to 2$. Let $\phi \underline{T}_\chi \to \underline{T}_\theta$ be an isomorphism. Consider $\underline{S}^{\chi(\alpha)}_\alpha$. Consider two distinct quadrilaterals q_0,q_1 from $\underline{S}^{\chi(\alpha)}_\alpha$. Suppose that ϕ maps αq_0 into $\underline{S}^{\theta(\beta)}_\beta$, and that it maps αq_1 into $\underline{S}^{\theta(\gamma)}_\gamma$. If $\gamma \neq \beta$, then $\phi(\alpha q_0)$ and $\phi(\alpha q_1)$ are not connected in $\underline{T}_\theta$. But $\underline{S}^{\chi(\alpha)}_\alpha$ is nice, and hence quadrilateral connected, a contradiction. Thus $\gamma = \beta$, and so $\underline{S}^{\chi(\alpha)}_\alpha$ and $\underline{S}^{\theta(\beta)}_\beta$ are isomorphic, hence equal.

If $\chi \neq \theta$, this is impossible, so $\underline{T}_\chi$ and $\underline{T}_\theta$ are not-isomorphic.

If $\chi = \theta$, then ϕ (restricted to $\underline{S}^{\chi(\alpha)}_\alpha$) must be an automorphism of $\underline{S}^{\chi(\alpha)}_\alpha$. Thus ϕ must be a combination of automorphisms on all components, and there are exactly 2^{λ} such combinations. $\square$

References

[DR] J.Doyen, A.Rosa, *An updated bibliography of Steiner systems* in *Topics on Steiner systems* (Ed. C.C.Lindner and A. Rosa), Ann. Discrete Math. 7(1980), 317-349.

[F] F.Franek, *Isomorphism of Infinite Steiner Triple Systems*, submitted for publication (being refereed); meanwhile available as Technical Report No. 87-06, Dept. of Comp. Sci. and Systems, McMaster university, Hamilton, Ont.

[N] J.Nesetril, *On infinite precise objects*, Matematica Slovaca 28(1978),253-260.

[Si] W.Sierpinski,*Sur un probleme de triades*, C.R.Soc.Sci.Varsovie 33-38(1946),13-16.

[So] B.Sobocinski, *A theorem of Sierpinski on triads and the axiom of choice*, Notre Dame J. Formal Logic 5(1964),51-58.

[Vu] V.Vuckovic, *Note on a theorem of W.Sierpinski*, Notre Dame J. Formal Logic 6(19865), 180-182.

[GGP] M.J.Grannell, T.S.Griggs, J.S.Phelan, *Countably infinite Steiner triple systems*, to appear in Ars Combinatoria.

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