Collagory Notes, Version 1

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March 2010

Abstract

These notes are a current snapshot of the development of the theory of *collagories*, which are defined essentially as “distributive allegories without zero morphisms” and named for their close relationship with the adhesive categories currently popular as a foundation for the categorical double-pushout approach to graph transformation.

We argue that, thanks to their relation-algebraic flavour, collagories provide a more accessible and more flexible setting. One contributing factor to this is that the universal characterisations of pushouts and pullbacks in categories can be replaced with the local characterisations of tabulations and co-tabulations in collagories.

We document accessibility by showing ways to construct collagories of semi-unary algebras, which allow natural representations in particular of graph structures, also with fixed label sets.

Via the local ordering on homsets, collagories have a simple 2-categorical structure, and we use this to show that co-tabulations are equivalent to lax colimits of difunctional morphisms, and co-tabulations arising from spans of mappings are equivalent to bipushouts, which satisfy stronger conditions than just pushouts of mappings.

Finally, we consider Van-Kampen squares, the central ingredient of the definition of adhesive categories, and obtain an interesting characterisation of Van-Kampen squares in collagories.

This research has been supported by the National Science and Engineering Research Council (NSERC), Canada

Software Quality Research Laboratory

McMaster University

SQRL Report No. 57

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Chapter 1

Introduction and Basic Definitions

1.1 Introduction

The co-tabulation characterisation of pushouts, which essentially goes back to Kawahara [Kaw90], therefore has a *precise* categorical counterpart in bipushouts, and, even more closely, in lax colimits of difunctional morphisms in an allegory context.

One of the hallmarks of the relation-algebraic approach to graph transformation [Kaw90, Kah01, Kah04] is that it allows an abstract characterisation of the gluing condition for the double pushout approach. Nevertheless, the categorical approach to graph transformation has continued to use the node-and-edge-based formulation of the gluing condition even in the handbook chapter [CMR+97]. Recently, the literature of the categorical approach, starting essentially with [EPPH06] has adopted the "adhesive categories" of Lack and Sobociński [LS04], where however the details of the gluing condition are completely sidestepped.

Although the toposes of graph structures that give rise to the relational categories used in the relational approach are examples of adhesive categories, the latter also include, for example, categories of pointed sets, which do not give rise to distributive allegories due to the failure of the zero law.

We introduce collagories by dropping the zero law (and the necessary existence of least morphisms) from distributive allegories, and show that this still produces a relational formalism that can accommodate the necessary tools for graph transformation, and furthermore relates nicely with adhesive categories.

After the basic category, allegory, and collagory definitions in the remaining sections of this chapter we turn to the definitions and properties of tabulations and co-tabulations in Chapter 2. In Chapter 3, we then use a simple generalisation of standard Σ-algebra construction for signatures containing only unary function symbols and constant symbols. This way, we obtain collagories of "relational Σ-algebra homomorphisms" on top of simpler base collagories, as for example the collagory *Rel* of sets and relations. We show when this construction also preserve bitabularity, and since constant symbols enable for example modelling pointed sets, we obtain collagories that do not correspond to toposes.

In Chapter 4 we return to study co-tabulations, now in the context of considering collagories as simple 2-categories via the local homset ordering. Here, we prove that co-tabulations are exactly equivalent to bicolimits of difunctional morphisms, and that furthermore cotabulations of spans of mappings are equivalent to bipushouts.

Finally, we show in Chapter 5 that mappings in bitabular collagories form an adhesive category, and produce some novel characterisations of van Kampen squares in this setting.

The central defintions of Sections 1.3, 2.2 and 2.3, the results of Chapter 3, and weaker versions of the results in Sect. 5.2 have been published as [Kah09a], and in a longer version as the report [Kah09b], which is essentially superseded by these current notes. The results of Chapter 4 together with newer results from Chapter 5 are summarised will appear in the proceedings of GT-VMT 2010.

1.2 Categories, Allegories

This section only serves to fix notation and terminology for standard concepts, see [FS90, SS93, Kah04]. Like Freyd and Scedrov and an increasing number of categorists, we denote composition in "diagram order" not only in relation-algebraic contexts, where this is customary, but also in the context of categories. We will always use the infix operator ";" to make composition explicit: $R ; S = \mathcal{A} \xrightarrow{R} \mathcal{B} \xrightarrow{S} \mathcal{C}$.

Definition 1.2.1 A *category* $\mathbf{C}$ is a tuple $(\mathsf{Obj}_{\mathbf{C}}, \mathsf{Mor}_{\mathbf{C}}, \mathsf{src}, \mathsf{trg}, \mathbb{I}, ;)$ where

- $\mathsf{Obj}_{\mathbf{C}}$ is a collection of *objects*.
- $\mathsf{Mor}_{\mathbf{C}}$ is a collection of *arrows* or *morphisms*.
- src (resp. trg) maps each morphism to its source (resp. target) object.
  Instead of $\mathsf{src}(f) = \mathcal{A} \wedge \mathsf{trg}(f) = \mathcal{B}$ we write $f : \mathcal{A} \to \mathcal{B}$.
  The collection of all morphisms f with $f : \mathcal{A} \to \mathcal{B}$ is denoted as $\mathsf{Mor}_{\mathbf{C}}[\mathcal{A}, \mathcal{B}]$ and also called a *homset*.
- ";" is the binary *composition* operator, and composition of two morphisms $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B}' \to \mathcal{C}$ is defined iff $\mathcal{B} = \mathcal{B}'$, and then $(f ; g) : \mathcal{A} \to \mathcal{C}$; composition is associative.
- $\mathbb{I}$ associates with every object $\mathcal{A}$ a morphism $\mathbb{I}_{\mathcal{A}}$ which is both a right and left unit for composition. □

Definition 1.2.2 An *ordered category* is a category $\mathbf{C}$ such that

- for each two objects $\mathcal{A}$ and $\mathcal{B}$, the relation $\sqsubseteq_{\mathcal{A},\mathcal{B}}$ is a partial order on $\mathsf{Mor}_{\mathbf{C}}[\mathcal{A}, \mathcal{B}]$ (the indices will usually be omitted), and
- composition is monotonic with respect to $\sqsubseteq$ in both arguments. □

Definition 1.2.3 An *upper-semilattice category* is an ordered category where

- each homset is an upper semilattice with binary join $\sqcup$,
- composition distributes over binary joins from both sides. □

For homsets that have least or greatest elements, we introduce corresponding notation:

Definition 1.2.4 In an ordered category, for each two objects $\mathcal{A}$ and $\mathcal{B}$ we introduce the following notions:

- If the homset $\mathsf{Mor}_{\mathbf{C}}[\mathcal{A}, \mathcal{B}]$ contains a greatest element, this is denoted $\mathbb{T}_{\mathcal{A},\mathcal{B}}$.
- If the homset $\mathsf{Mor}_{\mathbf{C}}[\mathcal{A}, \mathcal{B}]$ contains a least element, this is denoted $\perp\!\!\!\perp_{\mathcal{A},\mathcal{B}}$. □

For these extremal morphisms and for identities we frequently omit indices where these can be induced from the context.

Definition 1.2.5 An *ordered category with converse*, or *OCC*, is an ordered category such that

- each morphism $R : \mathcal{A} \to \mathcal{B}$ has a *converse* $R^{\smile} : \mathcal{B} \to \mathcal{A}$,
- the *involution equations* hold for all $R : \mathcal{A} \to \mathcal{B}$ and $S : \mathcal{B} \to \mathcal{C}$:

$$(R^{\smile})^{\smile} = R \qquad \mathbb{I}_{\mathcal{A}}^{\smile} = \mathbb{I}_{\mathcal{A}} \qquad (R \mathbin{;} S)^{\smile} = S^{\smile} \mathbin{;} R^{\smile}$$

- conversion is monotonic with respect to $\sqsubseteq$. □

Many standard properties of relations can be characterised in the context of OCCs [Kah04]:

Definition 1.2.6 A morphism $R : \mathcal{A} \to \mathcal{B}$ in an OCC is called:

- *univalent* iff $R^{\smile} \mathbin{;} R \sqsubseteq \mathbb{I}_{\mathcal{B}}$,
- *total* iff $\mathbb{I}_{\mathcal{A}} \sqsubseteq R \mathbin{;} R^{\smile}$,
- *injective* iff $R \mathbin{;} R^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{A}}$,
- *surjective* iff $\mathbb{I}_{\mathcal{B}} \sqsubseteq R^{\smile} \mathbin{;} R$,
- a *mapping* iff it is univalent and total,
- *bijective* iff it is injective and surjective,
- *difunctional* iff $R \mathbin{;} R^{\smile} \mathbin{;} R \sqsubseteq R$. (See [SS93, 4.4] for more about difunctionality). □

For an OCC $\mathbf{C}$, we write $\mathsf{Map}\,\mathbf{C}$ for the sub-category of $\mathbf{C}$ that contains only the mappings as arrows.

The following definition, based on allegories by Freyd and Scedrov, is already applicable in OCCs.

Definition 1.2.7 [FS90, 2.15] An object $\mathcal{U}$ in an OCC is a *partial unit* if $\mathbb{I}_{\mathcal{U}} = \mathbb{T}_{\mathcal{U},\mathcal{U}}$. The object $\mathcal{U}$ is a *unit* if, further, every object is the source of a total morphism targeted at $\mathcal{U}$. An OCC is said to be *unitary* if it has a unit. □

We use the symbol "$\mathbb{1}$" for an arbitrary but fixed unit object.

Difunctionality plays an important rôle in our theories; a concrete relation, understood as a Boolean matrix, is difunctional iff it can be rearranged into "loose block-diagonal form", with full rectangular blocks such that there is no overlap between different blocks in either direction. (See [SS93, 4.4] for more about difunctionality).

Definition 1.2.8 A morphism $R : \mathcal{A} \to \mathcal{B}$ in an OCC is called *difunctional* iff $R \mathbin{;} R^{\smile} \mathbin{;} R \sqsubseteq R$. □

Definition 1.2.9 For a morphism $R : \mathcal{A} \to \mathcal{B}$ in an allegory, we define its *difunctional closure* $R^{\boxast} : \mathcal{A} \to \mathcal{B}$ as the least difunctional morphism containing R (if this exists), and we further define $R^{\triangleright} : \mathcal{A} \to \mathcal{A}$ and $R^{\triangleleft} : \mathcal{B} \to \mathcal{B}$ as:

$$R^{\triangleright} := \mathbb{I} \sqcup R^{\boxast} \mathbin{;} (R^{\boxast})^{\smile} \qquad \text{and} \qquad R^{\triangleleft} := \mathbb{I} \sqcup (R^{\boxast})^{\smile} \mathbin{;} R^{\boxast} \ . \qquad \square$$

For endomorphisms, there are a few additional properties of interest:

Definition 1.2.10 A morphism $R : \mathcal{A} \to \mathcal{A}$ in an OCC is called:

- *reflexive* iff $\mathbb{I} \sqsubseteq R$,
- *transitive* iff $R \mathbin{;} R \sqsubseteq R$,
- *co-reflexive* or a *sub-identity* iff $R \sqsubseteq \mathbb{I}_{\mathcal{A}}$,
- *symmetric* iff $R^{\smile} \sqsubseteq R$,
- an *equivalence* iff it is symmetric, reflexive and transitive. □

Lemma 1.2.11 If $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is a span and $P^{\smile} \mathbin{;} Q$ is difunctional, then $P \mathbin{;} P^{\smile} \mathbin{;} Q \mathbin{;} Q^{\smile}$ is idempotent.

If P and Q are moreover total, then $P \mathbin{;} P^{\smile} \mathbin{;} Q \mathbin{;} Q^{\smile}$ is an equivalence.

Proof: The first claim is immediate: $P \mathbin{;} P^{\smile} \mathbin{;} Q \mathbin{;} Q^{\smile} \mathbin{;} P \mathbin{;} P^{\smile} \mathbin{;} Q \mathbin{;} Q^{\smile} = P \mathbin{;} P^{\smile} \mathbin{;} Q \mathbin{;} Q^{\smile}$.

For the second claim, reflexivity is obvious from totality, and the first claim implies transitivity, and, together with totality, also symmetry:

$$Q \mathbin{;} Q^{\smile} \mathbin{;} P \mathbin{;} P^{\smile} = \mathbb{I}_{\mathcal{A}} \mathbin{;} Q \mathbin{;} Q^{\smile} \mathbin{;} P \mathbin{;} P^{\smile} \mathbin{;} \mathbb{I}_{\mathcal{A}} \sqsubseteq P \mathbin{;} P^{\smile} \mathbin{;} Q \mathbin{;} Q^{\smile} \mathbin{;} P \mathbin{;} P^{\smile} \mathbin{;} Q \mathbin{;} Q^{\smile} = P \mathbin{;} P^{\smile} \mathbin{;} Q \mathbin{;} Q^{\smile} \quad \square$$

If, in the second case, $P \mathbin{;} P^{\smile}$ and $Q \mathbin{;} Q^{\smile}$ are themselves equivalences, then $P \mathbin{;} P^{\smile} \mathbin{;} Q \mathbin{;} Q^{\smile}$ is therefore their least upper bound among the equivalences on $\mathcal{A}$.

Lemma 1.2.12 If R and S are equivalences on $\mathcal{A}$ that commute, that is, where $R \mathbin{;} S = S \mathbin{;} R$ holds, then $R \mathbin{;} S$ is an equivalence again.

Proof: Reflexivity and symmetry are obvious; for transitivity we use commutation twice:

$$R \mathbin{;} S \mathbin{;} R \mathbin{;} S = R \mathbin{;} S \mathbin{;} S \mathbin{;} R = R \mathbin{;} S \mathbin{;} R = R \mathbin{;} R \mathbin{;} S = R \mathbin{;} S \qquad \square$$

While Freyd and Scedrov in their definition of allegories [FS90, 2.11] derive the homset ordering from the meet operation, we define allegories on top of ordered categories — the composition operator has higher precedence than all other binary operators.

Definition 1.2.13 An *allegory* is an OCC such that

- each homset is a lower semilattice with binary meet $\sqcap$.
- for all $Q : \mathcal{A} \to \mathcal{B}$, $R : \mathcal{B} \to \mathcal{C}$, and $S : \mathcal{A} \to \mathcal{C}$, the *modal rule* holds:

$$Q \mathbin{;} R \sqcap S \sqsubseteq (Q \sqcap S \mathbin{;} R^{\smile}) \mathbin{;} R \ . \qquad \square$$

The most well-known allegory is the category *Rel* of sets with relations and standard relational operations. Logical theories give rise to allegories of *derived predicates* [FS90, App. B]. A simpler case of that are the allegories arising from Σ-algebras (over some signature Σ) as objects, and with "relational Σ-homomorphisms", i.e. bisimulations in the sense of [Kah04], as morphisms.

In allegories, one can define domain and range operators:

Definition 1.2.14 For every morphism $R : \mathcal{A} \leftrightarrow \mathcal{B}$ in an allegory, we define $\mathsf{dom}\, R : \mathcal{A} \leftrightarrow \mathcal{A}$ and $\mathsf{ran}\, R : \mathcal{B} \leftrightarrow \mathcal{B}$ as:

$$\mathsf{dom}\, R := \mathbb{I}_{\mathcal{A}} \sqcap R \mathbin{;} R^{\smile} \qquad\qquad \mathsf{ran}\, R := \mathbb{I}_{\mathcal{B}} \sqcap R^{\smile} \mathbin{;} R \qquad \square$$

1.3 Collagories

κόλλα: glue

In Freyd and Scedrov's treatment, although allegories are not required to have zero-ary meets, distributive allegories are required to have zero-ary joins (least elements) together with distributivity of composition over them, that is, the zero law $\perp\!\!\!\perp \mathbin{;} R = \perp\!\!\!\perp$. We define an intermediate concept that does not assume anything about zero-ary joins:

Definition 1.3.1 A *collagory* is an allegory that is also an upper-semilattice category such that binary meet and join turn each homset into a distributive lattice. $\square$

Requiring least morphisms satisfying zero laws turns collagories into distributive allegories, which still heave a much weaker theory than relations in a topos, so graph structures (unary algebras) with relational graph homomorphism in particular also form collagories. In Chapter 3 below, we will show how to construct collagories of generalised algebras over signatures with only unary function symbols and constant symbols.

For adding Kleene star, we use Kozen's axioms [Koz94]:

Definition 1.3.2 A *Kleene collagory* is a collagory where, on homsets of endomorphisms, there is an additional unary operation $\_^*$ which satisfies the following axioms for all $R : \mathcal{A} \to \mathcal{A}$, $Q : \mathcal{B} \to \mathcal{A}$, and $S : \mathcal{A} \to \mathcal{C}$:

$$\begin{array}{rcccrcll}
 & & & & R^* & = & \mathbb{I}_{\mathcal{A}} \sqcup R \sqcup R^* \mathbin{;} R^* & \text{recursive star definition} \\
Q \mathbin{;} R & \sqsubseteq & Q & \Rightarrow & Q \mathbin{;} R^* & \sqsubseteq & Q & \text{right induction} \\
R \mathbin{;} S & \sqsubseteq & S & \Rightarrow & R^* \mathbin{;} S & \sqsubseteq & S & \text{left induction}
\end{array} \qquad \square$$

Proposition 1.3.3 In a Kleene collagory, all difunctional closures exist, and:

$$R^{\rhd\!\!\rhd} = (R \mathbin{;} R^{\smile})^* \;, \qquad R^{\lhd\!\!\lhd} = (R^{\smile} \mathbin{;} R)^* \;, \qquad R^{\boxplus} = R^{\rhd\!\!\rhd} \mathbin{;} R = R \mathbin{;} R^{\lhd\!\!\lhd} \;. \qquad \square$$

Alternatively, we also can fore-go the Kleene star and directly axiomatise difunctional closure:

Definition 1.3.4 A *difunctionally closed collagory* is a collagory where, there is an additional unary operation $\_^{\boxast}$ which satisfies the following axioms for all $R : \mathcal{A} \to \mathcal{B}$, $Q : \mathcal{C} \to \mathcal{A}$, and $S : \mathcal{B} \to \mathcal{C}$: $Q' : \mathcal{C} \to \mathcal{B}$, and $S' : \mathcal{A} \to \mathcal{C}$:

$$\begin{array}{ccccccccccccl}
 & & & & & & R^{\boxast} & = & R \sqcup R^{\boxast} ; (R^{\boxast})^{\smile} ; R^{\boxast} & & & & \text{recursive definition} \\
Q ; R & \sqsubseteq & Q' & \wedge & Q' ; R^{\smile} ; R & \sqsubseteq & Q' & \Rightarrow & Q ; R^{\boxast} & \sqsubseteq & Q' & & \text{right induction} \\
R ; S & \sqsubseteq & S' & \wedge & R ; R^{\smile} ; S' & \sqsubseteq & S' & \Rightarrow & R^{\boxast} ; S & \sqsubseteq & S' & & \text{left induction}
\end{array}$$

□

Proposition 1.3.5 In a difunctionally closed collagory, the operation $\_^{\boxast}$ produces difunctional closures.

Proof: Containment $R \sqsubseteq R^{\boxast}$ and difunctionality $R^{\boxast} ; (R^{\boxast})^{\smile} ; R^{\boxast} \sqsubseteq R^{\boxast}$ follow directly from the recursive definition.

For minimality, assume that C is difunctional with $R \sqsubseteq C$. Then we have $\mathbb{I} ; R \sqsubseteq C$ and $C ; R^{\smile} ; R \sqsubseteq C ; C^{\smile} ; C \sqsubseteq C$ and therefore, with the right induction rule, $R^{\boxast} = \mathbb{I} ; R^{\boxast} \sqsubseteq C$. □

Proposition 1.3.6 A Kellene collagory is dinfuctionally closed, with $R^{\boxast} := (R ; R^{\smile})^* ; R = R ; (R^{\smile} ; R)^*$.

Proof: Recursive definition:

$$\begin{aligned}
R \sqcup R^{\boxast} ; (R^{\boxast})^{\smile} ; R^{\boxast} &= R \sqcup (R ; R^{\smile})^* ; R ; R^{\smile} ; (R ; R^{\smile})^* ; (R ; R^{\smile})^* ; R \\
&= R \sqcup (R ; R^{\smile})^* ; R ; R^{\smile} ; R \\
&= \mathbb{I} ; R \sqcup (R ; R^{\smile})^+ ; R \\
&= (R ; R^{\smile})^* ; R \\
&= R^{\boxast}
\end{aligned}$$

For left induction, assume $Q ; R \sqsubseteq Q'$ and $Q' ; R^{\smile} ; R \sqsubseteq Q'$. Then left induction of Kleene star gives us

$$Q' ; R^{\smile} ; R^* \sqsubseteq Q' \ ,$$

and with this we obtain the conclusion of left induction for difunctional closure:

$$Q ; R^{\boxast} = Q ; R ; (R^{\smile} ; R)^* \sqsubseteq Q' ; (R^{\smile} ; R)^* \sqsubseteq Q'$$

Right induction can be shown analogously. □

Chapter 2

Tabulations and Co-Tabulations

2.1 Introduction

Central to the connection between pullbacks and pushouts in categories of mappings on the one hand and constructions in relational theories on the other hand is the fact that a square of mappings commutes iff the "relation" induced by the source span is contained in that induced by the target co-span.

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "P"'] \arrow[dr, "Q"] & \\ \mathcal{B} \arrow[dr, "R"'] & & \mathcal{C} \arrow[dl, "S"] \\ & \mathcal{D} & \end{tikzcd}$$

Lemma 2.1.1 [FS90, 2.146] Given a square of mappings in an OCC as drawn above, we have $P ; R = Q ; S$ iff $P^{\smile} ; Q \sqsubseteq R ; S^{\smile}$. $\square$

Proof: Freyd and Scedrov formulated this in the allegory context, but their proof needs neither meet nor the modal rule:

If $P ; R = Q ; S$, then

$$\begin{array}{rcll} P^{\smile} ; Q & \sqsubseteq & P^{\smile} ; Q ; S ; S^{\smile} & S \text{ total} \\ & = & P^{\smile} ; P ; R ; S^{\smile} & \text{assumption} \\ & \sqsubseteq & R ; S^{\smile} & P \text{ univalent} \end{array}$$

Conversely, if $P^{\smile} ; Q \sqsubseteq R ; S^{\smile}$, then:

$$\begin{array}{rcll} P ; R & \sqsubseteq & Q ; Q^{\smile} ; P ; R & Q \text{ total} \\ & \sqsubseteq & Q ; S ; R^{\smile} ; R & \text{assumption} \\ & \sqsubseteq & Q ; S & R \text{ univalent} \end{array}$$

Since both sides of this inclusion are mappings, it shows the equality $P ; R = Q ; S$. $\square$

We introduce a name for the case where the inclusion in Lemma 2.1.1 turns into an equality:

Definition 2.1.2 In an OCC, a square of morphisms as drawn above is said to *cross-commute*, or to satisfy the property of *cross-commutativity*, iff $P^{\smile} ; Q = R ; S^{\smile}$. □

Lemma 2.1.1 provides a first hint that in the relational setting, the identity of the two mappings P and Q does not matter when looking for a pushout of the span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ — we only need to consider the diagonal $P^{\smile} ; Q$. Dually, when looking for a pullback of the co-span $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$, only $R ; S^{\smile}$ needs to be considered. The gap between the two ways of calculating the horizontal diagonal can be significant since $R ; S^{\smile}$ is always difunctional.

Producing the result span of a pullback (respectively the result co-span of a pushout) from the horizontal diagonal alone is, in some sense, a generalisation of Freyd and Scedrov's splitting of idempotents; [Kah04] contains more discussion of this aspect.

We now first present tabulations and some of their properties, and then turn to co-tabulations in Sect. 2.3. In collagories, least morphisms need not exist; if they do exist, they satisfy only a subset of the properties that zero morphisms satisfy in distributive allegories, and we collect a few of these properties in Sect. 2.4. Since direct sums (Sect. 2.5) are defined via least morphisms, the detailed porperties of least morphisms are especially important in that context.

2.2 Tabulations

Definition 2.2.1 [FS90, 2.14] In an allegory, let a morphism $V : \mathcal{B} \to \mathcal{C}$ be given. The span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ of *mappings* P and Q is called a *tabulation of* V iff the following equations hold:

$$P^{\smile} ; Q = V \qquad P ; P^{\smile} \sqcap Q ; Q^{\smile} = \mathbb{I}_{\mathcal{A}} \ . \qquad \square$$

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "P"'] \arrow[dr, "Q"] & \\ \mathcal{B} \arrow[rr, "V"] & & \mathcal{C} \end{tikzcd}$$

The following equivalent characterisation provided by [Kah04] has the advantage that it is fully equational, without the implicit inclusion conditions in the requirement that P and Q are mappings. This frequently facilitates calculations. Notice that $\mathbb{I} \sqcap V ; V^{\smile} = \mathsf{dom}\, V$; we use the expanded form to emphasise the duality with Prop. 2.3.3 below.

Proposition 2.2.2 In an allegory, the span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is a tabulation of $V : \mathcal{B} \to \mathcal{C}$ if and only if the following equations hold:

$$P^{\smile} ; Q = V \qquad \begin{aligned} P^{\smile} ; P &= \mathbb{I} \sqcap V ; V^{\smile} \\ Q^{\smile} ; Q &= \mathbb{I} \sqcap V^{\smile} ; V \end{aligned} \qquad P ; P^{\smile} \sqcap Q ; Q^{\smile} = \mathbb{I}_{\mathcal{A}} \ . \qquad \square$$

Tabulations in an allegory are unique up to isomorphism (this uses the modal rule), and include the following special cases:

- In a tabulation of a sub-identity, both tabulation morphisms are the induced *sub-object* injection [FS90, 2.145].
- We can define a *direct product* of $\mathcal{A}$ and $\mathcal{B}$ to be a tabulation $\mathcal{A} \xleftarrow{\pi} \mathcal{P} \xrightarrow{\rho} \mathcal{B}$ of a $\mathbb{T}_{\mathcal{A},\mathcal{B}}$, provided that greatest morphism exists.
  The resulting direct product definition differs from that of [SS93] in extending naturally to "empty" objects (e.g., empty sets) by not demanding surjectivity of the projections, but only
  $$\pi^{\smile} ; \pi = \mathsf{dom}\, \mathbb{T}_{\mathcal{A},\mathcal{B}} \qquad\qquad \rho^{\smile} ; \rho = \mathsf{ran}\, \mathbb{T}_{\mathcal{A},\mathcal{B}} \;.$$
- If a co-span $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ of mappings is given, then each tabulation of $R ; S^{\smile}$ (which need not exist) is a *pullback* in $\mathsf{Map}\,\mathbf{A}$ [FS90, 2.147].
  For a tabular allegory $\mathbf{A}$, this implies that each pullback in $\mathsf{Map}\,\mathbf{A}$ is isomorphic to a tabulation, and therefore is itself a tabulation. However, if $\mathbf{A}$ is not tabular, then a co-span $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ of mappings for which no tabulation of $R ; S^{\smile}$ exists may still have a pullback in $\mathsf{Map}\,\mathbf{A}$, which then cannot be a tabulation.

If an allegory is known to have all direct products and subobjects, then these can be used to construct a tabulation for each morphism.

The following properties are immediate consequences of Prop. 2.2.2:

Corollary 2.2.3 If $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is a tabulation for $V : \mathcal{D} \to \mathcal{E}$, then the following hold:

1. $P \sqsubseteq Q ; V^{\smile}$ and $Q \sqsubseteq P ; V$
2. $\mathsf{ran}\, P = \mathsf{dom}\, V$ and $\mathsf{ran}\, Q = \mathsf{ran}\, V$.

From Lemma 1.2.11, we immediately have:

Lemma 2.2.4 If $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is a tabulation for $V : \mathcal{D} \to \mathcal{E}$, and V is difunctional, then $P ; P^{\smile} ; Q ; Q^{\smile}$ is an equivalence. □

Furthermore, we have the following special cases:

Lemma 2.2.5 If $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is a tabulation for $V : \mathcal{D} \to \mathcal{E}$, then:

- V is univalent iff $P ; P^{\smile} \sqsubseteq Q ; Q^{\smile}$;
- V is univalent iff P is injective;
- V is injective iff $Q ; Q^{\smile} \sqsubseteq P ; P^{\smile}$;
- V is injective iff Q is injective.

Proof: We only show the first two items since the last two follow analogously.

For the first item, we have:

$$\begin{array}{rcll} V^{\smile} ; V \sqsubseteq \mathbb{I} & \Leftrightarrow & Q^{\smile} ; P ; P^{\smile} ; Q \sqsubseteq \mathbb{I} & \text{first tabulation equation} \\ & \Leftrightarrow & P ; P^{\smile} \sqsubseteq Q ; Q^{\smile} & Q \text{ mapping} \end{array}$$

The last inclusion implies, with the second tabulation equation, that P is injective:

$$P ; P^{\smile} = P ; P^{\smile} \sqcap Q ; Q^{\smile} = \mathbb{I}$$

Conversely, if P is injective, then, since Q is univalent, $V = P^{\smile} ; Q$ is univalent, too. □

Lemma 2.2.6 If a co-span $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ of mappings is given with R injective, and $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is a tabulation for $R \mathbin{;} S^{\smile}$, then Q is injective, too.

PROOF: $R \mathbin{;} S^{\smile}$ is injective since R is injective and S is univalent, so, with Lemma 2.2.5, Q is injective, too. □

Lemma 2.2.7 If

- $\mathcal{B} \xleftarrow{P_1} \mathcal{A} \xrightarrow{Q_1} \mathcal{C}$ is a tabulation for $V_1 : \mathcal{B} \to \mathcal{C}$,
- $\mathcal{C} \xleftarrow{P_2} \mathcal{E} \xrightarrow{Q_2} \mathcal{D}$ is a tabulation for $V_2 : \mathcal{C} \to \mathcal{D}$,
- $\mathcal{A} \xleftarrow{P} \mathcal{F} \xrightarrow{Q} \mathcal{E}$ is a tabulation for $V : \mathcal{D} \to \mathcal{E}$ with $V := Q_1 \mathbin{;} P_2^{\smile}$, and
- $V_1^{\smile} \mathbin{;} V_1 \sqcap V_2 \mathbin{;} V_2^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{C}}$,

then $\mathcal{B} \xleftarrow{P ; P_1} \mathcal{F} \xrightarrow{Q ; Q_2} \mathcal{D}$ is a tabulation for $V_1 \mathbin{;} V_2$.

$$\begin{tikzcd}
 & & \mathcal{F} \arrow[dl, "P"'] \arrow[dr, "Q"] & & \\
 & \mathcal{A} \arrow[rr, "V"] \arrow[dl, "P_1"'] \arrow[dr, "Q_1"] & & \mathcal{E} \arrow[dl, "P_2"'] \arrow[dr, "Q_2"] & \\
\mathcal{B} \arrow[rr, "V_1"] & & \mathcal{C} \arrow[rr, "V_2"] & & \mathcal{D}
\end{tikzcd}$$

PROOF: Commutativity follows easily from the assumptions:

$$\begin{aligned}
P_1^{\smile} \mathbin{;} P^{\smile} \mathbin{;} Q \mathbin{;} Q_2 &= P_1^{\smile} \mathbin{;} V \mathbin{;} Q_2 \\
&= P_1^{\smile} \mathbin{;} Q_1 \mathbin{;} P_2^{\smile} \mathbin{;} Q_2 \\
&= V_1 \mathbin{;} V_2
\end{aligned}$$

In preparation for the second equation, we first see:

$$\begin{array}{lll}
& P \mathbin{;} P_1 \mathbin{;} P_1^{\smile} \mathbin{;} P^{\smile} \sqcap Q \mathbin{;} Q_2 \mathbin{;} Q_2^{\smile} \mathbin{;} Q^{\smile} & \\
\sqsubseteq & P \mathbin{;} Q_1 \mathbin{;} V_1^{\smile} \mathbin{;} V_1 \mathbin{;} Q_1^{\smile} \mathbin{;} P^{\smile} \sqcap Q \mathbin{;} P_2 \mathbin{;} V_2 \mathbin{;} V_2^{\smile} \mathbin{;} P_2^{\smile} \mathbin{;} Q^{\smile} & \text{Corollary 2.2.3} \\
= & P \mathbin{;} Q_1 \mathbin{;} V_1^{\smile} \mathbin{;} V_1 \mathbin{;} Q_1^{\smile} \mathbin{;} P^{\smile} \sqcap P \mathbin{;} Q_1 \mathbin{;} V_2 \mathbin{;} V_2^{\smile} \mathbin{;} Q_1^{\smile} \mathbin{;} P^{\smile} & P \mathbin{;} Q_1 = Q \mathbin{;} P_2 \\
= & P \mathbin{;} Q_1 \mathbin{;} (V_1^{\smile} \mathbin{;} V_1 \sqcap V_2 \mathbin{;} V_2^{\smile}) \mathbin{;} Q_1^{\smile} \mathbin{;} P^{\smile} & P \mathbin{;} Q_1 \text{ univalent} \\
\sqsubseteq & P \mathbin{;} Q_1 \mathbin{;} Q_1^{\smile} \mathbin{;} P^{\smile} & \text{Assumption} \\
= & P \mathbin{;} Q_1 \mathbin{;} Q_1^{\smile} \mathbin{;} P^{\smile} \sqcap Q \mathbin{;} P_2 \mathbin{;} P_2^{\smile} \mathbin{;} Q^{\smile} & P \mathbin{;} Q_1 = Q \mathbin{;} P_2
\end{array}$$

The resulting inclusion is used in the last step to show the second tabulation equation:

$$\begin{array}{ll}
\mathbb{I}_{\mathcal{F}} = & \{ \text{ tabulation for } V \} \\
& P \mathbin{;} P^{\smile} \sqcap Q \mathbin{;} Q^{\smile} \\
= & \{ \text{ tabulations for } V_1,\ V_2 \} \\
& P \mathbin{;} (P_1 \mathbin{;} P_1^{\smile} \sqcap Q_1 \mathbin{;} Q_1^{\smile}) \mathbin{;} P^{\smile} \sqcap Q \mathbin{;} (P_2 \mathbin{;} P_2^{\smile} \sqcap Q_2 \mathbin{;} Q_2^{\smile}) \mathbin{;} Q^{\smile} \\
= & \{ P,\ Q \text{ univalent } \} \\
& P \mathbin{;} P_1 \mathbin{;} P_1^{\smile} \mathbin{;} P^{\smile} \sqcap P \mathbin{;} Q_1 \mathbin{;} Q_1^{\smile} \mathbin{;} P^{\smile} \sqcap Q \mathbin{;} P_2 \mathbin{;} P_2^{\smile} \mathbin{;} Q^{\smile} \sqcap Q \mathbin{;} Q_2 \mathbin{;} Q_2^{\smile} \mathbin{;} Q^{\smile} \\
= & \{ \text{ above } \} \\
& P \mathbin{;} P_1 \mathbin{;} P_1^{\smile} \mathbin{;} P^{\smile} \sqcap Q \mathbin{;} Q_2 \mathbin{;} Q_2^{\smile} \mathbin{;} Q^{\smile}
\end{array}$$

□

The precondition $V_1^{\smile} \mathbin{;} V_1 \sqcap V_2 \mathbin{;} V_2^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{C}}$ here corresponds to the condition pointed out by Bruni and Gadducci [BG03] for preservation of composition by the standard conversion of relations to spans of total functions, namely that for all $b \in \mathcal{B}$ and $d \in \mathcal{D}$ there exists at most one $c \in \mathcal{C}$ for which $b\,V_1\,c$ and $c\,V_2\,d$.

Note that we also have $V_1^{\smile} \mathbin{;} V_1 \sqcap V_2 \mathbin{;} V_2^{\smile} = Q_1^{\smile} \mathbin{;} Q_1 \sqcap P_2^{\smile} \mathbin{;} P_2$:

$$
\begin{array}{cll}
 & V_1^{\smile} \mathbin{;} V_1 \sqcap V_2 \mathbin{;} V_2^{\smile} & \\
= & V_1^{\smile} \mathbin{;} V_1 \sqcap V_2 \mathbin{;} V_2^{\smile} \sqcap \mathbb{I}_{\mathcal{C}} & \text{assumption } V_1^{\smile} \mathbin{;} V_1 \sqcap V_2 \mathbin{;} V_2^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{C}} \\
= & Q_1^{\smile} \mathbin{;} P_1 \mathbin{;} P_1^{\smile} \mathbin{;} Q_1 \sqcap P_2^{\smile} \mathbin{;} Q_2 \mathbin{;} Q_2^{\smile} \mathbin{;} P_2 \sqcap \mathbb{I}_{\mathcal{C}} & \text{tabulation commutativity} \\
= & \mathsf{ran}\,(P_1^{\smile} \mathbin{;} Q_1) \sqcap \mathsf{ran}\,(Q_2^{\smile} \mathbin{;} P_2) & \text{Def. 1.2.14} \\
= & \mathsf{ran}\,(\mathsf{ran}\,(P_1^{\smile}) \mathbin{;} Q_1) \sqcap \mathsf{ran}\,(\mathsf{ran}\,(Q_2^{\smile}) \mathbin{;} P_2) & \text{locality of } \mathsf{ran} \\
= & \mathsf{ran}\,Q_1 \sqcap \mathsf{ran}\,P_2 & P_1 \text{ and } Q_2 \text{ total} \\
= & Q_1^{\smile} \mathbin{;} Q_1 \sqcap P_2^{\smile} \mathbin{;} P_2 & Q_1 \text{ and } P_2 \text{ univalent}
\end{array}
$$

We show the well-known composition property of pullbacks in the tabulation formulation:

Lemma 2.2.8 Let the following commuting diagram of mappings be given:

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{Tab1}"] & \mathcal{B} \arrow[r, "g"] \arrow[d, "b"'] \arrow[dr, phantom, "\mathrm{Tab2}"] & \mathcal{C} \arrow[d, "c"'] \\
\mathcal{D} \arrow[r, "m"] & \mathcal{E} \arrow[r, "n"] & \mathcal{F}
\end{tikzcd}
$$

If $\mathcal{E} \xleftarrow{b} \mathcal{B} \xrightarrow{g} \mathcal{C}$ is a tabulation for $n \mathbin{;} c^{\smile}$, then $\mathcal{D} \xleftarrow{a} \mathcal{A} \xrightarrow{f} \mathcal{B}$ is a tabulation for $m \mathbin{;} b^{\smile}$ iff $\mathcal{D} \xleftarrow{a} \mathcal{A} \xrightarrow{f;g} \mathcal{C}$ is a tabulation for $m \mathbin{;} n \mathbin{;} b^{\smile}$.

Proof: Assume that $\mathcal{E} \xleftarrow{b} \mathcal{B} \xrightarrow{g} \mathcal{C}$ is a tabulation for $n \mathbin{;} c^{\smile}$, that is,

$$b^{\smile} \mathbin{;} g = n \mathbin{;} c^{\smile} \qquad \text{and} \qquad b \mathbin{;} b^{\smile} \sqcap g \mathbin{;} g^{\smile} = \mathbb{I}_{\mathcal{B}}\ . \tag{Tab2}$$

"$\Rightarrow$": If also $\mathcal{D} \xleftarrow{a} \mathcal{A} \xrightarrow{f} \mathcal{B}$ is a tabulation for $m \mathbin{;} b^{\smile}$, that is,

$$a^{\smile} \mathbin{;} f = m \mathbin{;} b^{\smile} \qquad \text{and} \qquad a \mathbin{;} a^{\smile} \sqcap f \mathbin{;} f^{\smile} = \mathbb{I}_{\mathcal{A}}\ , \tag{Tab1}$$

then cross-commutativity composes easily: $a^{\smile} \mathbin{;} f \mathbin{;} g = m \mathbin{;} b^{\smile} \mathbin{;} g = m \mathbin{;} n \mathbin{;} c^{\smile}$, and we obtain:

$$
\begin{array}{cll}
 & a \mathbin{;} a^{\smile} \sqcap f \mathbin{;} g \mathbin{;} g^{\smile} \mathbin{;} f^{\smile} & \\
= & a \mathbin{;} a^{\smile} \sqcap a \mathbin{;} m \mathbin{;} m^{\smile} \mathbin{;} a^{\smile} \sqcap f \mathbin{;} g \mathbin{;} g^{\smile} \mathbin{;} f^{\smile} & m \text{ total} \\
= & a \mathbin{;} a^{\smile} \sqcap f \mathbin{;} b \mathbin{;} b^{\smile} \mathbin{;} f^{\smile} \sqcap f \mathbin{;} g \mathbin{;} g^{\smile} \mathbin{;} f^{\smile} & \text{(Tab1) with Lemma 2.1.1} \\
= & a \mathbin{;} a^{\smile} \sqcap f \mathbin{;} (b \mathbin{;} b^{\smile} \sqcap g \mathbin{;} g^{\smile}) \mathbin{;} f^{\smile} & f \text{ univalent} \\
= & a \mathbin{;} a^{\smile} \sqcap f \mathbin{;} f^{\smile} & \text{(Tab2)} \\
= & \mathbb{I}_{\mathcal{A}} & \text{(Tab1)}
\end{array}
$$

This shows that $\mathcal{D} \xleftarrow{a} \mathcal{A} \xrightarrow{f;g} \mathcal{C}$ is a tabulation for $m \mathbin{;} n \mathbin{;} b^{\smile}$.

"$\Leftarrow$": Now assume that $\mathcal{D} \xleftarrow{a} \mathcal{A} \xrightarrow{f;g} \mathcal{C}$ is a tabulation for $m \mathbin{;} n \mathbin{;} b^{\smile}$:

$$a^{\smile} \mathbin{;} f \mathbin{;} g = m \mathbin{;} b^{\smile} \qquad \text{and} \qquad a \mathbin{;} a^{\smile} \sqcap f \mathbin{;} g \mathbin{;} g^{\smile} \mathbin{;} f^{\smile} = \mathbb{I}_{\mathcal{A}} \;, \tag{Tab}$$

Then:

$$\begin{array}{rcll}
a \mathbin{;} a^{\smile} \sqcap f \mathbin{;} f^{\smile} & = & a \mathbin{;} a^{\smile} \sqcap f \mathbin{;} (b \mathbin{;} b^{\smile} \sqcap g \mathbin{;} g^{\smile}) \mathbin{;} f^{\smile} & \text{(Tab2)} \\
& \sqsubseteq & a \mathbin{;} a^{\smile} \sqcap f \mathbin{;} g \mathbin{;} g^{\smile} \mathbin{;} f^{\smile} & \\
& = & \mathbb{I}_{\mathcal{A}} & \text{(Tab)}
\end{array}$$

For cross-commutativity, we need the assumption that the left square commutes:

$$\begin{array}{rcll}
a^{\smile} \mathbin{;} f & = & a^{\smile} \mathbin{;} f \mathbin{;} (b \mathbin{;} b^{\smile} \sqcap g \mathbin{;} g^{\smile}) & \text{(Tab2)} \\
& = & a^{\smile} \mathbin{;} (f \mathbin{;} b \mathbin{;} b^{\smile} \sqcap f \mathbin{;} g \mathbin{;} g^{\smile}) & f \text{ univalent} \\
& = & a^{\smile} \mathbin{;} (a \mathbin{;} m \mathbin{;} b^{\smile} \sqcap f \mathbin{;} g \mathbin{;} g^{\smile}) & \text{commutativity} \\
& = & m \mathbin{;} b^{\smile} \sqcap a^{\smile} \mathbin{;} f \mathbin{;} g \mathbin{;} g^{\smile} & f \text{ univalent} \\
& = & m \mathbin{;} b^{\smile} \sqcap m \mathbin{;} n \mathbin{;} c^{\smile} \mathbin{;} g^{\smile} & \text{(Tab)} \\
& = & m \mathbin{;} b^{\smile} \sqcap m \mathbin{;} n \mathbin{;} n^{\smile} \mathbin{;} b^{\smile} & \text{(Tab2) with Lemma 2.1.1} \\
& = & m \mathbin{;} (\mathbb{I}_{\mathcal{B}} \sqcap n \mathbin{;} n^{\smile} \mathbin{;} b^{\smile} & m,\ b \text{ univalent} \\
& = & m \mathbin{;} b^{\smile} & n \text{ total}
\end{array}$$

□

2.3 Co-tabulations

While a tabulation can be seen as a certain kind of decomposition of an arbitrary morphism in an allegory (which are all co-difunctional) into a span, the dual of a tabulation is then a certain kind of decomposition of a difunctional morphism in a collagory into a co-span.

In this context, a stronger version of Lemma 2.1.1 is worth keeping in mind:

Lemma 2.3.1 Given a square of mappings in an OCC as drawn below, and existence of the difunctional closure of $P^{\smile} \mathbin{;} Q$, we have $P \mathbin{;} R = Q \mathbin{;} S$ iff $(P^{\smile} \mathbin{;} Q)^{\circledast} \sqsubseteq R \mathbin{;} S^{\smile}$.

$$\begin{tikzcd}
& \mathcal{A} \arrow[dl, "P"'] \arrow[dr, "Q"] & \\
\mathcal{B} \arrow[dr, "R"'] & & \mathcal{C} \arrow[dl, "S"] \\
& \mathcal{D} &
\end{tikzcd}$$

Proof: The "if" direction follows immediately from $P^{\smile} \mathbin{;} Q \sqsubseteq (P^{\smile} \mathbin{;} Q)^{\circledast}$ and the "if" direction of Lemma 2.1.1.

For "only if", assume $P \mathbin{;} R = Q \mathbin{;} S$. Then $P^{\smile} \mathbin{;} Q \sqsubseteq R \mathbin{;} S^{\smile}$ by Lemma 2.1.1, and

$$\begin{array}{rcll}
R \mathbin{;} S^{\smile} \mathbin{;} Q^{\smile} \mathbin{;} P \mathbin{;} P^{\smile} \mathbin{;} Q & = & R \mathbin{;} R^{\smile} \mathbin{;} P^{\smile} \mathbin{;} P \mathbin{;} P^{\smile} \mathbin{;} Q & \text{commutativity} \\
& = & R \mathbin{;} R^{\smile} \mathbin{;} P^{\smile} \mathbin{;} Q & P \text{ unival.} \\
& = & R \mathbin{;} S^{\smile} \mathbin{;} Q^{\smile} \mathbin{;} Q & \text{commutativity} \\
& \sqsubseteq & R \mathbin{;} S^{\smile} & Q \text{ unival.}
\end{array}$$

By left-induction for difunctional closure we therefore have

$$(P^{\smile} \,;\, Q)^{\boxast} \sqsubseteq R \,;\, S^{\smile} \;. \qquad \square$$

Although the formal material here is dual to that in Sect. 2.2, we still spell it out in full detail for reference and better intuition.

Definition 2.3.2 [Kah04] In a collagory, let a morphism $W : \mathcal{B} \to \mathcal{C}$ be given. The co-span $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ of *mappings* R and S is called a *co-tabulation of* W iff the following equations hold:

$$R \,;\, S^{\smile} = W \qquad\qquad R^{\smile} \,;\, R \sqcup S^{\smile} \,;\, S = \mathbb{I}_{\mathcal{D}} \;. \qquad \square$$

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "W"] \arrow[dr, "R"'] & & \mathcal{C} \arrow[dl, "S"] \\ & \mathcal{D} & \end{tikzcd}$$

The first equation implies $W \,;\, W^{\smile} \,;\, W = R \,;\, S^{\smile} \,;\, S \,;\, R^{\smile} \,;\, R \,;\, S^{\smile} \sqsubseteq R \,;\, S^{\smile} = W$ (using univalence of R and S), so if W has a co-tabulation, it has to be difunctional.

Furthermore, from univalence of R and S we also obtain $R^{\smile} \,;\, W = R^{\smile} \,;\, R \,;\, S^{\smile} \sqsubseteq S^{\smile}$ and $W \,;\, S = R \,;\, S^{\smile} \,;\, S \sqsubseteq R$.

Co-tabulations also have an equivalent characterisation that does not involve the mapping concept explicitly and is perfectly "bi-dual" to the tabulation characterisation in Prop. 2.2.2:

Proposition 2.3.3 In a collagory, the co-span $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation of $W : \mathcal{B} \to \mathcal{C}$ iff the following equations hold:

$$R \,;\, S^{\smile} = W \qquad \begin{array}{rcl} R \,;\, R^{\smile} & = & \mathbb{I} \sqcup W \,;\, W^{\smile} \\ S \,;\, S^{\smile} & = & \mathbb{I} \sqcup W^{\smile} \,;\, W \end{array} \qquad R^{\smile} \,;\, R \sqcup S^{\smile} \,;\, S = \mathbb{I}_{\mathcal{D}} \;. \qquad \square$$

In a collagory, we have the following special cases of co-tabulations:

- In a co-tabulation of an equivalence relation, both R and S are the induced *quotient* projections.
- We can define a *direct sum* of $\mathcal{A}$ and $\mathcal{B}$ to be a co-tabulation of $\perp\!\!\!\perp_{\mathcal{A},\mathcal{B}}$, if that least morphism exists.

If direct sums and quotients are available, then a co-tabulation can be constructed for each difunctional morphism.

To establish the relationship between the relation-algebraic co-tabulation definition and the universal characterisation of pushouts in categories, we first establish a generalised factorisation property for co-tabulations:

Lemma 2.3.4 In a collagory $\mathbf{C}$, let $W : \mathcal{B} \leftrightarrow \mathcal{C}$ be a difunctional morphism.

If the cospan $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation of W, and if the cospan $\mathcal{B} \xrightarrow{R'} \mathcal{D}' \xleftarrow{S'} \mathcal{C}$ in $\mathbf{C}$ consists of morphisms that satisfy

$$W \,;\, S' \sqsubseteq R' \qquad \text{and} \qquad W^{\smile} \,;\, R' \sqsubseteq S' \;,$$

then $U : \mathcal{D} \to \mathcal{D}'$ with $U := R^{\smile} ; R' \sqcup S^{\smile} ; S'$ is a morphism in $\mathbf{C}$ such that $R ; U = R'$ and $S ; U = S'$.

If R' and S' are univalent, then so is U.

If R' and S' are total, then so is U.

Proof: Factorisation follows easily from the assumptions:

$$\begin{array}{rcl} R ; U &=& R ; R^{\smile} ; R' \sqcup R ; S^{\smile} ; S' \;=\; (\mathbb{I} \sqcup W ; W^{\smile}) ; R' \sqcup W ; S' \;=\; R' \\ S ; U &=& S ; R^{\smile} ; R' \sqcup S ; S^{\smile} ; S' \;=\; W^{\smile} ; R' \sqcup (\mathbb{I} \sqcup W^{\smile} ; W) ; S' \;=\; S' \end{array}$$

Univalence follows from factorisation and univalence of R' and S':

$$U^{\smile} ; U = (R'^{\smile} ; R \sqcup S'^{\smile} ; S) ; U = R'^{\smile} ; R ; U \sqcup S'^{\smile} ; S ; U = R'^{\smile} ; R' \sqcup S'^{\smile} ; S' \sqsubseteq \mathbb{I}$$

Totality of U uses totality of R' and S', and the last co-tabulation condition:

$$\begin{array}{rcll} U ; U^{\smile} &\sqsupseteq& R^{\smile} ; R' ; R'^{\smile} ; R \sqcup S^{\smile} ; S' ; S'^{\smile} ; S & \text{Definition of } U \\ &\sqsupseteq& R^{\smile} ; R \sqcup S^{\smile} ; S & \text{Totality of } R' \text{ and } S' \\ &\sqsupseteq& \mathbb{I} & R,\ S \text{ co-tabulation} \end{array}$$

□

This helps to show that co-tabulations are unique up to isomorphism (for the factorisation of difunctional $W = R;S^{\smile}$ into *surjective* mappings this has already been shown in [SS93, 4.4.10]):

Theorem 2.3.5 In a collagory, let $W : \mathcal{B} \leftrightarrow \mathcal{C}$ be a difunctional morphism.

If the cospans $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ and $\mathcal{B} \xrightarrow{R'} \mathcal{D}' \xleftarrow{S'} \mathcal{C}$ are both co-tabulations for W, then there is a bijective mapping $U : \mathcal{D} \to \mathcal{D}'$ such that $R' = R ; U$ and $S' = S ; U$.

Proof: With the co-tabulation conditions for $\mathcal{B} \xrightarrow{R'} \mathcal{D}' \xleftarrow{S'} \mathcal{C}$ and univalence of R' and S' we obtain:

$$W ; S' = R' ; S'^{\smile} ; S' \sqsubseteq R' \qquad \text{and} \qquad W^{\smile} ; R' = S' ; R'^{\smile} ; R' \sqsubseteq S' \ .$$

With Lemma 2.3.4 we know that $U := R^{\smile} ; R' \sqcup S^{\smile} ; S'$ is a mapping that factorises R' and S'.

By the same argument for $U^{\smile}$, we obtain that U is also bijective. □

Although cocone commutativity is weakened in co-tabulations in comparison with that of OC-colimits, we still obtain the isotony property:

Lemma 2.3.6 In a collagory $\mathbf{C}$, let $W : \mathcal{B} \leftrightarrow \mathcal{C}$ be a difunctional morphism.

If the cospan $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation of W, and $U, U' : \mathcal{D} \to \mathcal{D}'$ are two morphisms in $\mathbf{C}$ with $R ; U \sqsubseteq R ; U'$ and $S ; U \sqsubseteq S ; U'$, then $U \sqsubseteq U'$.

Proof: From the assumptions, we obtain

$$R^{\smile} ; R ; U \sqsubseteq R^{\smile} ; R ; U' \qquad \text{and} \qquad S^{\smile} ; S ; U \sqsubseteq S^{\smile} ; S ; U' \ ,$$

and by combining these and using the last co-tabulation property, we see:

$$\begin{array}{rcl} U &=& (R^{\smile} ; R \sqcup S^{\smile} ; S) ; U \\ &=& R^{\smile} ; R ; U \sqcup S^{\smile} ; S ; U \\ &\sqsubseteq& R^{\smile} ; R ; U' \sqcup S^{\smile} ; S ; U' \\ &=& (R^{\smile} ; R \sqcup S^{\smile} ; S) ; U' \\ &=& U' \end{array}$$

□

Lemma 2.3.7 In a collagory $\mathbf{C}$, let $W_1 : \mathcal{B} \leftrightarrow \mathcal{C}$ and $W_2 : \mathcal{C} \leftrightarrow \mathcal{E}$ be a difunctional morphism.

If the cospan $\mathcal{B} \xrightarrow{R_1} \mathcal{D} \xleftarrow{S_1} \mathcal{C}$ is a co-tabulation of W_1, the cospan $\mathcal{C} \xrightarrow{R_2} \mathcal{F} \xleftarrow{S_2} \mathcal{E}$ is a co-tabulation of W_2, and the cospan $\mathcal{D} \xrightarrow{R} \mathcal{G} \xleftarrow{S} \mathcal{F}$ is a co-tabulation of $W := (S_1^\smile \mathbin{;} R_2)^{\boxast}$, then the cospan $\mathcal{B} \xrightarrow{R_1 ; R} \mathcal{G} \xleftarrow{S_2 ; S} \mathcal{E}$ satisfies

$$R_1 \mathbin{;} R \mathbin{;} S^\smile \mathbin{;} S_2^\smile = (W_1 \mathbin{;} W_2)^{\boxast} \ .$$

If furthermore

$$\mathbb{I}_{\mathcal{C}} \sqsubseteq W_1^\smile \mathbin{;} W_1 \sqcup W_2 \mathbin{;} W_2^\smile \ , \tag{$*$}$$

then it also satisfies

$$R^\smile \mathbin{;} R_1^\smile \mathbin{;} R_1 \mathbin{;} R \sqcup S^\smile \mathbin{;} S_2^\smile \mathbin{;} S_2 \mathbin{;} S = \mathbb{I}_{\mathcal{G}} \ ,$$

i.e., it is a co-tabulation of $(W_1 \mathbin{;} W_2)^{\boxast}$.

$$\begin{tikzcd}
\mathcal{B} \arrow[rr, "W_1"] \arrow[dr, "R_1"'] & & \mathcal{C} \arrow[rr, "W_2"] \arrow[dl, "S_1"] \arrow[dr, "R_2"'] & & \mathcal{E} \arrow[dl, "S_2"] \\
& \mathcal{D} \arrow[rr, "W"'] \arrow[dr, "R"'] & & \mathcal{F} \arrow[dl, "S"] & \\
& & \mathcal{G} & &
\end{tikzcd}$$

Proof:

$$\begin{aligned}
R_1 \mathbin{;} R \mathbin{;} S^\smile \mathbin{;} S_2^\smile &= R_1 \mathbin{;} W \mathbin{;} S_2^\smile \\
&= R_1 \mathbin{;} (S_1^\smile \mathbin{;} R_2)^{\boxast} \mathbin{;} S_2^\smile \\
&= W_1 \mathbin{;} (R_2 \mathbin{;} R_2^\smile \mathbin{;} S_1 \mathbin{;} S_1^\smile)^* \mathbin{;} W_2 \\
&= W_1 \mathbin{;} ((\mathbb{I} \sqcup W_2 \mathbin{;} W_2^\smile) \mathbin{;} (\mathbb{I} \sqcup W_1^\smile \mathbin{;} W_1))^* \mathbin{;} W_2 \\
&= (W_1 \mathbin{;} W_2)^{\boxast} && W_1,\ W_2 \text{ difunctional}
\end{aligned}$$

$$\begin{aligned}
\mathbb{I}_{\mathcal{G}} &= R^\smile \mathbin{;} R \sqcup S^\smile \mathbin{;} S \\
&= R^\smile \mathbin{;} (R_1^\smile \mathbin{;} R_1 \sqcup S_1^\smile \mathbin{;} S_1) \mathbin{;} R \sqcup S^\smile \mathbin{;} (R_2^\smile \mathbin{;} R_2 \sqcup S_2^\smile \mathbin{;} S_2) \mathbin{;} S \\
&= R^\smile \mathbin{;} R_1^\smile \mathbin{;} R_1 \mathbin{;} R \sqcup R^\smile \mathbin{;} S_1^\smile \mathbin{;} S_1 \mathbin{;} R \sqcup S^\smile \mathbin{;} R_2^\smile \mathbin{;} R_2 \mathbin{;} S \sqcup S^\smile \mathbin{;} S_2^\smile \mathbin{;} S_2 \mathbin{;} S \\
&= R^\smile \mathbin{;} R_1^\smile \mathbin{;} R_1 \mathbin{;} R \sqcup R^\smile \mathbin{;} S_1^\smile \mathbin{;} R_2 \mathbin{;} S \sqcup S^\smile \mathbin{;} S_2^\smile \mathbin{;} S_2 \mathbin{;} S && S_1 \mathbin{;} R = R_2 \mathbin{;} S \\
&= R^\smile \mathbin{;} R_1^\smile \mathbin{;} R_1 \mathbin{;} R \sqcup S^\smile \mathbin{;} S_2^\smile \mathbin{;} S_2 \mathbin{;} S && \text{see below}
\end{aligned}$$

The last equality relies on the following inclusion:

$$\begin{aligned}
R^\smile \mathbin{;} S_1^\smile \mathbin{;} R_2 \mathbin{;} S &\sqsubseteq R^\smile \mathbin{;} S_1^\smile \mathbin{;} (W_1^\smile \mathbin{;} W_1 \sqcup W_2 \mathbin{;} W_2^\smile) \mathbin{;} R_2 \mathbin{;} S && (*) \\
&= R^\smile \mathbin{;} S_1^\smile \mathbin{;} W_1^\smile \mathbin{;} W_1 \mathbin{;} S_1 \mathbin{;} R \sqcup S^\smile \mathbin{;} R_2^\smile \mathbin{;} W_2 \mathbin{;} W_2^\smile \mathbin{;} R_2 \mathbin{;} S && S_1 \mathbin{;} R = R_2 \mathbin{;} S \\
&\sqsubseteq R^\smile \mathbin{;} R_1^\smile \mathbin{;} R_1 \mathbin{;} R \sqcup S^\smile \mathbin{;} R_2^\smile \mathbin{;} W_2 \mathbin{;} W_2^\smile \mathbin{;} R_2 \mathbin{;} S && W_1 \mathbin{;} S_1 \sqsubseteq R_1 \\
&\sqsubseteq R^\smile \mathbin{;} R_1^\smile \mathbin{;} R_1 \mathbin{;} R \sqcup S^\smile \mathbin{;} S_2^\smile \mathbin{;} S_2 \mathbin{;} S && W_2^\smile \mathbin{;} R_2 \sqsubseteq S_2
\end{aligned}$$

□

A co-tabulation for a difunctional closure $Z^{\boxast}$ satisfies the following equations:

$$R ; S^{\smile} = Z^{\boxast} \qquad R ; R^{\smile} = Z^{\rhd} \qquad S ; S^{\smile} = Z^{\lhd} \qquad R^{\smile} ; R \sqcup S^{\smile} ; S = \mathbb{I}_{\mathcal{D}} \ .$$

This was introduced as a *gluing for* the morphism Z in [Kah01]. Kawahara is the first to have characterised pushouts relation-algebraically in essentially this way [Kaw90]; he used relation-algebraic operations on relations arising in toposes.

In the following, we will relate tabulations and co-tabulations on the one hand with pushouts and pullbacks on the other hand, and in order to make this discussion less awkward with repect to formulations, we introduce the following convention that extends the use of the words "tabulation" and "co-tabulation" from triangle diagrams to encompass also square diagrams:

Convention 2.3.8 For a square of morphisms as drawn to the right, we say that

$$\begin{tikzcd} \bullet \arrow[r, "Q"] \arrow[d, "P"'] & \bullet \arrow[d, "S"] \\ \bullet \arrow[r, "R"] & \bullet \end{tikzcd}$$

- it *is a tabulation* iff $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is a tabulation for $R ; S^{\smile}$,
- it *is a (direct) co-tabulation* iff $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation for $P^{\smile} ; Q$,
- it *is a gluing* iff $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a gluing for $P^{\smile} ; Q$, that is, if it is a co-tabulation for $(P^{\smile} ; Q)^{\boxast}$. □

Definition 2.3.9 If an allegory has a tabulation for each morphism, we call it *tabular*.

If a collagory has a co-tabulation for each morphism, we call it *co-tabular*, and if it is furthermore tabular, we call it *bi-tabular*. □

2.4 Least Morphisms in Collagories

We collect now a few properties of least morphisms that hold independent of the zero laws.

Lemma 2.4.1 If the least morphism $\perp\!\!\!\perp_{\mathcal{A},\mathcal{B}}$ exists, and $F : \mathcal{B} \leftrightarrow \mathcal{C}$ is univalent, then $\perp\!\!\!\perp_{\mathcal{A},\mathcal{C}}$ exists, too, and

$$\perp\!\!\!\perp_{\mathcal{A},\mathcal{B}} ; F = \perp\!\!\!\perp_{\mathcal{A},\mathcal{C}}$$

Proof: For any $R : \mathcal{A} \to \mathcal{C}$ we have $\perp\!\!\!\perp_{\mathcal{A},\mathcal{B}} \sqsubseteq R ; F^{\smile}$, and therefore $\perp\!\!\!\perp_{\mathcal{A},\mathcal{B}} ; F \sqsubseteq R ; F^{\smile} ; F \sqsubseteq R$ using univalence of F, so $\perp\!\!\!\perp_{\mathcal{A},\mathcal{B}} ; F$ is a least morphism. □

2.5 Direct Sums in Collagories

Direct sums can be defined as co-tabulations for least morphisms, but since in collagories, least morphisms are not assumed to satisfy any zero laws, not all properties one usually expects of direct sums carry over to the collagory setting.

Definition 2.5.1 For two objects $\mathcal{A}$ and $\mathcal{B}$ in a collagory for which the least morphism $\perp\!\!\!\perp_{\mathcal{A},\mathcal{B}}$ exists, a cospan $\mathcal{A} \xrightarrow{\iota} \mathcal{S} \xleftarrow{\kappa} \mathcal{B}$ is a *direct sum* iff it is a co-tabulation for $\perp\!\!\!\perp_{\mathcal{A},\mathcal{B}}$. □

Therefore, if $\mathcal{A}$ and $\mathcal{B}$ have a direct sum, the least element $\perp\!\!\!\perp_{\mathcal{A},\mathcal{B}}$ needs to exist, and needs to be difunctional.

We shall frequently use (implicit, partial) choices of direct sums with the following notation:

$$\mathcal{A} \xrightarrow{\iota_{\mathcal{A},\mathcal{B}}} \mathcal{A}+\mathcal{B} \xleftarrow{\kappa_{\mathcal{A},\mathcal{B}}} \mathcal{B}$$

Furthermore, given two morphisms $R : \mathcal{A} \to \mathcal{C}$ and $S : \mathcal{B} \to \mathcal{D}$, and the direct sums $\mathcal{A}+\mathcal{B}$ and $\mathcal{C}+\mathcal{D}$, we define:

$$\begin{array}{rcl} \mathbb{X}_{\mathcal{A},\mathcal{B}} & := & \iota_{\mathcal{A},\mathcal{B}}^{\smile} ; \kappa_{\mathcal{B},\mathcal{A}} \sqcup \kappa_{\mathcal{A},\mathcal{B}}^{\smile} ; \iota_{\mathcal{B},\mathcal{A}} \\ R+S & := & \iota_{\mathcal{A},\mathcal{B}}^{\smile} ; R ; \iota_{\mathcal{C},\mathcal{D}} \sqcup \kappa_{\mathcal{A},\mathcal{B}}^{\smile} ; S ; \kappa_{\mathcal{C},\mathcal{D}} \end{array}$$

When using direct sums, we need to be careful since, although we do have $\iota_{\mathcal{A},\mathcal{B}} ; \kappa_{\mathcal{A},\mathcal{B}}^{\smile} = \perp\!\!\!\perp_{\mathcal{A},\mathcal{B}}$ from the co-tabulation properties, the least morphism $\perp\!\!\!\perp_{\mathcal{A},\mathcal{B}}$ does not necessarily satisfy any zero laws.

For example, in general, we only have:

Lemma 2.5.2 For a direct sum $\mathcal{A}+\mathcal{B}$ and two morphisms $R : \mathcal{A} \to \mathcal{C}$ and $S : \mathcal{B} \to \mathcal{C}$:

$$\begin{array}{rcl} \iota_{\mathcal{A},\mathcal{B}} ; (\iota_{\mathcal{A},\mathcal{B}}^{\smile} ; R \sqcup \kappa_{\mathcal{A},\mathcal{B}}^{\smile} ; S) & = & R \sqcup \perp\!\!\!\perp_{\mathcal{A},\mathcal{B}} ; S \\ \kappa_{\mathcal{A},\mathcal{B}} ; (\iota_{\mathcal{A},\mathcal{B}}^{\smile} ; R \sqcup \kappa_{\mathcal{A},\mathcal{B}}^{\smile} ; S) & = & S \sqcup \perp\!\!\!\perp_{\mathcal{B},\mathcal{A}} ; R \end{array}$$

Proof: We only show the first equation:

$$\begin{array}{rcll} \iota_{\mathcal{A},\mathcal{B}} ; (\iota_{\mathcal{A},\mathcal{B}}^{\smile} ; R \sqcup \kappa_{\mathcal{A},\mathcal{B}}^{\smile} ; S) & = & \iota_{\mathcal{A},\mathcal{B}} ; \iota_{\mathcal{A},\mathcal{B}}^{\smile} ; R \sqcup \iota_{\mathcal{A},\mathcal{B}} ; \kappa_{\mathcal{A},\mathcal{B}}^{\smile} ; S & \text{join distributivity} \\ & = & R \sqcup \perp\!\!\!\perp_{\mathcal{A},\mathcal{B}} ; S & \text{co-tabulation properties} \end{array}$$ $\square$

Lemma 2.5.3 For a direct sum $\mathcal{A}+\mathcal{B}$ and two morphisms $R : \mathcal{A} \to \mathcal{C}$ and $S : \mathcal{B} \to \mathcal{C}$:

- If S is univalent, then $\iota_{\mathcal{A},\mathcal{B}} ; (\iota_{\mathcal{A},\mathcal{B}}^{\smile} ; R \sqcup \kappa_{\mathcal{A},\mathcal{B}}^{\smile} ; S) = R$.
- If R is univalent, then $\kappa_{\mathcal{A},\mathcal{B}} ; (\iota_{\mathcal{A},\mathcal{B}}^{\smile} ; R \sqcup \kappa_{\mathcal{A},\mathcal{B}}^{\smile} ; S) = S$.

Proof: Again, we show only the first item:

$$\begin{array}{rcll} \iota_{\mathcal{A},\mathcal{B}} ; (\iota_{\mathcal{A},\mathcal{B}}^{\smile} ; R \sqcup \kappa_{\mathcal{A},\mathcal{B}}^{\smile} ; S) & = & R \sqcup \perp\!\!\!\perp_{\mathcal{A},\mathcal{B}} ; S & \text{Lemma 2.5.2} \\ & = & R \sqcup \perp\!\!\!\perp_{\mathcal{A},\mathcal{C}} & \text{Lemma 2.4.1} \\ & = & R & \text{join with least morphism} \end{array}$$ $\square$

From that, we obtain some useful properties of $\mathbb{X}$ and $+$:

$$\begin{array}{rcll} \iota_{\mathcal{A},\mathcal{B}} ; \mathbb{X}_{\mathcal{A},\mathcal{B}} & = & \iota_{\mathcal{A},\mathcal{B}} ; (\iota_{\mathcal{A},\mathcal{B}}^{\smile} ; \kappa_{\mathcal{B},\mathcal{A}} \sqcup \kappa_{\mathcal{A},\mathcal{B}}^{\smile} ; \iota_{\mathcal{B},\mathcal{A}}) & \\ & = & \kappa_{\mathcal{B},\mathcal{A}} & \text{Lemma 2.5.3} \end{array}$$

and

$$\begin{aligned}
(R+S) ; \mathbb{X}_{\mathcal{C},\mathcal{D}} &= (\iota^{\smile}_{\mathcal{A},\mathcal{B}} ; R ; \iota_{\mathcal{C},\mathcal{D}} \sqcup \kappa^{\smile}_{\mathcal{A},\mathcal{B}} ; S ; \kappa_{\mathcal{C},\mathcal{D}}) ; \mathbb{X}_{\mathcal{C},\mathcal{D}} \\
&= \iota^{\smile}_{\mathcal{A},\mathcal{B}} ; R ; \kappa_{\mathcal{D},\mathcal{C}} \sqcup \kappa^{\smile}_{\mathcal{A},\mathcal{B}} ; S ; \iota_{\mathcal{D},\mathcal{C}} \\
&= \mathbb{X}_{\mathcal{A},\mathcal{B}} ; \kappa^{\smile}_{\mathcal{B},\mathcal{A}} ; R ; \kappa_{\mathcal{D},\mathcal{C}} \sqcup \mathbb{X}_{\mathcal{A},\mathcal{B}} ; \iota^{\smile}_{\mathcal{B},\mathcal{A}} ; S ; \iota_{\mathcal{D},\mathcal{C}} \\
&= \mathbb{X}_{\mathcal{A},\mathcal{B}} ; (\kappa^{\smile}_{\mathcal{B},\mathcal{A}} ; R ; \kappa_{\mathcal{D},\mathcal{C}} \sqcup \iota^{\smile}_{\mathcal{B},\mathcal{A}} ; S ; \iota_{\mathcal{D},\mathcal{C}}) \\
&= \mathbb{X}_{\mathcal{A},\mathcal{B}} ; (S+R)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{X}_{\mathcal{B},\mathcal{A}} ; (R+S) ; \mathbb{X}_{\mathcal{C},\mathcal{D}} &= \mathbb{X}_{\mathcal{B},\mathcal{A}} ; (\iota^{\smile}_{\mathcal{A},\mathcal{B}} ; R ; \kappa_{\mathcal{D},\mathcal{C}} \sqcup \kappa^{\smile}_{\mathcal{A},\mathcal{B}} ; S ; \iota_{\mathcal{D},\mathcal{C}}) \\
&= \kappa^{\smile}_{\mathcal{B},\mathcal{A}} ; R ; \kappa_{\mathcal{D},\mathcal{C}} \sqcup \iota^{\smile}_{\mathcal{B},\mathcal{A}} ; S ; \iota_{\mathcal{D},\mathcal{C}} \\
&= S+R
\end{aligned}$$

Also:

$$\begin{aligned}
\iota_{\mathcal{A},\mathcal{B}} ; (R+S) &= \iota_{\mathcal{A},\mathcal{B}} ; (\iota^{\smile}_{\mathcal{A},\mathcal{B}} ; R ; \iota_{\mathcal{C},\mathcal{D}} \sqcup \kappa^{\smile}_{\mathcal{A},\mathcal{B}} ; S ; \kappa_{\mathcal{C},\mathcal{D}}) \\
&= R ; \iota_{\mathcal{C},\mathcal{D}} \sqcup \perp\!\!\!\perp_{\mathcal{A},\mathcal{B}} ; S ; \kappa_{\mathcal{C},\mathcal{D}} \\
\kappa_{\mathcal{A},\mathcal{B}} ; (R+S) &= \kappa_{\mathcal{A},\mathcal{B}} ; (\iota^{\smile}_{\mathcal{A},\mathcal{B}} ; R ; \iota_{\mathcal{C},\mathcal{D}} \sqcup \kappa^{\smile}_{\mathcal{A},\mathcal{B}} ; S ; \kappa_{\mathcal{C},\mathcal{D}}) \\
&= S ; \kappa_{\mathcal{C},\mathcal{D}} \sqcup \perp\!\!\!\perp_{\mathcal{B},\mathcal{A}} ; R ; \iota_{\mathcal{C},\mathcal{D}}
\end{aligned}$$

and

$$\begin{aligned}
& (P+Q) ; (R+S) \\
= \;& (\iota^{\smile}_{\mathcal{A},\mathcal{B}} ; P ; \iota_{\mathcal{C},\mathcal{D}} \sqcup \kappa^{\smile}_{\mathcal{A},\mathcal{B}} ; Q ; \kappa_{\mathcal{C},\mathcal{D}}) ; (R+S) \\
= \;& \iota^{\smile}_{\mathcal{A},\mathcal{B}} ; P ; \iota_{\mathcal{C},\mathcal{D}} ; (R+S) \sqcup \kappa^{\smile}_{\mathcal{A},\mathcal{B}} ; Q ; \kappa_{\mathcal{C},\mathcal{D}} ; (R+S) \\
= \;& \iota^{\smile}_{\mathcal{A},\mathcal{B}} ; P ; (R ; \iota_{\mathcal{E},\mathcal{F}} \sqcup \perp\!\!\!\perp_{\mathcal{C},\mathcal{D}} ; S ; \kappa_{\mathcal{E},\mathcal{F}}) \sqcup \kappa^{\smile}_{\mathcal{A},\mathcal{B}} ; Q ; (S ; \kappa_{\mathcal{E},\mathcal{F}} \sqcup \perp\!\!\!\perp_{\mathcal{D},\mathcal{C}} ; R ; \iota_{\mathcal{E},\mathcal{F}}) \\
= \;& \iota^{\smile}_{\mathcal{A},\mathcal{B}} ; P ; R ; \iota_{\mathcal{E},\mathcal{F}} \sqcup \iota^{\smile}_{\mathcal{A},\mathcal{B}} ; P ; \perp\!\!\!\perp_{\mathcal{C},\mathcal{D}} ; S ; \kappa_{\mathcal{E},\mathcal{F}} \sqcup \kappa^{\smile}_{\mathcal{A},\mathcal{B}} ; Q ; S ; \kappa_{\mathcal{E},\mathcal{F}} \sqcup \kappa^{\smile}_{\mathcal{A},\mathcal{B}} ; Q ; \perp\!\!\!\perp_{\mathcal{D},\mathcal{C}} ; R ; \iota_{\mathcal{E},\mathcal{F}} \\
= \;& (P ; R + Q ; S) \sqcup \iota^{\smile}_{\mathcal{A},\mathcal{B}} ; P ; \perp\!\!\!\perp_{\mathcal{C},\mathcal{D}} ; S ; \kappa_{\mathcal{E},\mathcal{F}} \sqcup \kappa^{\smile}_{\mathcal{A},\mathcal{B}} ; Q ; \perp\!\!\!\perp_{\mathcal{D},\mathcal{C}} ; R ; \iota_{\mathcal{E},\mathcal{F}}
\end{aligned}$$

Lemma 2.5.4 (inspired by [BG03, Def. 4.1]) Let $\mathcal{A} \xrightarrow{\iota} \mathcal{S} \xleftarrow{\kappa} \mathcal{B}$ be a direct sum of $\mathcal{A}$ and $\mathcal{B}$.

1. If $R : \mathcal{S} \to \mathcal{S}$ is symmetric and transitive, then $\iota ; R ; \kappa^{\smile}$ is difunctional.
2. If $W : \mathcal{A} \to \mathcal{B}$ is difunctional, then $S := \iota^{\smile} ; W ; (W^{\smile} ; \iota \sqcup \kappa) \sqcup \kappa^{\smile} ; W^{\smile} ; (\iota \sqcup W ; \kappa)$ is symmetric and transitive, and $\iota ; S ; \kappa^{\smile} = W$.

Proof: The first item is shown by direct calculation:

$$\begin{aligned}
\iota ; R ; \kappa^{\smile} ; \kappa ; R^{\smile} ; \iota^{\smile} ; \iota ; R ; \kappa^{\smile} &\sqsubseteq \iota ; R ; R^{\smile} ; R ; \kappa^{\smile} && \iota,\ \kappa \text{ univalent} \\
&\sqsubseteq \iota ; R ; R ; R ; \kappa^{\smile} && R \text{ symmetric} \\
&\sqsubseteq \iota ; R ; \kappa^{\smile} && R \text{ transitive}
\end{aligned}$$

For the second item, S is obviously symmetric. Even if the zero law does not hold, and without assuming anything about $\perp\!\!\!\perp_{\mathcal{A},\mathcal{B}}$ except that it is a least morphism, we have

$$\begin{aligned}
\iota ; \kappa^{\smile} &= \perp\!\!\!\perp_{\mathcal{A},\mathcal{B}} \sqsubseteq W \\
\iota ; \iota^{\smile} &= \mathbb{I} \sqcup \perp\!\!\!\perp_{\mathcal{A},\mathcal{B}} ; \perp\!\!\!\perp^{\smile}_{\mathcal{A},\mathcal{B}} \sqsubseteq \mathbb{I} \sqcup W ; W^{\smile} \\
\kappa ; \kappa^{\smile} &= \mathbb{I} \sqcup \perp\!\!\!\perp^{\smile}_{\mathcal{A},\mathcal{B}} ; \perp\!\!\!\perp_{\mathcal{A},\mathcal{B}} \sqsubseteq \mathbb{I} \sqcup W^{\smile} ; W \ .
\end{aligned}$$

Together with difunctionality of W and injectivity and totality of ι and κ, this implies:

$$\begin{array}{rcl}
(W^{\smile} ; \iota \sqcup \kappa) ; \iota^{\smile} & = & W^{\smile} \\
W ; (W^{\smile} ; \iota \sqcup \kappa) ; \kappa^{\smile} & = & W \\
W^{\smile} ; (\iota \sqcup W ; \kappa) ; \iota^{\smile} & = & W^{\smile} \\
(\iota \sqcup W ; \kappa) ; \kappa^{\smile} & = & W
\end{array}$$

These equations, together with difunctionality of W, allow us to derive transitivity of S:

$$\begin{array}{rcl}
S ; S & = & \iota^{\smile} ; W ; (W^{\smile} ; \iota \sqcup \kappa) ; \iota^{\smile} ; W ; (W^{\smile} ; \iota \sqcup \kappa) \\
& & \sqcup\, \iota^{\smile} ; W ; (W^{\smile} ; \iota \sqcup \kappa) ; \kappa^{\smile} ; W^{\smile} ; (\iota \sqcup W ; \kappa) \\
& & \sqcup\, \kappa^{\smile} ; W^{\smile} ; (\iota \sqcup W ; \kappa) ; \iota^{\smile} ; W ; (W^{\smile} ; \iota \sqcup \kappa) \\
& & \sqcup\, \kappa^{\smile} ; W^{\smile} ; (\iota \sqcup W ; \kappa) ; \kappa^{\smile} ; W^{\smile} ; (\iota \sqcup W ; \kappa) \\
& = & \iota^{\smile} ; W ; W^{\smile} ; W ; (W^{\smile} ; \iota \sqcup \kappa) \\
& & \sqcup\, \iota^{\smile} ; W ; W^{\smile} ; (\iota \sqcup W ; \kappa) \\
& & \sqcup\, \kappa^{\smile} ; W^{\smile} ; W ; (W^{\smile} ; \iota \sqcup \kappa) \\
& & \sqcup\, \kappa^{\smile} ; W^{\smile} ; W ; W^{\smile} ; (\iota \sqcup W ; \kappa) \\
& = & \iota^{\smile} ; W ; (W^{\smile} ; \iota \sqcup \kappa) \sqcup \kappa^{\smile} ; W^{\smile} ; (\iota \sqcup W ; \kappa) \\
& = & S
\end{array}$$

The equations above also give us:

$$\begin{array}{rcl}
\iota ; S ; \kappa^{\smile} & = & \iota ; \iota^{\smile} ; W ; (W^{\smile} ; \iota \sqcup \kappa) ; \kappa^{\smile} \sqcup \iota ; \kappa^{\smile} ; W^{\smile} ; (\iota \sqcup W ; \kappa) ; \kappa^{\smile} \\
& = & \iota ; \iota^{\smile} ; W \sqcup \iota ; \kappa^{\smile} ; W^{\smile} ; W \\
& = & \iota ; (\iota^{\smile} \sqcup \kappa^{\smile} ; W^{\smile}) ; W \\
& = & W
\end{array}$$

□

The corresponding statements for equivalences easily follow from this.

2.6 The Gluing Condition in Collagories

We can now state a relational variant of the gluing condition, first introduced by Kawahara [Kaw90]:

Definition 2.6.1 Let two morphisms[1] $\Phi : \mathcal{G} \to \mathcal{L}$ and $X : \mathcal{L} \to \mathcal{A}$ in a collagory with pseudo-complements on subidentities be given.[2]

[1] Note that "X" is a capital "χ".

[2] *Pseudo-complements* are residuation of meet in lower semilattice categories; where pseudo-complements exist, we denote the pseudo-complement or R with respect to S as $R \to S$, and we have:

$$X \sqcap R \sqsubseteq S \qquad \Leftrightarrow \qquad X \sqsubseteq (R \to S)$$

For example, the pseudo-complement of a subgraph R of a graph G with respect to another subgraph S consists of all nodes of G that are in S or not in R, and all edges in S or not in R that are also nor incident with nodes in R. Intuitively, $R \to S$ therefore is G with the parts or R outside S removed, and then also all dangling edges removed.

- We say that the *identification condition* holds iff $X ; X^{\smile} \sqsubseteq \mathbb{I} \sqcup (\mathsf{ran}\,\Phi) ; X ; X^{\smile} ; \mathsf{ran}\,\Phi$.
- We say that the *dangling condition* holds iff $\mathsf{ran}\,X \sqcup (\mathsf{ran}\,X \rightarrow \mathsf{ran}\,(\Phi ; X)) = \mathbb{I}$. □

$$
\begin{tikzcd}
\mathcal{L} \arrow[d, "X"'] & \mathcal{G} \arrow[l, "\Phi"'] \arrow[d, dotted, "\Xi"] \\
\mathcal{A} & \mathcal{H} \arrow[l, dotted, "\Psi"]
\end{tikzcd}
$$

The proofs that the gluing condition is sufficient for the existence of a pushout complement [Kaw90], and that injectivity of Φ is sufficient for unambiguity of the pushout complement [Kah01] carry over to the collagory setting, but are outside the scope of this paper.

Another related condition is important in the context of the single-pushout approach:

Definition 2.6.2 In an allegory, we call X *conflict-free for* Φ iff $\mathsf{ran}\,(\Phi ; X ; X^{\smile}) \sqsubseteq \mathsf{ran}\,\Phi$. □

For a node-and-edges-level formulation of conflict-freeness it is well-known that the induced single-pushout squares have a total embedding of the right-hand side into the application graph [Löw90, Cor. 3.18.5]. The component-free formulation above was first given in [Kah01], where it is also shown (Thm. 5.4.11) that a restricting derivation step for a conflict-free redex produces a pushout of partial functions.

Chapter 3

Algebraic Collagory Constructions

3.1 Collagories of Semi-Unary Algebras and Bisimulations

In [Kah01, Kah04], relational homomorphisms between unary algebras have been shown to form a distributive allegory. In this section we generalise this result to collagories by allowing constant symbols and in turn dropping the zero law requirement.

Most of the mathematical content of this section has been presented and proven in more detail in [Kah01, Kah04]. Besides the proof of Theorem 3.1.6, also the reformulation using the sort-indexed product category and the forgetful functor $\mathcal{U}_\Sigma$ is new.

Definition 3.1.1 A *signature* is a tuple $(\mathcal{S}, \mathcal{F}, \mathsf{src}, \mathsf{trg})$ consisting of

- a set $\mathcal{S}$ of *sorts*,
- a set $\mathcal{F}$ of *function symbols*,
- a mapping $\mathsf{src} : \mathcal{F} \to \mathcal{S}^*$ associating with every function symbol the list of its *source sorts*, and
- a mapping $\mathsf{trg} : \mathcal{F} \to \mathcal{S}$ associating with every function symbol its *target sort*.

Such a signature is called *semi-unary* if $\mathsf{length}(\mathsf{src}(f)) \leq 1$ for each $f : \mathcal{F}$, and *unary* if $\mathsf{length}(\mathsf{src}(f)) = 1$ for each $f : \mathcal{F}$. □

For a function symbol $f : \mathcal{F}$, we usually employ the shorthand "$f : s_1 \times \cdots \times s_n \to t$" instead of the rather verbose "$\mathsf{src}(f) = \langle s_1, \ldots, s_n \rangle$ and $\mathsf{trg}(f) = t$". For a zero-ary function symbol, also called *constant symbol*, we write "$f : \mathbb{1} \to t$".

The following example signatures will be used for discussion and results in sections 3.2 and 3.3:

$$\begin{aligned} \mathsf{sigGraph} := \langle\ & \textbf{sorts: } \mathsf{V}, \mathsf{E} \\ & \textbf{ops: } \mathbf{D}s, \mathbf{D}t : \mathsf{E} \to \mathsf{V} \\ \rangle\ & \end{aligned} \qquad \begin{aligned} \mathsf{sigPointedSet} := \langle\ & \textbf{sorts: } \mathsf{S} \\ & \textbf{ops: } \mathbf{D}point : \mathbb{1} \to \mathsf{S} \\ \rangle\ & \end{aligned}$$

$$\begin{array}{lll} \mathsf{sigPoint} := & \langle & \textbf{sorts: } \mathsf{P} \\ & & \textbf{ops: } \\ & \rangle & \end{array} \qquad \begin{array}{lll} \mathsf{sigPointed} := & \langle & \textbf{sorts: } \mathsf{P}, \mathsf{O} \\ & & \textbf{ops: } \mathbf{D}p : \mathsf{P} \to \mathsf{O} \\ & \rangle & \end{array}$$

$$\begin{array}{lll} \mathsf{sigType} := & \langle & \textbf{sorts: } \mathsf{T} \\ & & \textbf{ops: } \\ & \rangle & \end{array} \qquad \begin{array}{lll} \mathsf{sigTyped} := & \langle & \textbf{sorts: } \mathsf{O}, \mathsf{T} \\ & & \textbf{ops: } \mathbf{D}t : \mathsf{O} \to \mathsf{T} \\ & \rangle & \end{array}$$

$$\begin{array}{lll} \mathsf{sigNELabels} := & \langle & \textbf{sorts: } \mathsf{NL}, \mathsf{EL} \\ & & \textbf{ops: } \\ & \rangle & \end{array} \qquad \begin{array}{llll} \mathsf{sigLGraph} := & \langle & \textbf{sorts: } & \mathsf{N}, \mathsf{E}, \mathsf{NL}, \mathsf{EL} \\ & & \textbf{ops: } & \mathbf{D}s, \mathbf{D}t : \mathsf{E} \to \mathsf{N}, \\ & & & \mathbf{D}n : \mathsf{N} \to \mathsf{NL}, \\ & & & \mathbf{D}e : \mathsf{E} \to \mathsf{EL} \\ & \rangle & & \end{array}$$

Definition 3.1.2 For a set $\mathcal{S}$ (of *sorts*) and a category $\mathbf{C}$, we define $\mathbf{C}^{\mathcal{S}}$, the *$\mathcal{S}$-indexed product category of* $\mathbf{C}$, as follows:

- an object $\mathcal{A}$ of $\mathbf{C}^{\mathcal{S}}$ consists of $\mathbf{C}$-objects $s^{\mathcal{A}}$ for every $s : \mathcal{S}$;
- a morphism $\Phi : \mathcal{A} \to \mathcal{B}$ of $\mathbf{C}^{\mathcal{S}}$ is an $\mathcal{S}$-indexed family of $\mathbf{C}$-morphisms $\Phi = (\Phi_s)_{s:\mathcal{S}}$ such that $\Phi_s : s^{\mathcal{A}} \to s^{\mathcal{B}}$ for every sort $s : \mathcal{S}$.
- composition $;^{\mathcal{S}}$ and identities $\mathbb{I}^{\mathcal{S}}$ are defined component-wise;
- if $\mathbf{C}$ is an allegory, then inclusion $\sqsubseteq^{\mathcal{S}}$, meet $\sqcap^{\mathcal{S}}$ and converse are defined component-wise;
- if $\mathbf{C}$ is a collagory, then join $\sqcup^{\mathcal{S}}$ is defined component-wise. □

One easily verifies that the resulting $\mathcal{S}$-indexed product categories, allegories, and collagories all satisfy the respective axioms.

When defining Σ-algebras in the presence of binary function symbols, we need several technical conditions on direct products [Kah01, Def. 3.1.12]; for the current study, we can do without direct products (at the cost of some duplication of formalisation for unary and zero-ary function symbols), but we still need OCCs for the characterisation of mappings:

Definition 3.1.3 Given a semi-unary signature $\Sigma = (\mathcal{S}, \mathcal{F}, \mathsf{src}, \mathsf{trg})$ and an OCC $\mathbf{C}$, which has to have a unit $\mathbb{1}$ if Σ contains constant symbols, an *abstract* Σ*-algebra* over $\mathbf{C}$ consists of the following items:

- an object $\mathcal{A}$ of $\mathbf{C}^{\mathcal{S}}$,
- for every function symbol $f{:}\mathcal{F}$ with $f : s \to t$ a mapping $f^{\mathcal{A}} : s^{\mathcal{A}} \to t^{\mathcal{A}}$ in $\mathbf{C}$.
- for every constant symbol $c{:}\mathcal{F}$ with $c : \mathbb{1} \to t$ a mapping $c^{\mathcal{A}} : \mathbb{1} \to t^{\mathcal{A}}$ in $\mathbf{C}$. □

It is important to note that, where we use sets as carriers, we have no restriction to non-empty sets — unlike most of the universal algebra literature.

Since we use this definition to construct an allegory with abstract Σ-algebras as objects, the generality of discussing *abstract* Σ-algebras over allegories allows us to stack this construction at no cost at all, with possibly different signatures at every level, building for example graphs where the nodes and edges are hypergraphs and hypergraph morphisms.

The morphisms in allegories of Σ-algebras have to behave "essentially like relations", and so it is only natural that we consider a relational generalisation of conventional (functional) Σ-homomorphisms. For arbitrary signatures, this has been presented in [Kah01]. For unary signatures, one naturally starts with defining L-simulations satisfying $\Phi_s^{\smile} ; f^{\mathcal{A}} \sqsubseteq f^{\mathcal{B}} ; \Phi_t^{\smile}$ according to de Roever and Engelhardt [dRE98], and then proceeds to L-simulations for which their converse is an L-simulation, too; these are called "bisimulations" in [Kah04].

Definition 3.1.4 Let a signature $\Sigma = (\mathcal{S}, \mathcal{F}, \mathsf{src}, \mathsf{trg})$, an allegory $\mathbf{C}$, and two abstract Σ-algebras $\mathcal{A}$ and $\mathcal{B}$ over $\mathbf{C}$ be given.

A *Σ-bisimulation from $\mathcal{A}$ to $\mathcal{B}$* is a $\mathbf{C}^{\mathcal{S}}$-morphisms from $\mathcal{A}$ to $\mathcal{B}$ such that for every function symbol $f \in \mathcal{F}$ with $f : s \to t$ and every constant symbol $c \in \mathcal{F}$ with $c : \mathbb{1} \to t$ the following inclusions hold:

$$\Phi_s ; f^{\mathcal{B}} \sqsubseteq f^{\mathcal{A}} ; \Phi_t, \qquad \text{and} \qquad c^{\mathcal{B}} \sqsubseteq c^{\mathcal{A}} ; \Phi_t \ . \qquad \square$$

In the allegory $\mathbf{C}$, this gives rise to the following sub-commuting diagrams (including one for the n-ary case):

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "c^{\mathcal{A}}"] \arrow[d, "\mathbb{I}_{\mathbb{1}}"'] \arrow[dr, phantom, "\sqsubseteq"] & t^{\mathcal{A}} \arrow[d, "\Phi_t"] \\
\mathbb{1} \arrow[r, "c^{\mathcal{B}}"'] & t^{\mathcal{B}}
\end{tikzcd}
\qquad
\begin{tikzcd}
s^{\mathcal{A}} \arrow[r, "f^{\mathcal{A}}"] \arrow[d, "\Phi_s"'] \arrow[dr, phantom, "\sqsubseteq"] & t^{\mathcal{A}} \arrow[d, "\Phi_t"] \\
s^{\mathcal{B}} \arrow[r, "f^{\mathcal{B}}"'] & t^{\mathcal{B}}
\end{tikzcd}
\qquad
\begin{tikzcd}
s_1^{\mathcal{A}} \times \cdots \times s_n^{\mathcal{A}} \arrow[r, "g^{\mathcal{A}}"] \arrow[d, "\Phi_{s_1} \times \cdots \times \Phi_{s_n}"'] \arrow[dr, phantom, "\sqsubseteq"] & t^{\mathcal{A}} \arrow[d, "\Phi_t"] \\
s_1^{\mathcal{B}} \times \cdots \times s_n^{\mathcal{B}} \arrow[r, "g^{\mathcal{B}}"'] & t^{\mathcal{B}}
\end{tikzcd}$$

Using Σ-algebras over $\mathbf{C}$ as objects and Σ-bisimulations as morphisms defines a category $\mathbf{C}^{\Sigma}$ with an obvious "underlying" functor $\mathcal{U}_{\Sigma} : \mathbf{C}^{\Sigma} \to \mathbf{C}^{\mathcal{S}}$.

This "forgetful" functor $\mathcal{U}_{\Sigma}$ is faithful. If $\mathbf{C}$ is an allegory, then $\mathcal{U}_{\Sigma}$ reflects inclusion, meets and converse in the sense that these can be defined for $\mathbf{C}^{\Sigma}$ via their $\mathcal{U}_{\Sigma}$ images. Therefore, $\mathbf{C}^{\Sigma}$ is an allegory, too [Kah01, Thm. 3.2.6].

We may observe a few simple facts:

- If $\mathbf{C}$ contains an initial object $\varnothing$, and Σ contains no constants, then we obtain an initial object $\mathbb{O}_{\Sigma}$ in $\mathbf{C}^{\Sigma}$ by choosing $s^{\mathbb{O}_{\Sigma}} = \varnothing$ for each sort s and $f^{\mathbb{O}_{\Sigma}} = \mathbb{I}_{\varnothing}$ for each function symbol f.
- If $\mathbf{C}$ contains a unit $\mathbb{1}$, then we obtain a unit $\mathbb{1}_{\Sigma}$ in $\mathbf{C}^{\Sigma}$ by choosing $s^{\mathbb{1}_{\Sigma}} = \mathbb{1}$ for each sort s and $f^{\mathbb{1}_{\Sigma}} = \mathbb{I}_{\mathbb{1}}$ for each function symbol f.

Conventional Σ-algebra homomorphisms are just mappings in the allegory Rel^{Σ} of concrete Σ-algebras over the allegory *Rel* of sets and concrete relations.

If Σ contains a constant symbol, then even if the allegory $\mathbf{C}$ has least morphisms, then least homomorphisms in $\mathbf{C}^{\mathcal{S}}$ are not generally in the range of $\mathcal{U}_{\Sigma}$, and even if $\mathbf{C}^{\Sigma}$ does have least morphisms, the zero law will in general not hold for them, no matter whether it holds in $\mathbf{C}$.

If Σ contains a function symbol of arity at least 2, then even if $\mathbf{C}$ is an upper-semilattice category, then $\mathcal{U}_\Sigma$ does not reflect joins, in the sense that $\mathcal{U}_\Sigma(\Phi) \sqcup^{\mathcal{S}} \mathcal{U}_\Sigma(\Psi)$ is not necessarily in the range of $\mathcal{U}_\Sigma$. Furthermore, even if $\mathbf{C}^\Sigma$ has joins, composition will, in presence of function symbols of arity at least 2, in general not distribute over these joins (since non-empty joins do not distribute over the product $\times$ occurring in the homomorphism condition) so $\mathbf{C}^\Sigma$ will not be an upper-semilattice category.

For semi-unary signatures, however, $\mathcal{U}_\Sigma$ does reflect joins:

Lemma 3.1.5 If $\mathbf{C}$ is an upper-semilattice category, Σ is a semi-unary signature, and $\Phi, \Psi : \mathcal{A} \to \mathcal{B}$ are two Σ-bisimulations, then $\Phi \sqcup^{\mathcal{S}} \Psi$ is a Σ-bisimulation, too, and is the join in $\mathbf{C}^\Sigma$ of Φ and Ψ, that is, $\Phi \sqcup^\Sigma \Psi = \Phi \sqcup^{\mathcal{S}} \Psi$.

PROOF: We need to check the bisimulation conditions for unary function symbols $f : s \to t$ and for constant symbols $c : \mathbb{1} \to t$:

$$\begin{array}{rcl}
(\Phi \sqcup^{\mathcal{S}} \Psi)_s \mathbin{;} f^{\mathcal{B}} & = & (\Phi_s \sqcup \Psi_s) \mathbin{;} f^{\mathcal{B}} \;=\; \Phi_s \mathbin{;} f^{\mathcal{B}} \sqcup \Psi_s \mathbin{;} f^{\mathcal{B}} \\
& \sqsubseteq & f^{\mathcal{A}} \mathbin{;} \Phi_t \sqcup f^{\mathcal{A}} \mathbin{;} \Psi_t \;=\; f^{\mathcal{A}} \mathbin{;} (\Phi_t \sqcup \Psi_t) \;=\; f^{\mathcal{A}} \mathbin{;} (\Phi \sqcup^{\mathcal{S}} \Psi)_t \\
c^{\mathcal{B}} & \sqsubseteq & c^{\mathcal{A}} \mathbin{;} \Phi_t \sqcup c^{\mathcal{A}} \mathbin{;} \Psi_t \;=\; c^{\mathcal{A}} \mathbin{;} (\Phi_t \sqcup \Psi_t) \;=\; c^{\mathcal{A}} \mathbin{;} (\Phi \sqcup^{\mathcal{S}} \Psi)_t
\end{array}$$

The equation $\Phi \sqcup^\Sigma \Psi = \Phi \sqcup^{\mathcal{S}} \Psi$ follows from the reflection of inclusion by $\mathcal{U}_\Sigma$. □

Given the closure of Σ-bisimulations under the converse, meet, and join operations in $\mathbf{C}^{\mathcal{S}}$, properties of $\mathbf{C}$-morphisms for these operations are inherited by Σ-bisimulations because of the component-wise definitions, and we obtain:

Theorem 3.1.6 If Σ is a semi-unary signature and $\mathbf{C}$ is a collagory, then $\mathbf{C}^\Sigma$ is a collagory, too. □

If $\mathbf{C}$ has tabulations (respectively co-tabulations), the sort-indexed product category $\mathbf{C}^{\mathcal{S}}$ obviously has tabulations (respectively co-tabulations), too, and they can be calculated component-wise. Perhaps surprisingly, these can be extended to the collagory $\mathbf{C}^\Sigma$ of bisimulations between Σ-algebras without problems; we just need to provide definitions for the function symbols of the "new" objects, and verify all relevant conditions:

Theorem 3.1.7 If $\Sigma = (\mathcal{S}, \mathcal{F}, \mathsf{src}, \mathsf{trg})$ is a semi-unary signature and $\mathbf{C}$ is an allegory, and $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is a tabulation in $\mathbf{C}^{\mathcal{S}}$ of the Σ-bisimulation $V : \mathcal{B} \to \mathcal{C}$, i.e., for each sort $s : \mathcal{S}$, $\mathcal{B} \xleftarrow{P_s} \mathcal{A} \xrightarrow{Q_s} \mathcal{C}$ is a tabulation of $V_s : s^{\mathcal{B}} \to s^{\mathcal{C}}$, then we define for each function symbol $f : s \to t$ and each constant symbol $c : \mathbb{1} \to t$ in Σ:

$$\begin{array}{rcl}
f^{\mathcal{A}} & := & P_s \mathbin{;} f^{\mathcal{B}} \mathbin{;} P_t^{\smile} \sqcap Q_s \mathbin{;} f^{\mathcal{C}} \mathbin{;} Q_t^{\smile} \\
c^{\mathcal{A}} & := & c^{\mathcal{B}} \mathbin{;} P_t^{\smile} \sqcap c^{\mathcal{C}} \mathbin{;} Q_t^{\smile}
\end{array}$$

Then $\mathcal{A}$ turns into a Σ-algebra and P and Q are Σ-bisimulations, too, so $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is a tabulation in $\mathbf{C}^\Sigma$.

Proof: We first show the bisimulation conditions for P; those for Q follow analogously:

$$\begin{array}{rcll}
P_s ; f^{\mathcal{B}} & = & (P_s \sqcap Q_s ; Q_s^{\smile} ; P_s) ; f^{\mathcal{B}} & Q_s \text{ total} \\
& = & (P_s \sqcap Q_s ; V_s^{\smile}) ; f^{\mathcal{B}} & \text{tabulation of } V_s \\
& \sqsubseteq & P_s ; f^{\mathcal{B}} \sqcap Q_s ; V_s^{\smile} ; f^{\mathcal{B}} & \text{meet-subdistributivity} \\
& \sqsubseteq & P_s ; f^{\mathcal{B}} \sqcap Q_s ; f^{\mathcal{C}} ; V_t^{\smile} & V^{\smile} \text{ bisimulation} \\
& = & P_s ; f^{\mathcal{B}} \sqcap Q_s ; f^{\mathcal{C}} ; Q_t^{\smile} ; P_t & \text{tabulation of } V_t \\
& = & (P_s ; f^{\mathcal{B}} ; P_t^{\smile} \sqcap Q_s ; f^{\mathcal{C}} ; Q_t^{\smile}) ; P_t & P_t \text{ univalent} \\
& = & f^{\mathcal{A}} ; P_t & \text{Def. } f^{\mathcal{A}} \\
c^{\mathcal{B}} & = & c^{\mathcal{B}} \sqcap c^{\mathcal{C}} ; V_t^{\smile} & V^{\smile} \text{ bisimulation} \\
& = & c^{\mathcal{B}} \sqcap c^{\mathcal{C}} ; Q_t^{\smile} ; P_t & \text{tabulation of } V_t \\
& = & (c^{\mathcal{B}} ; P_t^{\smile} \sqcap c^{\mathcal{C}} ; Q_t^{\smile}) ; P_t & P_t \text{ univalent} \\
& = & c^{\mathcal{A}} ; P_t & \text{Def. } c^{\mathcal{A}}
\end{array}$$

Next we show that $f^{\mathcal{A}}$ and $c^{\mathcal{A}}$ are univalent:

$$\begin{array}{rcll}
(f^{\mathcal{A}})^{\smile} ; f^{\mathcal{A}} & = & (P_t ; (f^{\mathcal{B}})^{\smile} ; P_s^{\smile} \sqcap Q_t ; (f^{\mathcal{C}})^{\smile} ; Q_s^{\smile}) & \\
& & \quad ; (P_s ; f^{\mathcal{B}} ; P_t^{\smile} \sqcap Q_s ; f^{\mathcal{C}} ; Q_t^{\smile}) & \\
& \sqsubseteq & P_t ; (f^{\mathcal{B}})^{\smile} ; P_s^{\smile} ; P_s ; f^{\mathcal{B}} ; P_t^{\smile} \sqcap & \\
& & Q_t ; (f^{\mathcal{C}})^{\smile} ; Q_s^{\smile} ; Q_s ; f^{\mathcal{C}} ; Q_t^{\smile} & \\
& \sqsubseteq & P_t ; (f^{\mathcal{B}})^{\smile} ; f^{\mathcal{B}} ; P_t^{\smile} \sqcap Q_t ; (f^{\mathcal{C}})^{\smile} ; f^{\mathcal{C}} ; Q_t^{\smile} & P_s,\ Q_s \text{ univalent} \\
& \sqsubseteq & P_t ; P_t^{\smile} \sqcap Q_t ; Q_t^{\smile} & f^{\mathcal{B}},\ f^{\mathcal{C}} \text{ univalent} \\
& = & \mathbb{I}_{t^{\mathcal{A}}} & \text{tabulation of } V_t \\
(c^{\mathcal{A}})^{\smile} ; c^{\mathcal{A}} & = & (P_t ; (c^{\mathcal{B}})^{\smile} \sqcap Q_t ; (c^{\mathcal{C}})^{\smile}) ; (c^{\mathcal{B}} ; P_t^{\smile} \sqcap c^{\mathcal{C}} ; Q_t^{\smile}) & \\
& \sqsubseteq & P_t ; (c^{\mathcal{B}})^{\smile} ; c^{\mathcal{B}} ; P_t^{\smile} \sqcap Q_t ; (c^{\mathcal{C}})^{\smile} ; c^{\mathcal{C}} ; Q_t^{\smile}) & \\
& \sqsubseteq & P_t ; P_t^{\smile} \sqcap Q_t ; Q_t^{\smile} & c^{\mathcal{B}},\ c^{\mathcal{C}} \text{ univalent} \\
& = & \mathbb{I}_{t^{\mathcal{A}}} & \text{tabulation of } V_t
\end{array}$$

For showing totality of $f^{\mathcal{A}}$ and $c^{\mathcal{A}}$, we use all the above:

$$\begin{array}{rcll}
f^{\mathcal{A}} ; (f^{\mathcal{A}})^{\smile} & = & f^{\mathcal{A}} ; (P_t ; (f^{\mathcal{B}})^{\smile} ; P_s^{\smile} \sqcap Q_t ; (f^{\mathcal{C}})^{\smile} ; Q_s^{\smile}) & \\
& = & f^{\mathcal{A}} ; P_t ; (f^{\mathcal{B}})^{\smile} ; P_s^{\smile} \sqcap f^{\mathcal{A}} ; Q_t ; (f^{\mathcal{C}})^{\smile} ; Q_s^{\smile} & f^{\mathcal{A}} \text{ univalent} \\
& \sqsupseteq & P_s ; f^{\mathcal{B}} ; (f^{\mathcal{B}})^{\smile} ; P_s^{\smile} \sqcap Q_s ; f^{\mathcal{C}} ; (f^{\mathcal{C}})^{\smile} ; Q_s^{\smile} & P,\ Q \text{ bisim.} \\
& \sqsupseteq & P_s ; P_s^{\smile} \sqcap Q_s ; Q_s^{\smile} & f^{\mathcal{B}},\ f^{\mathcal{C}} \text{ total} \\
& = & \mathbb{I}_{s^{\mathcal{A}}} & \text{tabulation of } V_s \\
c^{\mathcal{A}} ; (c^{\mathcal{A}})^{\smile} & = & c^{\mathcal{A}} ; (P_t ; (c^{\mathcal{B}})^{\smile} \sqcap Q_t ; (c^{\mathcal{C}})^{\smile}) & \\
& = & c^{\mathcal{A}} ; P_t ; (c^{\mathcal{B}})^{\smile} \sqcap c^{\mathcal{A}} ; Q_t ; (c^{\mathcal{C}})^{\smile} & c^{\mathcal{A}} \text{ univalent} \\
& = & c^{\mathcal{B}} ; (c^{\mathcal{B}})^{\smile} \sqcap c^{\mathcal{C}} ; (c^{\mathcal{C}})^{\smile} & P,\ Q \text{ bisim.} \\
& \sqsupseteq & \mathbb{I}_{\mathbb{1}} & c^{\mathcal{B}},\ c^{\mathcal{C}} \text{ total}
\end{array}$$

□

Theorem 3.1.8 If $\Sigma = (\mathcal{S}, \mathcal{F}, \mathsf{src}, \mathsf{trg})$ is a semi-unary signature and $\mathbf{C}$ is a collagory, and $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation in $\mathbf{C}^{\mathcal{S}}$ of the Σ-bisimulation $W : \mathcal{B} \to \mathcal{C}$, i.e., for each sort $s : \mathcal{S}$, $\mathcal{B} \xrightarrow{R_s} \mathcal{D} \xleftarrow{S_s} \mathcal{C}$ is a tabulation of $W_s : s^{\mathcal{B}} \to s^{\mathcal{C}}$, then we define for each function symbol $f : s \to t$ and each constant symbol $c : \mathbb{1} \to t$ in Σ:

$$\begin{aligned} f^{\mathcal{D}} &:= R_s^{\smile} ; f^{\mathcal{B}} ; R_t \sqcup S_s^{\smile} ; f^{\mathcal{C}} ; S_t \\ c^{\mathcal{D}} &:= c^{\mathcal{B}} ; R_t \sqcup c^{\mathcal{C}} ; S_t \end{aligned}$$

Then $\mathcal{D}$ turns into a Σ-algebra and R and S are Σ-bisimulations, too, so $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation in $\mathbf{C}^{\Sigma}$.

PROOF: We first show the bisimulation conditions for R; those for S follow analogously:

$$\begin{array}{rcll} R_s ; f^{\mathcal{D}} & = & R_s ; (R_s^{\smile} ; f^{\mathcal{B}} ; R_t \sqcup S_s^{\smile} ; f^{\mathcal{C}} ; S_t) & \text{Def. } f^{\mathcal{A}} \\ & = & R_s ; R_s^{\smile} ; f^{\mathcal{B}} ; R_t \sqcup R_s ; S_s^{\smile} ; f^{\mathcal{C}} ; S_t & \text{join distr.} \\ & = & (\mathbb{I}_{s^{\mathcal{B}}} \sqcup W_s ; W_s^{\smile}) ; f^{\mathcal{B}} ; R_t \sqcup W_s ; f^{\mathcal{C}} ; S_t & \text{co-tabulation of } W_s \\ & = & f^{\mathcal{B}} ; R_t \sqcup W_s ; W_s^{\smile}) ; f^{\mathcal{B}} ; R_t \sqcup W_s ; f^{\mathcal{C}} ; S_t & \text{join distr.} \\ & \sqsubseteq & f^{\mathcal{B}} ; R_t \sqcup W_s ; f^{\mathcal{C}} ; W_t^{\smile} ; R_t \sqcup W_s ; f^{\mathcal{C}} ; S_t & W^{\smile} \text{ bisimulation} \\ & \sqsubseteq & f^{\mathcal{B}} ; R_t \sqcup W_s ; f^{\mathcal{C}} ; S_t & \text{co-tabulation of } W_t \\ & \sqsubseteq & f^{\mathcal{B}} ; R_t \sqcup f^{\mathcal{B}} ; W_t ; S_t & W \text{ bisimulation} \\ & \sqsubseteq & f^{\mathcal{B}} ; R_t & \text{co-tabulation of } W_t \\ c^{\mathcal{D}} & = & c^{\mathcal{B}} ; R_t \sqcup c^{\mathcal{C}} ; S_t & \text{Def. } c^{\mathcal{D}} \\ & = & c^{\mathcal{B}} ; R_t \sqcup c^{\mathcal{B}} ; W_t^{\smile} ; S_t & W^{\smile} \text{ bisimulation} \\ & = & c^{\mathcal{B}} ; R_t & \text{co-tabulation of } W_t \end{array}$$

Totality and univalence of $c^{\mathcal{D}}$ follows immediately from $c^{\mathcal{D}} = c^{\mathcal{B}} ; R_t$ shown above; for $f^{\mathcal{D}}$, we easily obtain totality:

$$\begin{array}{rcll} f^{\mathcal{D}} ; (f^{\mathcal{D}})^{\smile} & = & (R_s^{\smile} ; f^{\mathcal{B}} ; R_t \sqcup S_s^{\smile} ; f^{\mathcal{C}} ; S_t) & \\ & & ; (R_t^{\smile} ; (f^{\mathcal{B}})^{\smile} ; R_s \sqcup S_t^{\smile} ; (f^{\mathcal{C}})^{\smile} ; S_s) & \\ & \sqsupseteq & R_s^{\smile} ; f^{\mathcal{B}} ; R_t ; R_t^{\smile} ; (f^{\mathcal{B}})^{\smile} ; R_s & \\ & & ; S_s^{\smile} ; f^{\mathcal{C}} ; S_t ; S_t^{\smile} ; (f^{\mathcal{C}})^{\smile} ; S_s & \\ & \sqsupseteq & R_s^{\smile} ; f^{\mathcal{B}} ; (f^{\mathcal{B}})^{\smile} ; R_s ; S_s^{\smile} ; f^{\mathcal{C}} ; (f^{\mathcal{C}})^{\smile} ; S_s & R_t,\ S_t \text{ total} \\ & \sqsupseteq & R_s^{\smile} ; R_s ; S_s^{\smile} ; S_s & f^{\mathcal{B}},\ f^{\mathcal{C}} \text{ total} \\ & = & \mathbb{I}_{s^{\mathcal{D}}} & \text{co-tabulation of } W_s \end{array}$$

Univalence of $f^{\mathcal{D}}$:

$$\begin{array}{rcll} (f^{\mathcal{D}})^{\smile} ; f^{\mathcal{D}} & = & (R_t^{\smile} ; (f^{\mathcal{B}})^{\smile} ; R_s \sqcup S_t^{\smile} ; (f^{\mathcal{C}})^{\smile} ; S_s) ; f^{\mathcal{D}} & \\ & = & R_t^{\smile} ; (f^{\mathcal{B}})^{\smile} ; R_s ; f^{\mathcal{D}} \sqcup S_t^{\smile} ; (f^{\mathcal{C}})^{\smile} ; S_s ; f^{\mathcal{D}} & \text{join distr.} \\ & \sqsubseteq & R_t^{\smile} ; (f^{\mathcal{B}})^{\smile} ; f^{\mathcal{B}} ; R_t \sqcup S_t^{\smile} ; (f^{\mathcal{C}})^{\smile} ; f^{\mathcal{C}} ; S_t & R,\ S \text{ bisimul.} \\ & \sqsubseteq & R_t^{\smile} ; R_t \sqcup S_t^{\smile} ; ; S_t & f^{\mathcal{B}},\ f^{\mathcal{C}} \text{ univalent} \\ & = & \mathbb{I}_{t^{\mathcal{D}}} & \text{co-tabulation of } W_t \end{array}$$

□

3.2 Reducts Along Signature Homomorphisms

While the concept of Σ-algebra is sufficient to capture, for example, unlabelled graphs as $\mathsf{sigGraph}$-algebras, categories of labelled graphs are frequently considered as having *fixed* label sets, which means that only certain sub-categories of $Set^{\mathsf{sigLGraph}}$ are considered.

We use the concept of *reducts* to formalise this in a general way. In the example, we consider the reduct of $Set^{\mathsf{sigLGraph}}$ to the sub-signature $\mathsf{sigNELabels}$. The fixed label sets under consideration form a one-object sub-category $\mathbf{K}$ of $Set^{\mathsf{sigNELabels}}$, and in order to obtain graphs labelled over these label sets, we restrict attention to objects in $Set^{\mathsf{sigLGraph}}$ for which the reduct lies in that sub-category $\mathbf{K}$.

The current section introduces and studies the reduct relator. This is employed in Sect. 3.3 to implement the restriction of Σ-algebra collagories via reduct-side sub-categories. This single construction principle for generating concrete bi-tabular collagories corresponds, as shown in Corollary 3.3.7, to several categorical constructions that are known for adhesive categories.

Definition 3.2.1 Let $\Sigma = (\mathcal{S}, \mathcal{F}, \mathsf{src}, \mathsf{trg})$ and $\Sigma_{\mathrm{R}} = (\mathcal{S}_{\mathrm{R}}, \mathcal{F}_{\mathrm{R}}, \mathsf{src}_{\mathrm{R}}, \mathsf{trg}_{\mathrm{R}})$ be two signatures, and let $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$ be a signature homomorphism.

For any Σ-algebra $\mathcal{A}$, such a signature homomorphism $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$ induces a Σ_{R}-algebra $\mathcal{A}{\downharpoonleft}\sigma$, the σ*-reduct of* $\mathcal{A}$, in the following way:

- For every sort $r : \mathcal{S}_{\mathrm{R}}$, its carrier is $r^{\mathcal{A}\downharpoonleft\sigma} = (\sigma\ r)^{\mathcal{A}}$;
- for every function symbol $f \in \mathcal{F}_{\mathrm{R}}$, its interpretation is $f^{\mathcal{A}\downharpoonleft\sigma} = (\sigma\ f)^{\mathcal{A}}$. □

It is easy to verify that $\mathcal{A}{\downharpoonleft}\sigma$ is indeed a Σ_{R}-algebra.

If $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$ is a sub-signature embedding, then we also call $\mathcal{A}{\downharpoonleft}\sigma$ the Σ_{R}-reduct of $\mathcal{A}$ and write also $\mathcal{A}{\downharpoonleft}\Sigma_{\mathrm{R}}$.

Since our signatures are a special case of sketches [BW99, Chapters 4,7,8,10], ${\downharpoonleft}\sigma$ is a special case of what Barr and Wells call "model category functor". We complete the definition and show that is is a relator:

Definition 3.2.2 For a signature homomorphism $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$, the σ-reduct of a $\mathbf{C}^{\mathcal{S}}$-morphism $\Phi = (\Phi_s)_{s:\mathcal{S}}$ is the $\mathbf{C}^{\mathcal{S}_{\mathrm{R}}}$-morphism $\Phi{\downharpoonleft}\sigma = ((\Phi{\downharpoonleft}\sigma)_r)_{r:\mathcal{S}_{\mathrm{R}}}$ with $(\Phi{\downharpoonleft}\sigma)_r := \Phi_{\sigma\ r}$ for every $r : \mathcal{S}_{\mathrm{R}}$. □

Proposition 3.2.3 For a signature homomorphism $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$, the σ-reduct of a Σ-bisimulation is a Σ_{R}-bisimulation.

Furthermore, the reduct operation ${\downharpoonleft}\sigma$ is an allegory relator from $\mathbf{C}^{\Sigma}$ to $\mathbf{C}^{\Sigma_{\mathrm{R}}}$ and therefore also a functor from $\mathsf{Map}\,(\mathbf{C}^{\Sigma})$ to $\mathsf{Map}\,(\mathbf{C}^{\Sigma_{\mathrm{R}}})$.

Proof: Bisimulation property: For any n-ary function symbol (we do not need the restriction to semi-unary signatures here) $f : r_1 \times \cdots \times r_n \to q$ in $\mathcal{F}_{\mathrm{R}}$:

$$\begin{array}{ll} & ((\Phi{\downharpoonleft}\sigma)_{r_1} \times \cdots \times (\Phi{\downharpoonleft}\sigma)_{r_n}) \,;\, f^{\mathcal{B}\downharpoonleft\sigma} \\ = & (\Phi_{\sigma\, r_1} \quad \times \cdots \times \Phi_{\sigma\, r_n} \quad) \,;\, (\sigma\, f)^{\mathcal{B}} \ \sqsubseteq\ (\sigma\, f)^{\mathcal{A}} \,;\, \Phi_{\sigma\, q} \ =\ f^{\mathcal{A}\downharpoonleft\sigma} \,;\, (\Phi{\downharpoonleft}\sigma)_q \end{array}$$

Preservation of identities:

$$\mathbb{I}_{\mathcal{A}}{\downharpoonleft}\sigma = ((\mathbb{I}_{s^{\mathcal{A}}})_{s\in\mathcal{S}}){\downharpoonleft}\sigma = ((\mathbb{I}_{(\sigma\, r)^{\mathcal{A}}})_{r\in\mathcal{S}_{\mathrm{R}}} = ((\mathbb{I}_{r^{\mathcal{A}\downharpoonleft\sigma}})_{r\in\mathcal{S}_{\mathrm{R}}} = \mathbb{I}_{\mathcal{A}\downharpoonleft\sigma}$$

Preservation of composition:

$$\begin{aligned}(\Phi ;^{\Sigma} \Psi)\lfloor\sigma &= ((\Phi ; \Psi)_{\sigma\, r})_{r\in\mathcal{S}_{\mathrm{R}}} = (\Phi_{\sigma\, r} ; \Psi_{\sigma\, r})_{r\in\mathcal{S}_{\mathrm{R}}} \\ &= (\Phi_{\sigma\, r})_{r\in\mathcal{S}_{\mathrm{R}}} ; (\Psi_{\sigma\, r})_{r\in\mathcal{S}_{\mathrm{R}}} = (\Phi\lfloor\sigma) ;^{\Sigma_{\mathrm{R}}} (\Psi\lfloor\sigma)\end{aligned}$$

Preservation of converse:

$$\Phi^{\smile}\lfloor\sigma = ((\Phi^{\smile})_{\sigma\, r})_{r\in\mathcal{S}_{\mathrm{R}}} = ((\Phi_{\sigma\, r})_{r\in\mathcal{S}_{\mathrm{R}}})^{\smile} = (\Phi\lfloor\sigma)^{\smile}$$

Preservation of meet:

$$\begin{aligned}(\Phi \sqcap^{\Sigma} \Psi)\lfloor\sigma &= ((\Phi \sqcap \Psi)_{\sigma\, r})_{r\in\mathcal{S}_{\mathrm{R}}} = (\Phi_{\sigma\, r} \sqcap \Psi_{\sigma\, r})_{r\in\mathcal{S}_{\mathrm{R}}} \\ &= (\Phi_{\sigma\, r})_{r\in\mathcal{S}_{\mathrm{R}}} \sqcap (\Psi_{\sigma\, r})_{r\in\mathcal{S}_{\mathrm{R}}} = (\Phi\lfloor\sigma) \sqcap^{\Sigma_{\mathrm{R}}} (\Psi\lfloor\sigma)\end{aligned}$$ □

Joins that are defined component-wise are preserved in the same way.

Obviously, the reduct relator is in general not full if σ is not injective on sorts.

If σ is injective, we can "replace in $\mathcal{A}$ its reduct part along a morphism to $\mathcal{A}\lfloor\sigma$", which will be useful in the next section:

Theorem 3.2.4 If $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$ is an injective signature homomorphism, then the reduct functor $\lfloor\sigma$ is a fibration [BW99, 12.1].

Proof: If $\mathcal{A}$ is an object in $\mathbf{C}^{\Sigma}$, $\mathcal{R}$ is an object in $\mathbf{C}^{\Sigma_{\mathrm{R}}}$, and $\phi : \mathcal{R} \to \mathcal{A}\lfloor\sigma$ is a morphism in $\mathbf{C}^{\Sigma_{\mathrm{R}}}$, then we construct an object $\mathcal{B}$ in $\mathbf{C}^{\Sigma}$ and a morphism $\psi : \mathcal{B} \to \mathcal{A}$ as follows:

- For every $s : \mathcal{S}$ outside the range of σ, we let $s^{\mathcal{B}} := s^{\mathcal{A}}$ and $\psi_s := \mathbb{I}_{s^{\mathcal{A}}}$.
- For every $r : \mathcal{S}_{\mathrm{R}}$, we let $(\sigma\, r)^{\mathcal{B}} := r^{\mathcal{R}}$ and $\psi_{\sigma\, r} := \phi_r$.
- For every $f : \mathcal{F}$ outside the range of σ, we let $f^{\mathcal{B}} := \psi_{\mathsf{src} f} ; f^{\mathcal{A}} ; \psi^{\smile}_{\mathsf{trg} f}$.
- For every $g : \mathcal{F}_{\mathrm{R}}$, we let $(\sigma\, g)^{\mathcal{B}} := \phi^{\smile}_{\mathsf{src} g} ; g^{\mathcal{R}} ; \phi_{\mathsf{trg} g}$.

Well-definedness is easily verified. We can now show that ψ is *cartesian for ϕ and $\mathcal{A}$*:

If $v : \mathcal{Z} \to \mathcal{A}$ in $\mathbf{C}^{\Sigma}$ and $h : \mathcal{Z}\lfloor\sigma \to \mathcal{R}$ such that $h ; \phi = v\lfloor\sigma$, then $w : \mathcal{Z} \to \mathcal{B}$ defined by

- for every $s : \mathcal{S}$ outside the range of σ, let $w_s := v_s$,
- for every $r : \mathcal{S}_{\mathrm{R}}$, let $w_{\sigma\, r} := h_r$,

obviously satisfies $w ; \psi = v$ and $w\lfloor\sigma = h$, and obviously is the unique such arrow. □

3.3 Reduct-Restricted Σ-Algebra Categories

In the following, let $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$ be an arbitrary but fixed signature homomorphism, and $\mathbf{K}$ a sub-category of $\mathbf{C}^{\Sigma_{\mathrm{R}}}$. We will further assume that $\mathbf{K}$ is contained in the image of $\lfloor\sigma$ — this restriction is not essential, but frequently allows more concise formulations.

Definition 3.3.1 The $\sigma, \mathbf{K}$-restriction of $\mathbf{C}^{\Sigma}$ contains exactly those objects and morphisms for which the image under $\lfloor\sigma$ is in $\mathbf{K}$.

We denote this restriction as $\mathbf{C}^{\sigma|_{\mathbf{K}}}$. □

Because relators preserve identities and composition, and $\mathbf{K}$ is a category, the restriction $\mathbf{C}^{\sigma|_{\mathbf{K}}}$ is a category again.

The technical importance of the assumption on $\mathbf{K}$ is that it provides surjectivity on homsets for the reduct relator:

Proposition 3.3.2 If $\mathbf{K}$ is contained in the image of $\lfloor\sigma$, then the restriction of $\lfloor\sigma$ to $\mathbf{C}^{\sigma|_{\mathbf{K}}}$ is a full relator. □

If σ is a sub-signature embedding, we also write $\mathbf{C}^{\Sigma|_{\mathbf{K}}}$ instead of $\mathbf{C}^{\sigma|_{\mathbf{K}}}$. If, in addition, the restriction category $\mathbf{K}$ contains only one object $\mathcal{L}$ and its identity, we also write $\mathbf{C}^{\Sigma|_{\mathcal{L}}}$.

This latter case covers in particular the situation where Σ_{R} contains only label sorts and $\mathcal{L}$ fixes the label interpretations, producing for example a category of labelled graphs with fixed label sets.

Note that every one-object-one-morphism category has all limits and colimits and is not only an allegory, but even a (trivial) relation algebra, and also a bi-tabular collagory. This therefore provides an important special case for many of the properties in the remainder of this paper.

Proposition 3.3.3 If $\mathbf{K}$ is a sub-allegory of $\mathbf{C}^{\Sigma_{\mathrm{R}}}$, then $\mathbf{C}^{\sigma|_{\mathbf{K}}}$ is an allegory.

Proof: Assume that $\Phi\lfloor\sigma$ and $\Psi\lfloor\sigma$ are in $\mathbf{K}$. Since $\mathbf{K}$ is closed under converse and meets, $\Phi^{\smile}\lfloor\sigma = (\Phi\lfloor\sigma)^{\smile}$ and $(\Phi \sqcap \Psi)\lfloor\sigma = (\Phi\lfloor\sigma) \sqcap (\Psi\lfloor\sigma)$ are in $\mathbf{K}$, too.

Therefore, $\mathbf{C}^{\sigma|_{\mathbf{K}}}$ is closed under converse and meets, too, and therefore is a sub-allegory of $\mathbf{C}^{\Sigma}$. □

Proposition 3.3.4 For semi-unary Σ, if $\mathbf{K}$ is a sub-collagory of $\mathbf{C}^{\Sigma_{\mathrm{R}}}$, then $\mathbf{C}^{\sigma|_{\mathbf{K}}}$ is a collagory.

Proof: Assume that $\Phi\lfloor\sigma$ and $\Psi\lfloor\sigma$ are in $\mathbf{K}$. With Lemma 3.1.5 and since $\mathbf{K}$ is closed under joins, the join $(\Phi \sqcup^{\Sigma} \Psi)\lfloor\sigma = (\Phi\lfloor\sigma) \sqcup^{\Sigma_{\mathrm{R}}} (\Psi\lfloor\sigma)$ is in $\mathbf{K}$, too.

So $\mathbf{C}^{\sigma|_{\mathbf{K}}}$ is closed under joins, too, and therefore is a sub-collagory of $\mathbf{C}^{\Sigma}$. □

This join preservation works in particular in the case where $\mathbf{K}$ is a one-object-one-morphism category, since in that case, non-empty joins in $\mathbf{K}$ are still inherited (trivially) from $\mathbf{C}^{\Sigma_{\mathrm{R}}}$.

Empty joins, i.e., least morphisms, however, are generally *not* inherited in the one-object-one-morphism category, since identity morphisms are rarely least morphisms in $\mathbf{C}^{\Sigma_{\mathrm{R}}}$. Therefore the zero law does in general not hold in $\mathbf{C}^{\sigma|_{\mathbf{K}}}$. A simple example for this arises in $Set^{\mathsf{sigPointed}|_{\{\bullet\}}}$, i.e., the allegory of relational homomorphisms between pointed sets: The presence of the point induces exactly the same counterexamples as the presence of a zero-ary function symbol, for example if $\mathsf{O}^{\mathcal{A}} = \{0, 1\}$, and the point (respectively the value of the constant) in $\mathcal{A}$ is 1, then $\perp\!\!\!\perp_{\mathsf{O}^{\mathcal{A}},\mathsf{O}^{\mathcal{A}}} = \{(1, 1)\}$ is a non-trivial closure of the non-inherited least element of $\mathbf{K}$, and with $R := \{(0, 1), (1, 1)\}$ we have $R \mathbin{;} \perp\!\!\!\perp = R \neq \perp\!\!\!\perp$.

Since the reduct relator $\lfloor\sigma$ distributes over all relevant operations, it also preserves tabulations and co-tabulations, i.e.:

- If the span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is a tabulation for the morphism $V : \mathcal{B} \to \mathcal{C}$ in $\mathbf{C}^{\Sigma}$, then the span $\mathcal{B}\lfloor\sigma \xleftarrow{P\lfloor\sigma} \mathcal{A}\lfloor\sigma \xrightarrow{Q\lfloor\sigma} \mathcal{C}\lfloor\sigma$ is a tabulation for $(V\lfloor\sigma) : \mathcal{B}\lfloor\sigma \to \mathcal{C}\lfloor\sigma$ in $\mathbf{C}^{\Sigma_{\mathrm{R}}}$.

- If the co-span $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation for the difunctional morphism $W : \mathcal{B} \to \mathcal{C}$ in $\mathbf{C}^{\Sigma}$, then the co-span $\mathcal{B}\lfloor\sigma \xrightarrow{R\lfloor\sigma} \mathcal{D}\lfloor\sigma \xleftarrow{S\lfloor\sigma} \mathcal{C}\lfloor\sigma$ is a co-tabulation for $(W\lfloor\sigma) : \mathcal{B}\lfloor\sigma \to \mathcal{C}\lfloor\sigma$ in $\mathbf{C}^{\Sigma_{\mathrm{R}}}$.

Theorem 3.3.5 For semi-unary Σ, if $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$ is injective, $\mathbf{K}$ is a sub-collagory of $\mathbf{C}^{\Sigma_{\mathrm{R}}}$, the morphism $V : \mathcal{B} \to \mathcal{C}$ has a tabulation $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ in $\mathbf{C}^{\Sigma}$, and $V\lfloor\sigma$ has a tabulation $\mathcal{B}\lfloor\sigma \xleftarrow{P_0} \mathcal{A}_0 \xrightarrow{Q_0} \mathcal{C}\lfloor\sigma$ in $\mathbf{K}$, then V also has a tabulation in $\mathbf{C}^{\sigma|_{\mathbf{K}}}$.

PROOF: Since tabulations in $\mathbf{C}^{\Sigma_{\mathrm{R}}}$ are unique up to isomorphism, there must be an isomorphism $\phi : \mathcal{A}_0 \to \mathcal{A}\lfloor\sigma$. According to Theorem 3.2.4, we obtain a cartesian morphism $\psi : \mathcal{A}_1 \to \mathcal{A}$ for ϕ and $\mathcal{A}$, and since this is also an isomorphism, $\mathcal{B} \xleftarrow{\psi;P} \mathcal{A}_1 \xrightarrow{\psi;Q} \mathcal{C}$ is a tabulation for V in $\mathbf{C}^{\sigma|_{\mathbf{K}}}$. □

The corresponding statement for co-tabulations is shown in the same way, so we obtain as result:

Theorem 3.3.6 For semi-unary Σ and an injective signature homomorphism $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$, if $\mathbf{C}$ is a bi-tabular collagory and if $\mathbf{K}$ is a bi-tabular sub-collagory of $\mathbf{C}^{\Sigma_{\mathrm{R}}}$, then $\mathbf{C}^{\sigma|_{\mathbf{K}}}$ is a bi-tabular collagory, too. □

This includes all the systematically constructed examples for adhesive categories provided by Lack and Sobociński [LS04], in particular the following uses of a one-object-one-morphism collagory $\mathbf{K}$:

Corollary 3.3.7 If $\mathbf{C}$ is a bi-tabular collagory, then the following are bi-tabular collagories, too:

- $\mathbf{C}^{\mathsf{sigPointed}|_{\mathcal{C}}}$ for any object $\mathcal{C}$ (conflating $\mathcal{C}$ in $\mathbf{C}$ with the $\mathsf{sigPoint}$-algebra that assigns $\mathcal{C}$ to the sort P) — this is equivalent to the co-slice category $\mathcal{C}/\mathbf{C}$,
- $\mathbf{C}^{\mathsf{sigTyped}|_{\mathcal{C}}}$ for any object — this is equivalent to the slice category $\mathbf{C}/\mathcal{C}$,
- node- and edge-labelled graphs considered as $\mathsf{sigLGraph}$-algebras with fixed node and edge label sets. □

Chapter 4

Co-Tabulations as Bicolimits and Lax Colimits

4.1 OC-Colimits: Bicolimits in Ordered Categories

Ordered categories are a simple example of 2-categories and bicategories: between two morphisms there is at most one two-cell, and there is a two-cell between two morphisms $R, S : \mathcal{A} \to \mathcal{B}$ iff $R \sqsubseteq S$.

Therefore, there is an invertible two-cell between R and S if and only if $R = S$.

The general notion of bicolimits takes as its point of departure a *diagram* defined via a functor from a category. We introduce a specialised variant of the definition used in [HS09] by restricting our attention to ordered categories.

Definition 4.1.1 Given a category $\mathbf{C}$, an (index) category $\mathbf{J}$, a functor $\mathbf{D} : \mathbf{J} \to \mathbf{C}$ defining a diagram, and an object $\mathcal{D}$, a *cocone* η from $\mathbf{D}$ to $\mathcal{D}$ consists of a morphism $\eta_{\mathcal{A}} : \mathbf{D}\,\mathcal{A} \to \mathcal{D}$ in $\mathbf{C}$ for each object $\mathcal{A}$ of $\mathbf{J}$, satisfying the following *cocone commutativity* condition:

$$\mathbf{D}\,F \mathbin{;} \eta_{\mathcal{B}} = \eta_{\mathcal{A}} \qquad \text{for each morphism } F : \mathcal{A} \to \mathcal{B} \text{ in } \mathbf{J}. \qquad \square$$

Definition 4.1.2 Given an ordered category $\mathbf{C}$, an (index) category $\mathbf{J}$, and a functor $\mathbf{D} : \mathbf{J} \to \mathbf{C}$, an *OC-colimit of* $\mathbf{D}$ is given by

- an object $\mathcal{D}$ of $\mathbf{C}$, and
- a cocone η from $\mathbf{D}$ to $\mathcal{D}$

satisfying the following conditions

1. *factorisation:* for any other object $\mathcal{D}'$ of $\mathbf{C}$ with cocone κ from $\mathbf{D}$ to $\mathcal{D}'$, there is a morphism $h : \mathcal{D} \to \mathcal{D}'$ in $\mathbf{C}$ with

$$\eta_{\mathcal{A}} \mathbin{;} h = \kappa_{\mathcal{A}} \qquad \text{for each object } \mathcal{A} \text{ in } \mathbf{J}.$$

2. *isotony:* for any other object $\mathcal{D}'$ of $\mathbf{C}$ and any two morphisms $h, h' : \mathcal{D} \to \mathcal{D}'$, if $\eta_{\mathcal{A}} \mathbin{;} h \sqsubseteq \eta_{\mathcal{A}} \mathbin{;} h'$ for all objects $\mathcal{A}$ in $\mathbf{J}$, then $h \sqsubseteq h'$. $\square$

Lemma 4.1.3 If the cocones η from $\mathbf{D}$ to $\mathcal{D}$ and κ from $\mathbf{D}$ to $\mathcal{D}'$ are both OC-colimits of $\mathbf{D}$, then there is an isomorphism $U : \mathcal{D} \to \mathcal{D}'$ such that

$$\eta_{\mathcal{A}} ; U = \kappa_{\mathcal{A}} \qquad \text{for each object } \mathcal{A} \text{ in } \mathbf{J}.$$

PROOF: Because of the factorisation property (in Def. 4.1.2) of η, we only need to show that U is an isomorphism.

From the factorisation property of κ, we obtain a morphism $V : \mathcal{D}' \to \mathcal{D}$ with, for each object $\mathcal{A}$ in $\mathbf{J}$:

$$\eta_{\mathcal{A}} = \kappa_{\mathcal{A}} ; V = \eta_{\mathcal{A}} ; U ; V$$

Using isotony of η with $h = \mathbb{I}_{\mathcal{D}}$ and $h' = U ; V$, we obtain $\mathbb{I} \sqsubseteq U ; V$; by swapping h and h' we obtain the converse inclusion, and therefore equality $U ; V = \mathbb{I}_{\mathcal{D}}$. In the same way, we also obtain $V ; U = \mathbb{I}_{\mathcal{D}'}$, so U must be an isomorphism (and V its inverse). □

Note that in an allegory, R is an isomorphism iff R is a bijective mapping [FS90, 2.135].

It is instructive to investigate OC-colimits for one-object one-morphism diagrams:

Lemma 4.1.4 Let $\mathbf{J} = \bullet$, and let the diagram functor $\mathbf{D}$ be defined by $\mathbf{D}\bullet = \mathcal{A}$. Then $R : \mathcal{A} \to \mathcal{D}$ is an OC-colimit for $\mathbf{D}$ iff R is an isomorphism.

PROOF: Any $R : \mathcal{A} \to \mathcal{D}$ is trivially a cocone from $\mathbf{D}$ to $\mathcal{D}$, and we have:

1. For any other object $\mathcal{D}'$ of $\mathbf{C}$ with $R' : \mathcal{A} \to \mathcal{D}'$, we can choose $h := S ; R'$ to obtain factorisiation if S is a right-inverse of R, that is, $R ; S = \mathbb{I}_{\mathcal{A}}$:

$$R ; h = R ; S ; R' = \mathbb{I}_{\mathcal{A}} ; R' = R'$$

2. For any other object $\mathcal{D}'$ of $\mathbf{C}$ and any two morphisms $h, h' : \mathcal{D} \to \mathcal{D}'$, if $R ; h \sqsubseteq R ; h'$, then we obtain isotony if Q is a left-inverse of R, that is, $Q ; R = \mathbb{I}_{\mathcal{D}}$:

$$h = \mathbb{I}_{\mathcal{D}} ; h = Q ; R ; h \sqsubseteq Q ; R ; h' = \mathbb{I}_{\mathcal{D}} ; h = h'$$

This shows that if R is iso, then it is an OC-colimit for $\mathbf{D}$.

Therefore, $\mathbb{I}_{\mathcal{A}}$ is an OC-colimit for $\mathbf{D}$.

Since OC-colimits are unique up to isomorphism by Lemma 4.1.3, for any given OC-colimit R' for $\mathbf{D}$ there is an isomorphism U factoring it over $\mathbb{I}_{\mathcal{A}}$ and we have $R' = \mathbb{I}_{\mathcal{A}} ; U = U$. □

4.2 Co-Tabulations Produce OC-Pushouts

A co-tabulation for a difunctional closure $Z^{\circledast}$ satisfies the following equations:

$$R ; S^{\smile} = Z^{\circledast} \qquad R ; R^{\smile} = Z^{\triangleright} \qquad S ; S^{\smile} = Z^{\triangleleft} \qquad R^{\smile} ; R \sqcup S^{\smile} ; S = \mathbb{I}_{\mathcal{D}} \ .$$

This was introduced as a *gluing for Z* in [Kah01]. Kawahara is the first to have characterised pushouts relation-algebraically in essentially this way [Kaw90]; he used relation-algebraic operations on relations arising in toposes.

Lemma 4.2.1 Let $\mathbf{C}$ be a collagory, and let $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ be a span in $\mathsf{Map}\,\mathbf{C}$, that is, P and Q are mappings.

If the cospan $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ in the collagory $\mathbf{C}$ is a co-tabulation for $W := (P^{\smile} ; Q)^{\boxast}$, then it is a cocone for $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ in $\mathsf{Map}\,\mathbf{C}$.

Proof: The co-tabulation properties imply that R and S are mappings. For commutativity, we first show one inclusion:

$$P ; R \sqsupseteq P ; R ; \mathrm{ran}\, S = P ; R ; S^{\smile} ; S = P ; (P^{\smile} ; Q)^{\boxast} ; S \sqsupseteq P ; P^{\smile} ; Q ; S \sqsupseteq Q ; S$$

The opposite inclusion is derived in the same way, so we have equality. $\square$

Lemma 4.2.2 Let $\mathbf{C}$ be a collagory, and let $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ be a span in $\mathsf{Map}\,\mathbf{C}$, that is, P and Q are mappings.

If the cospan $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ in the collagory $\mathbf{C}$ is a co-tabulation for $W := (P^{\smile} ; Q)^{\boxast}$, and $\mathcal{B} \xrightarrow{R'} \mathcal{D}' \xleftarrow{S'} \mathcal{C}$ is a cospan in $\mathbf{C}$ with $P ; R' = Q ; S'$, then $U : \mathcal{D} \to \mathcal{D}'$ with $U := R^{\smile} ; R' \sqcup S^{\smile} ; S'$ satisfies the factorisation property, i.e., $R ; U = R'$ and $S ; U = S'$, and U is a mapping if both R' and S' are.

Proof: Commutativity $P ; R' = Q ; S'$ together with univalence of P and Q implies

$$P^{\smile} ; Q ; S' = P^{\smile} ; P ; R' \sqsubseteq R' \qquad \text{and} \qquad Q^{\smile} ; P ; R' = Q^{\smile} ; Q ; S' \sqsubseteq S' \ .$$

Using left induction for difunctional closure, this gives us:

$$W ; S' = (P^{\smile} ; Q)^{\boxast} ; S' \sqsubseteq R' \qquad \text{and} \qquad W^{\smile} ; R' = (Q^{\smile} ; P)^{\boxast} ; R' \sqsubseteq S' \ .$$

With Lemma 2.3.4 we then know that $U := R^{\smile} ; R' \sqcup S^{\smile} ; S'$ is a morphism with $R ; U = R'$ and $S ; U = S'$, and a mapping if R' and S' are mappings. $\square$

$$\begin{tikzcd} \mathcal{C} \arrow[dr, "S"] & \\ & \mathcal{D} \\ \mathcal{B} \arrow[uu, "W"] \arrow[ur, "R"'] & \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{C} \arrow[dr, "S"] & \\ \mathcal{A} \arrow[ur, "Q"] \arrow[dr, "P"'] & & \mathcal{D} \\ & \mathcal{B} \arrow[uu, "W"] \arrow[ur, "R"'] & \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{C} \arrow[dr, "S"] \arrow[drr, "S'"] & & \\ \mathcal{A} \arrow[ur, "Q"] \arrow[dr, "P"'] & & \mathcal{D} \arrow[r, dotted, "U"] & \mathcal{D}' \\ & \mathcal{B} \arrow[uu, "W"] \arrow[ur, "R"] \arrow[urr, "R'"'] & & \end{tikzcd}$$

Theorem 4.2.3 Let $\mathbf{C}$ be a collagory, and let $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ be a span in $\mathsf{Map}\,\mathbf{C}$, that is, P and Q are mappings.

If the cospan $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ in $\mathbf{C}$ is a co-tabulation for $W := (P^{\smile} ; Q)^{\boxast}$, then that cospan is an OC-pushout for $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$.

Proof: Factorisation has been shown in Lemma 4.2.2, and isotony follows from Lemma 2.3.6. $\square$

The path to obtain this was in fact only a slight adaptation of the following fact which was shown directly in [Kah09b, Thm. 4.9] by adapting the argument of [Kah01, Thm. 5.3.5], and which now easily follows:

Corollary 4.2.4 Let $\mathbf{C}$ be a collagory, and let $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ be a span in $\mathsf{Map}\,\mathbf{C}$, that is, P and Q are mappings.

If the cospan $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ in the collagory $\mathbf{C}$ is a co-tabulation for $W := (P^{\smile} ; Q)^{\circledast}$, then it is a pushout for $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ in $\mathsf{Map}\,\mathbf{C}$. $\square$

For pushouts along injective mappings, the difunctional closure becomes trivial:

Lemma 4.2.5 If a span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ of mappings is given with Q injective, then $P^{\smile} ; Q$ is difunctional (and therefore $(P^{\smile} ; Q)^{\circledast} = P^{\smile} ; Q$).

Proof: Since P, as a mapping, is difunctional, we have

$$P^{\smile} ; Q ; Q^{\smile} ; P ; P^{\smile} ; Q = P^{\smile} ; P ; P^{\smile} ; Q = P^{\smile} ; Q \ . \qquad \square$$

Furthermore, co-tabulations preserve injectivity:

Lemma 4.2.6 If a span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ of mappings is given with Q injective, and $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation for $P^{\smile} ; Q$, then R is injective, too.

Proof: Using injectivity of Q and univalence of P in one of the equations from Prop. 2.3.3 gives us injectivity of R:

$$R ; R^{\smile} = \mathbb{I} \sqcup P^{\smile} ; Q ; (P^{\smile} ; Q)^{\smile} = \mathbb{I} \sqcup P^{\smile} ; Q ; Q^{\smile} ; P = \mathbb{I} \sqcup P^{\smile} ; P = \mathbb{I} \ . \qquad \square$$

With that, we can show that, essentially, a pushout over an injective mapping is also a pullback:

Lemma 4.2.7 If a span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ of mappings is given with Q injective, and $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation for $P^{\smile} ; Q$, then $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is also a tabulation for $R ; S^{\smile}$.

Proof: Cross-commutativity $R ; S^{\smile} = P^{\smile} ; Q$ is already contained in the co-tabulation conditions. Since Q is injective and P is total, we also obtain

$$P ; P^{\smile} \sqcap Q ; Q^{\smile} = P ; P^{\smile} \sqcap \mathbb{I}_{\mathcal{A}} = \mathbb{I}_{\mathcal{A}} \ . \qquad \square$$

4.3 Lax Colimits in OCCs

For lax cocones, we only need the concept of lax functor, which differs from the functor concept in that a *lax functor* $\mathbf{D}$ only needs to satisfy $\mathbb{I}_{\mathbf{D}\,\mathcal{A}} \sqsubseteq \mathbf{D}\,\mathbb{I}_{\mathcal{A}}$ and $(\mathbf{D}\,f) ; (\mathbf{D}\,g) \sqsubseteq \mathbf{D}(f ; g)$, see, e.g., [Stu05, Sect. 8, p. 37ff].

Again, we provide specialised definition of lax cocones and lax colimits for the ordered category case:

Definition 4.3.1 Given an *ordered* category $\mathbf{C}$, an (index) category $\mathbf{J}$, a lax functor $\mathbf{D} : \mathbf{J} \to \mathbf{C}$ defining a diagram, and an object $\mathcal{D}$, a *lax cocone* η from $\mathbf{D}$ to $\mathcal{D}$ consists of a morphism $\eta_{\mathcal{A}} : \mathbf{D}\,\mathcal{A} \to \mathcal{D}$ in $\mathbf{C}$ for each object $\mathcal{A}$ of $\mathbf{J}$, satisfying the following *cocone subcommutativity* condition:

$$\mathbf{D}\,F ; \eta_{\mathcal{B}} \sqsubseteq \eta_{\mathcal{A}} \qquad \text{for each morphism } F : \mathcal{A} \to \mathcal{B} \text{ in } \mathbf{J}. \qquad \square$$

Definition 4.3.2 Given an ordered category $\mathbf{C}$, an (index) category $\mathbf{J}$, and a lax functor $\mathbf{D} : \mathbf{J} \to \mathbf{C}$, a *lax colimit of* $\mathbf{D}$ is given by

- an object $\mathcal{D}$ of $\mathbf{C}$, and
- a lax cocone η from $\mathbf{D}$ to $\mathcal{D}$

satisfying the following conditions

1. *factorisation:* for any object $\mathcal{D}'$ of $\mathbf{C}$ with lax cocone κ from $\mathbf{D}$ to $\mathcal{D}'$, there is a morphism $U : \mathcal{D} \to \mathcal{D}'$ in $\mathbf{C}$ with
$$\eta_{\mathcal{A}} ; U = \kappa_{\mathcal{A}} \qquad \text{for each object } \mathcal{A} \text{ in } \mathbf{J},$$
2. *isotony:* for any object $\mathcal{D}'$ of $\mathbf{C}$ and any two morphisms $U, U' : \mathcal{D} \to \mathcal{D}'$, if $\eta_{\mathcal{A}} ; U \sqsubseteq \eta_{\mathcal{A}} ; U'$ for each object $\mathcal{A}$ in $\mathbf{J}$, then $U \sqsubseteq U'$. □

Proposition 4.3.3 If the lax cocones η from $\mathbf{D}$ to $\mathcal{D}$ and κ from $\mathbf{D}$ to $\mathcal{D}'$ both are lax colimits of $\mathbf{D}$, then they mutually factor over isomorphisms.

PROOF: Let $U : \mathcal{D} \to \mathcal{D}'$ and $V : \mathcal{D}' \to \mathcal{D}$ be the two factorisation morphisms, i.e., with
$$\eta_{\mathcal{A}} ; U = \kappa_{\mathcal{A}} \qquad \text{and} \qquad \kappa_{\mathcal{A}} ; V = \eta_{\mathcal{A}} \qquad \text{for each object } \mathcal{A} \text{ in } \mathbf{J}.$$
Then we also have, for each object $\mathcal{A}$ in $\mathbf{J}$,
$$\begin{array}{lllllll} \eta_{\mathcal{A}} ; U ; V & = & \kappa_{\mathcal{A}} ; V & = & \eta_{\mathcal{A}} & = & \eta_{\mathcal{A}} ; \mathbb{I}_{\mathcal{D}} \ , \\ \kappa_{\mathcal{A}} ; V ; U & = & \eta_{\mathcal{A}} ; U & = & \kappa_{\mathcal{A}} & = & \kappa_{\mathcal{A}} ; \mathbb{I}_{\mathcal{D}'} \ , \end{array}$$
and we obtain, from isotony, $U ; V = \mathbb{I}_{\mathcal{D}}$ and $V ; U = \mathbb{I}_{\mathcal{D}'}$, that is, U and V are inverse isomorphisms. □

Instead of considering lax colimits in general ordered categories, we consider a specialised variant for diagrams in OCCs. We want to make sure that for each morphism in the index category, its converse is in the index category too, and we currently see no harm in demanding not only that, but even that the index category is an OCC.

We consider "$\bullet \to \bullet$" to denote an OCC with the homset from the first object $\mathcal{A}$ to the second, different object $\mathcal{B}$ containing exactly one morphism, say F, from $\mathcal{A}$ to $\mathcal{B}$. As an OCC, it needs to also have $F^{\smile}$, which will be the only morphism from $\mathcal{B}$ to $\mathcal{A}$. Since in this OCC, also $F ; F^{\smile} ; F$ needs to exist as a morphism from $\mathcal{A}$ to $\mathcal{B}$, it has to be equal to F, which therefore is difunctional.

A lax functor $\mathbf{D}$ mapping $F : \mathcal{A} \to \mathcal{B}$ to $W : \mathcal{A}' \to \mathcal{B}'$ has to satisfy
$$W ; W^{\smile} ; W = \mathbf{D}\, F ; (\mathbf{D}\, F)^{\smile} ; \mathbf{D}\, F \sqsubseteq \mathbf{D}\,(F ; F^{\smile} ; F) = \mathbf{D}\, F = W \ ,$$
so it can map $\mathcal{F}$ only to difunctional morphisms.

Furthermore, if, for a lax cocone, its source $\mathbf{J}$ is considered as an OCC, this implies that for each morphism $F : \mathcal{A} \to \mathcal{B}$ in $\mathbf{J}$, also the converse morphism $F^{\smile} : \mathcal{B} \to \mathcal{A}$ needs to be considered. Such a lax cocone therefore automatically has to satisfy both the following conditions:
$$\left.\begin{array}{rcl} \mathbf{D}\, F ; \eta_{\mathcal{B}} & \sqsubseteq & \eta_{\mathcal{A}} \\ (\mathbf{D}\, F)^{\smile} ; \eta_{\mathcal{A}} & \sqsubseteq & \eta_{\mathcal{B}} \end{array}\right\} \qquad \text{for each morphism } F : \mathcal{A} \to \mathcal{B} \text{ in } \mathbf{J}.$$

Convention 4.3.4 Given a morphism $W : \mathcal{B} \to \mathcal{C}$ in the OCC $\mathbf{C}$, we will frequently identify W with the functor $\mathbf{D}$ mapping the single morphism explicitly mentioned in the OCC $\bullet \to \bullet$ to W.

(Since we are dealing with an OCC, that morphism also has a converse, which then must be mapped to $W^{\smile}$.) □

A lax cocone from W to $\mathcal{D}$ therefore is a cospan $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ satisfying $W \mathbin{;} S \sqsubseteq R$ and $W^{\smile} \mathbin{;} R \sqsubseteq S$.

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "S"] & \\
& \mathcal{D} \\
\mathcal{B} \arrow[uu, "W"] \arrow[ur, "R"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{C} \arrow[dr, "S"'] \arrow[drr, "S'"] & & \\
& \mathcal{D} \arrow[r, dotted, "U"] & \mathcal{D}' \\
\mathcal{B} \arrow[uu, "W"] \arrow[ur, "R"] \arrow[urr, "R'"'] & &
\end{tikzcd}$$

We explicitly state the definition of resulting special case of lax colimits:

Definition 4.3.5 An *OCC-colimit* of $W : \mathcal{B} \to \mathcal{C}$ in the OCC $\mathbf{C}$ is a lax cocone $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ from W to $\mathcal{D}$ satisfying the following conditions:

1. *factorisation:* for any object $\mathcal{D}'$ of $\mathbf{C}$ with lax cocone $\mathcal{B} \xrightarrow{R'} \mathcal{D}' \xleftarrow{S'} \mathcal{C}$ from W to $\mathcal{D}'$, there is a morphism $U : \mathcal{D} \to \mathcal{D}'$ in $\mathbf{C}$ with $R \mathbin{;} U = R'$ and $S \mathbin{;} U = S'$;
2. *isotony:* for any object $\mathcal{D}'$ of $\mathbf{C}$ and any two morphisms $U, U' : \mathcal{D} \to \mathcal{D}'$, if $R \mathbin{;} U \sqsubseteq R \mathbin{;} U'$ and $S \mathbin{;} U \sqsubseteq S \mathbin{;} U'$, then $U \sqsubseteq U'$. □

Lemma 4.3.6 Let $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ be a span in $\mathsf{Map}\,\mathbf{C}$, that is, P and Q are mappings.

If the cospan $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ in the OCC $\mathbf{C}$ is a cocone for $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$, then it is also a lax cocone from $P^{\smile} \mathbin{;} Q$ to $\mathcal{D}$.

If, furthermore, $W := (P^{\smile} \mathbin{;} Q)^{\boxast}$ exists, then that cospan is also a lax cocone from W to $\mathcal{D}$.

Proof: Commutativity $P \mathbin{;} R = Q \mathbin{;} S$ together with univalence of P and Q implies the lax cocone properties from $P^{\smile} \mathbin{;} Q$:

$$P^{\smile} \mathbin{;} Q \mathbin{;} S = P^{\smile} \mathbin{;} P \mathbin{;} R \sqsubseteq R \qquad \text{and} \qquad Q^{\smile} \mathbin{;} P \mathbin{;} R = Q^{\smile} \mathbin{;} Q \mathbin{;} S \sqsubseteq S \ .$$

Using left induction for $\_^{\boxast}$, this shows that $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is also a lax cocone from W to $\mathcal{D}$:

$$W \mathbin{;} S = (P^{\smile} \mathbin{;} Q)^{\boxast} \mathbin{;} S \sqsubseteq R \qquad \text{and} \qquad W^{\smile} \mathbin{;} R = (Q^{\smile} \mathbin{;} P)^{\boxast} \mathbin{;} R \sqsubseteq S \ . \qquad □$$

Lemma 4.3.7 Let $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ be a span in $\mathsf{Map}\,\mathbf{C}$, that is, P and Q are mappings.

If the cospan $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ in the OCC $\mathbf{C}$ is an OCC-colimit for $W := (P^{\smile} \mathbin{;} Q)^{\boxast}$, and $\mathcal{B} \xrightarrow{R'} \mathcal{D}' \xleftarrow{S'} \mathcal{C}$ is a cocone for $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$, that is, a cospan in $\mathbf{C}$ with $P \mathbin{;} R' = Q \mathbin{;} S'$, then there is a morphism $U : \mathcal{D} \to \mathcal{D}'$ that satisfies the factorisation property of OC-pushouts, i.e., $R \mathbin{;} U = R'$ and $S \mathbin{;} U = S'$.

$$
\begin{tikzcd}
& \mathcal{C} \arrow[dr, "S"] \arrow[drr, "S'"] & & \\
\mathcal{A} \arrow[ur, "Q"] \arrow[dr, "P"'] & & \mathcal{D} \arrow[r, dotted, "U"] & \mathcal{D}' \\
& \mathcal{B} \arrow[uu, "W"] \arrow[ur, "R"] \arrow[urr, "R'"'] & &
\end{tikzcd}
$$

Proof: From Lemma 4.3.6 we know that $\mathcal{B} \xrightarrow{R'} \mathcal{D}' \xleftarrow{S'} \mathcal{C}$ is also a lax cocone from W to $\mathcal{D}'$. From the factorisation property of OCC-colimits we then know that there is a morphism $U : \mathcal{D} \to \mathcal{D}'$ with $R \mathbin{;} U = R'$ and $S \mathbin{;} U = S'$. □

This, together with the fact that the isotony properties of OC-colimits and OCC-colimits coincide, immediately implies:

Theorem 4.3.8 If a cospan $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is an OCC-colimit for $(P^{\smile} \mathbin{;} Q)^{\boxast}$, then it is also an OC-pushout for $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$. □

Lemma 4.3.9 If, in an OCC, a cospan $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a lax cocone from $W := (P^{\smile} \mathbin{;} Q)^{\boxast}$ to $\mathcal{D}$, then it is also a cocone for $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$.

Proof: The lax cocone property here means that we have:

$$
\begin{aligned}
(P^{\smile} \mathbin{;} Q)^{\boxast} \mathbin{;} S &\sqsubseteq R \\
(Q^{\smile} \mathbin{;} P)^{\boxast} \mathbin{;} R &\sqsubseteq S
\end{aligned}
$$

This implies in particular:

$$
\begin{aligned}
P^{\smile} \mathbin{;} Q \mathbin{;} S &\sqsubseteq R \\
Q^{\smile} \mathbin{;} P \mathbin{;} R &\sqsubseteq S
\end{aligned}
$$

and, with totality of P and Q,

$$
\begin{aligned}
Q \mathbin{;} S &\sqsubseteq P \mathbin{;} P^{\smile} \mathbin{;} Q \mathbin{;} S &&\sqsubseteq P \mathbin{;} R \\
P \mathbin{;} R &\sqsubseteq Q \mathbin{;} Q^{\smile} \mathbin{;} P \mathbin{;} R &&\sqsubseteq Q \mathbin{;} S \ ,
\end{aligned}
$$

so we have equality $P \mathbin{;} R = Q \mathbin{;} S$ and therefore a cone for $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$. □

From this, we now easily obtain also the converse implication to Theorem 4.3.8:

Theorem 4.3.10 If, in an OCC, a cospan $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is an OC-pushout for $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$, then it is also an OCC-colimit for $W := (P^{\smile} \mathbin{;} Q)^{\boxast}$.

Proof: From Lemma 4.3.6 we know that $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is also a lax cocone from W to $\mathcal{D}$.

If $\mathcal{B} \xrightarrow{R'} \mathcal{D}' \xleftarrow{S'} \mathcal{C}$ is a lax cocone from W to $\mathcal{D}'$, then we know from Lemma 4.3.9 that it is also a cocone for $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$.

The theorem then follows since the appropriately instantiated factorisation and isotony properties of OCC-colimits for W then coincide with those for OC-colimits for $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$. □

4.4 OCC-Colimits are Co-Tabulations

Theorem 4.4.1 If a cospan $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ in a collagory is a co-tabulation of $W : \mathcal{B} \to \mathcal{C}$, then it is also an OCC-colimit for W.

PROOF: Immediate from Lemma 2.3.4 (factorisation) and Lemma 2.3.6 (isotony). □

Lemma 4.4.2 If, in an allegory, $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is an OCC-colimit for W, then

$$\begin{array}{rclcrcl} W^{\smile} ; R &=& S ; \mathsf{ran}\, R & \qquad & W^{\smile} ; R ; R^{\smile} &=& S ; R^{\smile} \\ W ; S &=& R ; \mathsf{ran}\, S & & W ; S ; S^{\smile} &=& R ; S^{\smile} \end{array}$$

PROOF: Let $R_0 = W ; S$ and $S_0 = S$. This defines a lax cocone $\mathcal{B} \xrightarrow{R_0} \mathcal{D} \xleftarrow{S_0} \mathcal{C}$ from W to $\mathcal{D}$, since:

$$\begin{array}{rcl} W^{\smile} ; R_0 &=& W^{\smile} ; W ; S \sqsubseteq W^{\smile} ; R \sqsubseteq S = S_0 \ ; \\ W ; S_0 &=& W ; S = R_0 \ . \end{array}$$

Then factorisation gives us a $U_0 : \mathcal{D} \to \mathcal{D}$ such that $R_0 = W ; S = R ; U_0$ and $S_0 = S = S ; U_0$. Since $R ; U_0 = W ; S \sqsubseteq R = R ; \mathbb{I}_{\mathcal{D}}$ and $S ; U_0 = S \sqsubseteq S ; \mathbb{I}_{\mathcal{D}}$, isotony gives us $U_0 \sqsubseteq \mathbb{I}_{\mathcal{D}}$.

So U_0 is a sub-identity, and $S = S;U_0$ implies $\mathsf{ran}\, S \sqsubseteq U_0$. Since composition of sub-identities is meet, we obtain the following (which implies $U_0 = \mathsf{ran}\, S$):

$$W ; S = W ; S ; \mathsf{ran}\, S = R ; U_0 ; \mathsf{ran}\, S = R ; \mathsf{ran}\, S$$

Analogously, $W^{\smile} ; R = S ; \mathsf{ran}\, R$ also holds, and these further imply

$$W^{\smile} ; R ; R^{\smile} = S ; R^{\smile} \qquad \text{and} \qquad W ; S ; S^{\smile} = R ; S^{\smile} \ .$$ □

Lemma 4.4.2 does not use difunctionality of W, and implies:

$$\begin{array}{rcl} W^{\smile} ; W ; W^{\smile} ; R &=& W^{\smile} ; W ; S ; \mathsf{ran}\, R \;=\; W^{\smile} ; R ; \mathsf{ran}\, S ; \mathsf{ran}\, R \\ &=& W^{\smile} ; R ; \mathsf{ran}\, S \;=\; S ; \mathsf{ran}\, R ; \mathsf{ran}\, S \;=\; S ; \mathsf{ran}\, R \;=\; W^{\smile} ; R \end{array}$$

and, analogously, $W;W^{\smile};W;S = W;S$. Therefore, even with a weaker concept of OCC-colimit, we would still have, in some sense, "almost-difunctionality" of W.

Lemma 4.4.2 did use allegory properties (for sub-identities); for showing the converse implication to Theorem 4.4.1 we need full collagories:

Theorem 4.4.3 If, in a collagory, $W : \mathcal{B} \to \mathcal{C}$ is a difunctional morphism and $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is an OCC-colimit for W, then it is also a co-tabulation for W.

PROOF: Let $R_1 = W$ and $S_1 = \mathbb{I}_{\mathcal{C}} \sqcup W^{\smile} ; W$. This defines a lax cocone $\mathcal{B} \xrightarrow{R_1} \mathcal{C} \xleftarrow{S_1} \mathcal{C}$ from W to $\mathcal{C}$, since (using difunctionality of W):

$$\begin{array}{rcl} W^{\smile} ; R_1 &=& W^{\smile} ; W \sqsubseteq S_1 \\ W ; S_1 &=& W ; (\mathbb{I}_{\mathcal{C}} \sqcup W^{\smile} ; W) \;=\; W \sqcup W ; W^{\smile} ; W \;=\; W \;=\; R_1 \end{array}$$

Then factorisation gives us a $U_1 : \mathcal{D} \to \mathcal{C}$ such that $R_1 = W = R ; U_1$ and $S_1 = \mathbb{I}_{\mathcal{C}} \sqcup W^{\smile} ; W = S ; U_1$. This immediately implies that S is total and U_1 is surjective, and with the same argument we also obtain that R is total.

We further have

$$\begin{array}{rcll} R ; S^{\smile} & = & W ; S ; S^{\smile} & \text{Lemma 4.4.2} \\ & \sqsupseteq & W & S \text{ total} \\ & = & R ; U_1 & \\ S ; S^{\smile} & \sqsupseteq & \mathbb{I}_{\mathcal{C}} \sqcup W^{\smile} ; R ; R^{\smile} ; W & S \text{ total, } W^{\smile} ; R \sqsubseteq S \\ & \sqsupseteq & \mathbb{I}_{\mathcal{C}} \sqcup W^{\smile} ; W & R \text{ total} \\ & = & S ; U_1 & \end{array}$$

With isotony, this implies $S^{\smile} \sqsupseteq U_1$, which produces $W = R ; U_1 \sqsubseteq R ; S^{\smile}$.

Now:

$$\begin{array}{rcl} R ; \mathsf{dom}\, U_1 & \sqsubseteq & R ; U_1 ; U_1^{\smile} \\ & = & W ; U_1^{\smile} \\ & = & W ; U_1^{\smile} ; \mathsf{dom}\, U_1 \\ & \sqsubseteq & R ; \mathsf{dom}\, U_1 \end{array}$$

which implies equality:

$$R ; \mathsf{dom}\, U_1 = R ; U_1 ; U_1^{\smile} = W ; U_1^{\smile} \ .$$

Similarly:

$$\begin{array}{rcl} S ; \mathsf{dom}\, U_1 & \sqsubseteq & S ; U_1 ; U_1^{\smile} \\ & = & (\mathbb{I}_{\mathcal{C}} \sqcup W^{\smile} ; W) ; U_1^{\smile} \\ & = & U_1^{\smile} \sqcup W^{\smile} ; W ; U_1^{\smile} \\ & = & U_1^{\smile} \sqcup W^{\smile} ; R ; \mathsf{dom}\, U_1 \\ & \sqsubseteq & U_1^{\smile} \sqcup S ; \mathsf{dom}\, U_1 \\ & = & S ; \mathsf{dom}\, U_1 \end{array}$$

which implies equality:

$$S ; \mathsf{dom}\, U_1 = S ; U_1 ; U_1^{\smile}$$

From these two equalities together, isotony produces:

$$\mathsf{dom}\, U_1 = U_1 ; U_1^{\smile} \ ,$$

so U_1 is also injective.

Furthermore, we have (using difunctionality of W):

$$\begin{array}{rcl} R ; \mathsf{dom}\, U_1 ; S^{\smile} & = & W ; U_1^{\smile} ; S^{\smile} \; = \; W ; (\mathbb{I}_{\mathcal{C}} \sqcup W^{\smile} ; W) \; = \; W \; = \; R ; U_1 \\ S ; \mathsf{dom}\, U_1 ; S^{\smile} & = & S ; U_1 ; U_1^{\smile} ; S^{\smile} \\ & = & (\mathbb{I}_{\mathcal{C}} \sqcup W^{\smile} ; W) ; (\mathbb{I}_{\mathcal{C}} \sqcup W^{\smile} ; W) \; = \; \mathbb{I}_{\mathcal{C}} \sqcup W^{\smile} ; W \; = \; S ; U_1 \ . \end{array}$$

With isotony this produces $\mathsf{dom}\ U_1 ; S^{\smile} = U_1$, which we use to define $S' : \mathcal{C} \to \mathcal{D}$ as:

$$S' := S ; \mathsf{dom}\ U_1 = U_1^{\smile}$$

From the above, we know that S' is a mapping, and that:

$$\begin{aligned} R ; S'^{\smile} &= R ; U_1 = W \\ S' ; S'^{\smile} &= S ; \mathsf{dom}\ U_1 ; U_1 = S ; U_1 = \mathbb{I}_{\mathcal{C}} \sqcup W^{\smile} ; W \end{aligned}$$

For easier reference, we explicitly present the results of mirroring the starting configuration made up by R_1 and S_1 above. Therefore, we obtain $U_2 : \mathcal{D} \to \mathcal{B}$ via factorisation of the lax cocone $\mathcal{B} \xrightarrow{R_2} \mathcal{B} \xleftarrow{S_2} \mathcal{C}$ formed by of $R_2 = \mathbb{I} \sqcup W ; W^{\smile}$ and $S_2 = W^{\smile}$, with $R_2 = \mathbb{I} \sqcup W ; W^{\smile} = R ; U_2$ and $S_2 = W^{\smile} = S ; U_2$, and we obtain first $R^{\smile} \sqsupseteq U_2$, and then

$$\begin{aligned} R ; \mathsf{dom}\ U_2 &= R ; U_2 ; U_2^{\smile} \\ S ; \mathsf{dom}\ U_2 &= S ; U_2 ; U_2^{\smile} = W^{\smile} ; U_2^{\smile} \end{aligned}$$

implying $\mathsf{dom}\ U_2 = U_2 ; U_2^{\smile}$ and $\mathsf{dom}\ U_2 ; R^{\smile} = U_2$, which we use to define

$$R' := R ; \mathsf{dom}\ U_2 \ .$$

R' is a mapping, and we have

$$\begin{aligned} R' ; S^{\smile} &= W \\ R' ; R'^{\smile} &= \mathbb{I}_{\mathcal{B}} \sqcup W ; W^{\smile} \end{aligned}$$

So far the results of the development of U_2 in parallel with the explicit development of U_1 above.

Now we relate U_2 and U_1, i.e., R' and S':

$$\begin{aligned} R ; \mathsf{dom}\ U_1 ; R^{\smile} &\sqsubseteq R ; U_1 ; U_1^{\smile} ; R^{\smile} \\ &= W ; W^{\smile} \\ &\sqsubseteq R ; R'^{\smile} \\ S ; \mathsf{dom}\ U_1 ; R^{\smile} &= S ; U_1 ; U_1^{\smile} ; R^{\smile} \\ &= (\mathbb{I}_{\mathcal{C}} \sqcup W^{\smile} ; W) ; W^{\smile} \\ &= W^{\smile} \\ &= S ; R'^{\smile} \end{aligned}$$

From this and the analogous symmetric development, isotony produces:

$$\begin{aligned} R ; \mathsf{dom}\ U_1 &\sqsubseteq R' \\ S ; \mathsf{dom}\ U_2 &\sqsubseteq S' \end{aligned}$$

This shows the following cyclic inclusion chain:

$$W = R ; S'^{\smile} = R ; \mathsf{dom}\ U_1 ; S'^{\smile} \sqsubseteq R' ; S'^{\smile} \sqsubseteq W \ ,$$

which contains the first co-tabulation condition:

$$W = R' ; S'^{\smile} \ .$$

We furthermore have:

$$\begin{array}{rcll}
R ; (R'^{\smile} ; R' \sqcup S'^{\smile} ; S') & = & R ; R'^{\smile} ; R' \sqcup R ; S'^{\smile} ; S' & \sqcup\text{-distributivity} \\
& = & (\mathbb{I} \sqcup W ; W^{\smile}) ; R' \sqcup W ; S' & R ; S'^{\smile} = W \\
& = & R' \sqcup W ; W^{\smile} ; R' \sqcup W ; S' & \sqcup\text{-distributivity} \\
& = & R' \sqcup W ; S' & W^{\smile} ; R' \sqsubseteq S' \\
& = & R' & W ; S' \sqsubseteq R'
\end{array}$$

and, analogously, $S ; (R'^{\smile} ; R' \sqcup S'^{\smile} ; S') = S$, which implies, with isotony, the last co-tabulation condition $R'^{\smile} ; R' \sqcup S'^{\smile} ; S' = \mathbb{I}_{\mathcal{D}}$.

This shows that R' and S' define a co-tabulation. According to Theorem 4.4.1, co-tabulations are OCC-colimits, and according to Prop. 4.3.3, OCC-colimits are unique up to isomorphism, so there must be an isomorphism $U : \mathcal{D} \to \mathcal{D}$ with $R' = R ; U$ and $S' = S ; U$, and therefore $R = R' ; U^{\smile}$ and $S = S' ; U^{\smile}$. Then:

$$\begin{array}{rcl}
R ; S^{\smile} & = & R' ; U^{\smile} ; U ; S'^{\smile} \ = \ R' ; S'^{\smile} \ = \ W \\
R ; R^{\smile} & = & R' ; U^{\smile} ; U ; R'^{\smile} \ = \ R' ; R'^{\smile} \ = \ \mathbb{I}_{\mathcal{B}} \sqcup W ; W^{\smile} \\
S ; S^{\smile} & = & S' ; U^{\smile} ; U ; S'^{\smile} \ = \ S' ; S'^{\smile} \ = \ \mathbb{I}_{\mathcal{C}} \sqcup W^{\smile} ; W \\
R^{\smile} ; R \sqcup S^{\smile} ; S & = & U ; R'^{\smile} ; R' ; U^{\smile} \sqcup U ; S'^{\smile} ; S' ; U^{\smile} \\
& = & U ; (R'^{\smile} ; R' \sqcup S'^{\smile} ; S') ; U^{\smile} \ = \ U ; \mathbb{I}_{\mathcal{D}} ; U^{\smile} \ = \ \mathbb{I}_{\mathcal{D}}
\end{array}$$

Therefore, the original OCC-colimit is a co-tabulation, too. □

Chapter 5

Van Kampen Squares and Adhesive Categories

5.1 Adhesive Categories and Van Kampen Setups

Adhesive categories as a more specific setting for double-pushout graph rewriting have been introduced by Lack and Sobociński [LS04, LS05]; the following two definitions are taken from there:

Definition 5.1.1 A *van Kampen square* (i) is a pushout which satisfies the following condition: given a commutative cube (ii) of which (i) forms the bottom face and the back faces are pullbacks (where $\mathcal{C}$ is considered to be in the back), the front faces are pullbacks if and only if the top face is a pushout.

$$\begin{tikzcd}
& \mathcal{C} \arrow[dl, "M"'] \arrow[dr, "F"] & \\
\mathcal{A} \arrow[dr, "G"'] & & \mathcal{B} \arrow[dl, "N"] \\
& \mathcal{D} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& \mathcal{C}' \arrow[rr, "f"] \arrow[dl, "m"'] \arrow[dd, "c"] & & \mathcal{B}' \arrow[dl, "n"'] \arrow[dd, "b"] \\
\mathcal{A}' \arrow[rr, "g"] \arrow[dd, "a"'] & & \mathcal{D}' \arrow[dd, "d"] & \\
& \mathcal{C} \arrow[rr, "F"] \arrow[dl, "M"] & & \mathcal{B} \arrow[dl, "N"] \\
\mathcal{A} \arrow[rr, "G"'] & & \mathcal{D} &
\end{tikzcd}$$

(i) (ii) □

Definition 5.1.2 A category $\mathbf{C}$ is said to be *adhesive* if

1. $\mathbf{C}$ has pushouts along monomorphisms;
2. $\mathbf{C}$ has pullbacks;
3. pushouts along monomorphisms are van Kampen squares. □

For more concise formulations, we define:

Definition 5.1.3 A *van Kampen setup* in a collagory $\mathbf{C}$ for a square as in Def. 5.1.1(i) is a commuting cube in $\mathsf{Map}\,\mathbf{C}$ as in Def. 5.1.1(ii) where the bottom square is a gluing and the two back squares are tabulations. □

For reference, we expand this into the implied equations:

Lemma 5.1.4 In a collagory $\mathbf{C}$, a van Kampen setup in $\mathsf{Map}\,\mathbf{C}$ means that the following hold:

Bottom gluing:

$$
\begin{array}{rclcrclcrcl}
G ; N^{\smile} & = & (M^{\smile} ; F)^{\boxast} & \quad & G^{\smile} ; G \sqcup N^{\smile} ; N & = & \mathbb{I}_{\mathcal{D}} & \quad & G ; G^{\smile} & = & (M^{\smile} ; F)^{\triangleright} \\
 & & & & & & & & N ; N^{\smile} & = & (M^{\smile} ; F)^{\triangleleft}
\end{array}
$$

Back tabulations:

$$
\begin{array}{rclcrclcrcl}
c^{\smile} ; m & = & M ; a^{\smile} & \quad & c ; c^{\smile} \sqcap m ; m^{\smile} & = & \mathbb{I}_{\mathcal{C}'} & \quad & c^{\smile} ; c & = & \mathbb{I}_{\mathcal{C}} \sqcap M ; a^{\smile} ; a ; M^{\smile} \\
 & & & & & & & & m^{\smile} ; m & = & \mathbb{I}_{\mathcal{A}'} \sqcap a ; M^{\smile} ; M ; a^{\smile} \\
c^{\smile} ; f & = & F ; b^{\smile} & \quad & c ; c^{\smile} \sqcap f ; f^{\smile} & = & \mathbb{I}_{\mathcal{C}'} & \quad & c^{\smile} ; c & = & \mathbb{I}_{\mathcal{C}} \sqcap F ; b^{\smile} ; b ; F^{\smile} \\
 & & & & & & & & f^{\smile} ; f & = & \mathbb{I}_{\mathcal{B}'} \sqcap b ; F^{\smile} ; F ; b^{\smile}
\end{array}
$$

Remaining commutative squares:

$$
\begin{array}{rclcrclcrcl}
m ; g & = & f ; n & \quad & g ; d & = & a ; G & \quad & n ; d & = & b ; N
\end{array}
$$

□

5.2 Maps in Collagories form Adhesive Categories

The equations in Lemma 5.1.4 are now used to prove the following:

Lemma 5.2.1 In a collagory, if the front squares of a van Kampen setup are tabulations, then the top square is a gluing. If furthermore $M^{\smile} ; F$ is difunctional, then $m^{\smile} ; f$ is difunctional, too.

Proof: Besides the assumptions in Lemma 5.1.4, we also have in particular the following equations for the two front tabulations:

$$
\begin{array}{rclcrcl}
g^{\smile} ; g & = & \mathbb{I}_{\mathcal{D}'} \sqcap d ; G^{\smile} ; G ; d^{\smile} & \quad & g ; g^{\smile} \sqcap a ; a^{\smile} & = & \mathbb{I}_{\mathcal{A}'} \\
n^{\smile} ; n & = & \mathbb{I}_{\mathcal{D}'} \sqcap d ; N^{\smile} ; N ; d^{\smile}
\end{array}
$$

These front tabulation equations alone are sufficient to propagate sharpness from the bottom co-tabulation to the top square:

$$
\begin{array}{rcll}
g^{\smile} ; g \sqcup n^{\smile} ; n & = & \mathbb{I}_{\mathcal{D}'} \sqcap (d ; G^{\smile} ; G ; d^{\smile} \sqcup d ; N^{\smile} ; N ; d^{\smile}) & \text{front tabulations} \\
 & = & \mathbb{I}_{\mathcal{D}'} \sqcap d ; (G^{\smile} ; G \sqcup N^{\smile} ; N) ; d^{\smile} & \text{join-distributivity} \\
 & = & \mathbb{I}_{\mathcal{D}'} \sqcap d ; \mathbb{I}_{\mathcal{D}} ; d^{\smile} & \text{bottom co-tabulation} \\
 & = & \mathbb{I}_{\mathcal{D}'} \sqcap d ; d^{\smile} & \text{identity law} \\
 & = & \mathbb{I}_{\mathcal{D}'} & d \text{ total}
\end{array}
$$

Lemma 2.3.1 give us the inclusion $(m^{\smile} ; f)^{\boxast} \sqsubseteq g ; n^{\smile}$ for the top square, so for gluing cross-commutativity we only need to show the opposite inclusion:

$$
\begin{array}{rcll}
g ; n^{\smile} & = & g ; n^{\smile} \sqcap g ; d ; d^{\smile} ; n^{\smile} & d \text{ total} \\
 & = & g ; n^{\smile} \sqcap a ; G ; N^{\smile} ; b^{\smile} & \text{front squares commute} \\
 & = & g ; n^{\smile} \sqcap a ; (M^{\smile} ; F)^{\boxast} ; b^{\smile} & \text{bottom co-tabulation} \\
 & = & (m^{\smile} ; f)^{\boxast} & \_^{\boxast} \text{ induction (see below)}
\end{array}
$$

The last step is based on the equation $g ; g^{\smile} \sqcap a ; a^{\smile} = \mathbb{I}_{\mathcal{A}'}$ arising from the front tabulation, and the following two inductive inclusions (for arbitrary $R : \mathcal{A} \to \mathcal{A}$ and $S : \mathcal{A} \to \mathcal{B}$):

$$\begin{array}{rcll}
g ; n^{\smile} \sqcap a ; R ; (M^{\smile} ; F) ; b^{\smile} & = & g ; n^{\smile} \sqcap a ; R ; M^{\smile} ; c^{\smile} ; f & \text{back tabulation} \\
& = & (g ; n^{\smile} ; f^{\smile} \sqcap a ; R ; M^{\smile} ; c^{\smile}) ; f & f \text{ univalent} \\
& = & (g ; g^{\smile} ; m^{\smile} \sqcap a ; R ; a^{\smile} ; m^{\smile}) ; f & \text{top, left commut.} \\
& = & (g ; g^{\smile} \sqcap a ; R ; a^{\smile}) ; (m^{\smile} ; f) & m \text{ univalent} \\
g ; g^{\smile} \sqcap a ; S ; (F^{\smile} ; M) ; a^{\smile} & = & g ; g^{\smile} \sqcap a ; S ; F^{\smile} ; c^{\smile} ; m & \text{back tabulation} \\
& = & g ; g^{\smile} \sqcap a ; S ; b^{\smile} ; f^{\smile} ; m & \text{left commutativity} \\
& = & (g ; g^{\smile} ; m^{\smile} \sqcap a ; S ; b^{\smile} ; f^{\smile}) ; m & m \text{ univalent} \\
& = & (g ; n^{\smile} ; f^{\smile} \sqcap a ; S ; b^{\smile} ; f^{\smile}) ; m & \text{top commutativity} \\
& = & (g ; n^{\smile} \sqcap a ; S ; b^{\smile}) ; (f^{\smile} ; m) & f \text{ univalent}
\end{array}$$

For the case where $M^{\smile} ; F$ is difunctional, the first of these equations implies, with $R := \mathbb{I}$, direct co-tabulation cross-commutativity $g ; n^{\smile} = m^{\smile} ; f$, and therewith difunctionality of the latter. □

Lemma 5.2.2 In a van Kampen setup where the top square is a gluing, the front squares cross-commute.

Proof: We only show cross-commutativity of the right front square, $d ; N^{\smile} = n^{\smile} ; b$, since the situation of the other front square is perfectly symmetric.

$$\begin{array}{rcll}
d ; N^{\smile} & = & (g^{\smile} ; g \sqcup n^{\smile} ; n) ; d ; N^{\smile} & \text{top gluing} \\
& = & g^{\smile} ; g ; d ; N^{\smile} \sqcup n^{\smile} ; n ; d ; N^{\smile} & \text{join distributivity} \\
& = & g^{\smile} ; a ; G ; N^{\smile} \sqcup n^{\smile} ; b ; N ; N^{\smile} & \text{front squares comm.} \\
& = & g^{\smile} ; a ; (M^{\smile} ; F)^{\boxed{*}} \sqcup n^{\smile} ; b ; (M^{\smile} ; F)^{\lhd\!*} & \text{bottom gluing} \\
& = & g^{\smile} ; a ; (M^{\smile} ; F)^{\boxed{*}} \sqcup n^{\smile} ; b ; (\mathbb{I}_{\mathcal{B}} \sqcup F^{\smile} ; M ; (M^{\smile} ; F)^{\boxed{*}}) & _^{\boxed{*}} \text{ properties} \\
& = & n^{\smile} ; b \sqcup (g^{\smile} ; a \sqcup n^{\smile} ; b ; F^{\smile} ; M) ; (M^{\smile} ; F)^{\boxed{*}} & \text{join distributivity} \\
& = & n^{\smile} ; b & \text{(see below)}
\end{array}$$

The last step holds by right induction for $\_^{\boxed{*}}$, since

$$\begin{array}{rll}
& (g^{\smile} ; a \sqcup n^{\smile} ; b ; F^{\smile} ; M) ; M^{\smile} ; F & \\
= & (g^{\smile} ; a \sqcup n^{\smile} ; f^{\smile} ; c ; M) ; M^{\smile} ; F & \text{back tabulation} \\
= & (g^{\smile} ; a \sqcup n^{\smile} ; f^{\smile} ; m ; a) ; M^{\smile} ; F & \text{left square commutes} \\
= & g^{\smile} ; a ; M^{\smile} ; F & n^{\smile} ; f^{\smile} ; m = g^{\smile} ; m^{\smile} ; m \sqsubseteq g^{\smile} \\
= & g^{\smile} ; m^{\smile} ; c ; F & \text{left tabulation} \\
= & n^{\smile} ; f^{\smile} ; f ; b & \text{top, back squares commute} \\
\sqsubseteq & n^{\smile} ; b & f \text{ univalent}
\end{array}$$

and, from that, also

$$n^{\smile} ; b ; F^{\smile} ; M ; M^{\smile} ; F \sqsubseteq (g^{\smile} ; a \sqcup n^{\smile} ; b ; F^{\smile} ; M) ; M^{\smile} ; F \sqsubseteq n^{\smile} ; b \ .$$

□

Lemma 5.2.3 In a van Kampen setup where the top square is a gluing, the front squares are tabulations iff the following holds:

$$m ; (m^{\smile} ; f)^{\boxast} ; f^{\smile} \sqcap c ; c^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{C}'} \tag{$*$}$$

Proof: We only show that the right front square is a tabulation, since the situation of the other front square is perfectly symmetric. cross-commutativity has been shown separately in Lemma 5.2.2. For the second tabulation condition, we first show that, for every $R : \mathcal{C}' \to \mathcal{C}'$:

$$\begin{array}{rcll}
f^{\smile} ; R ; f \sqcap b ; b^{\smile} & \sqsubseteq & b ; (b^{\smile} ; f^{\smile} ; R ; f ; b \sqcap \mathbb{I}_{\mathcal{B}}) ; b^{\smile} & \text{modal rules} \\
& = & b ; (F^{\smile} ; c^{\smile} ; R ; c ; F \sqcap \mathbb{I}_{\mathcal{B}}) ; b^{\smile} & \text{back square commutativity} \\
& \sqsubseteq & b ; \mathsf{ran}\, F ; b^{\smile} & \mathsf{ran}\ \text{properties} \\
& = & b ; F^{\smile} ; F ; b^{\smile} & F \text{ univalent}
\end{array}$$

This implies $f^{\smile} ; m ; m^{\smile} ; f \sqcap b ; b^{\smile} = f^{\smile} ; m ; m^{\smile} ; f \sqcap b ; F^{\smile} ; F ; b^{\smile}$, which we use below:

$$\begin{array}{rll}
& n ; n^{\smile} \sqcap b ; b^{\smile} = \mathbb{I}_{\mathcal{B}'} & \\
\Leftrightarrow & (m^{\smile} ; f)^{\triangleleft} \sqcap b ; b^{\smile} = \mathbb{I}_{\mathcal{B}'} & \text{top gluing} \\
\Leftrightarrow & (\mathbb{I}_{\mathcal{B}'} \sqcup f^{\smile} ; m ; (m^{\smile} ; f)^{\boxast}) \sqcap b ; b^{\smile} = \mathbb{I}_{\mathcal{B}'} & R^{\triangleleft} = \mathbb{I} \sqcup R^{\smile} ; R^{\boxast} \\
\Leftrightarrow & (\mathbb{I}_{\mathcal{B}'} \sqcap b ; b^{\smile}) \sqcup (f^{\smile} ; m ; (m^{\smile} ; f)^{\boxast} \sqcap b ; b^{\smile}) = \mathbb{I}_{\mathcal{B}'} & \text{lattice distributivity} \\
\Leftrightarrow & \mathbb{I}_{\mathcal{B}'} \sqcup (f^{\smile} ; m ; (m^{\smile} ; f)^{\boxast} \sqcap b ; b^{\smile}) = \mathbb{I}_{\mathcal{B}'} & b \text{ total} \\
\Leftrightarrow & f^{\smile} ; m ; (m^{\smile} ; f)^{\boxast} \sqcap b ; b^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{B}'} & \text{ordering} \\
\Leftrightarrow & f^{\smile} ; m ; (m^{\smile} ; f)^{\boxast} \sqcap b ; F^{\smile} ; F ; b^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{B}'} & \text{(above)} \\
\Leftrightarrow & f^{\smile} ; m ; (m^{\smile} ; f)^{\boxast} \sqcap f^{\smile} ; c ; c^{\smile} ; f \sqsubseteq \mathbb{I}_{\mathcal{B}'} & \text{back tabulation} \\
\Leftrightarrow & f^{\smile} ; (m ; (m^{\smile} ; f)^{\boxast} \sqcap f ; f^{\smile} ; c ; c^{\smile} ; f) \sqsubseteq \mathbb{I}_{\mathcal{B}'} & f \text{ univalent} \\
\Leftrightarrow & m ; (m^{\smile} ; f)^{\boxast} \sqcap f ; f^{\smile} ; c ; c^{\smile} ; f \sqsubseteq f & f \text{ mapping} \\
\Leftrightarrow & m ; (m^{\smile} ; f)^{\boxast} \sqcap c ; c^{\smile} ; f ; f^{\smile} ; f \sqsubseteq f & \text{Lemma 2.2.4} \\
\Leftrightarrow & m ; (m^{\smile} ; f)^{\boxast} \sqcap c ; c^{\smile} ; f \sqsubseteq f & f \text{ univalent} \\
\Leftrightarrow & (m ; (m^{\smile} ; f)^{\boxast} ; f^{\smile} \sqcap c ; c^{\smile}) ; f \sqsubseteq f & f \text{ univalent} \\
\Leftrightarrow & m ; (m^{\smile} ; f)^{\boxast} ; f^{\smile} \sqcap c ; c^{\smile} \sqsubseteq f ; f^{\smile} & f \text{ mapping} \\
\Leftrightarrow & m ; (m^{\smile} ; f)^{\boxast} ; f^{\smile} \sqcap c ; c^{\smile} \sqsubseteq f ; f^{\smile} \sqcap c ; c^{\smile} & \text{ordering} \\
\Leftrightarrow & m ; (m^{\smile} ; f)^{\boxast} ; f^{\smile} \sqcap c ; c^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{C}'} & \text{back tabulation}
\end{array}$$

This is exactly the condition ($*$). □

Let us name the diagonals of the top and bottom squares:

$$\begin{array}{rcccc}
q & := & m ; g & = & f ; n \\
Q & := & M ; G & = & F ; N
\end{array}$$

Now, since $m ; (m^{\smile} ; f)^{\boxast} ; f = m ; g ; n^{\smile} ; f = q ; q^{\smile}$, the condition ($*$) in Lemma 5.2.3 is equivalent to one of the conditions stating that $\mathcal{C} \xleftarrow{c} \mathcal{C}' \xrightarrow{q} \mathcal{D}$ is a tabulation for $Q ; d^{\smile}$:

$$q ; q^{\smile} \sqcap c ; c^{\smile} = \mathbb{I}_{\mathcal{C}'}$$

The other condition for this tabulation, cross-commutativity $c^{\smile} ; q = Q ; d^{\smile}$, follows (to choose one side) from the assumed cross-commutativity of the back square $c^{\smile} ; f = F ; b^{\smile}$ and the cross-commutativity of the right square $b^{\smile} ; n = N ; d^{\smile}$ which follows from the top gluing via Lemma 5.2.2 without use of $(*)$. Since these two tabulations together already produce the diagonal tabulation via Lemma 2.2.8, the "only if" aspect of Lemma 5.2.3 could be considered as irrelevant.

However, the advantage of the formulation of the condition $(*)$ in Lemma 5.2.3 lies in the fact that it is stated in terms of only m, f, and c, thanks to gluing cross-commutativity $g ; n^{\smile} = (m^{\smile} ; f)^{\circledast}$. With that, it is easy to see that the condition $(*)$ is equivalent to the following inclusion (or, equivalently, equation) in the lattice of equivalences on $\mathcal{C}'$:

$$(m ; m^{\smile} \vee f ; f^{\smile}) \wedge c ; c^{\smile} \leq \mathbb{I}_{\mathcal{C}'}$$

Since equivalence lattices are not necessarily distributive, we cannot derive this from the tabulation equations $m ; m^{\smile} \wedge c ; c^{\smile} = \mathbb{I}_{\mathcal{C}'}$ and $f ; f^{\smile} \wedge c ; c^{\smile} = \mathbb{I}_{\mathcal{C}'}$.

Theorem 5.2.4 In the category $\mathsf{Map}\,\mathbf{C}$ of maps over a bi-tabular collagory $\mathbf{C}$, *pushouts along injective maps are van Kampen squares.*

PROOF: Bi-tabularity guarantees that all pushouts in $\mathsf{Map}\,\mathbf{C}$ are gluings and all pullbacks in $\mathsf{Map}\,\mathbf{C}$ are tabulations.

Lemma 5.2.1 therefore shows the "only if" part of the definition of Van Kampen squares.

For the "if" part, assume a van Kampen setup where M is injective and the top square is a gluing. Lemma 2.2.6 implies that m is injective, too, which in turn implies difunctionality of $m^{\smile} ; f$ and the assumption $(*)$ of Lemma 5.2.3, which then shows that the front squares are tabulations, and therefore pullbacks. □

The main result of this section is now an immediate consequence of this theorem; note that we do not need difunctional (or transitive) closure for this:

Corollary 5.2.5 For a bi-tabular collagory $\mathbf{C}$ where all monos in $\mathsf{Map}\,\mathbf{C}$ are injective in $\mathbf{C}$, the mapping category $\mathsf{Map}\,\mathbf{C}$ is adhesive. □

(The restriction on monic mappings is necessary since there might, for example, be an object $\mathcal{A}$ in $\mathbf{C}$ for which the only mapping with target $\mathcal{A}$ is $\mathbb{I}_{\mathcal{A}}$; in that case, all mappings $f : \mathcal{A} \to \mathcal{B}$ would automatically be monos in $\mathsf{Map}\,\mathbf{C}$ regardless whether they are injective in $\mathbf{C}$. Note that f (together with identities) itself forms a tabulation and a co-tabulation for f.)

This result immediately makes the rewriting concepts and results from [LS04], including the local Church-Rosser theorem and the concurrency theorem, available for DPO rewriting defined via tabulations and co-tabulations in the context of collagories.

From Lemmas 5.2.1 and 5.2.3, we also directly obtain a characterisation of van Kampen squares in bitabular collagories:

Theorem 5.2.6 A gluing square (as in Def. 5.1.1(i)) in a bitabular collagory is van Kampen iff all its van Kampen setups (as in Def. 5.1.3) where the top square is a gluing satisfy the following:

$$m ; (m^{\smile} ; f)^{\circledast} ; f^{\smile} \sqcap c ; c^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{C}'} \qquad \square$$

5.3 Further Investigation of Van Kampen Squares

The following properties will be useful below:

Lemma 5.3.1 In a van Kampen setup where $M ; M^{\smile} \sqcap F ; F^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{C}}$, the following hold:

1. $f ; f^{\smile} \sqcap m ; m^{\smile} ; c ; c^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{C}'}$
2. $c ; c^{\smile} \sqcap m ; m^{\smile} ; f ; f^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{C}'}$

PROOF:

$$\begin{array}{lll}
1. & f ; f^{\smile} \sqcap m ; m^{\smile} ; c ; c^{\smile} & \\
= & f ; f^{\smile} \sqcap m ; a ; M^{\smile} ; c^{\smile} & \text{left tabulation} \\
= & f ; f^{\smile} \sqcap c ; M ; M^{\smile} ; c^{\smile} & \text{left commutativity} \\
= & f ; f^{\smile} \sqcap c ; (M ; M^{\smile} ; c^{\smile} \sqcap c^{\smile} ; f ; f^{\smile}) & \text{modal rule} \\
\sqsubseteq & f ; f^{\smile} \sqcap c ; (M ; M^{\smile} \sqcap c^{\smile} ; f ; f^{\smile} ; c) ; c^{\smile} & \text{modal rule} \\
= & f ; f^{\smile} \sqcap c ; (M ; M^{\smile} \sqcap F ; b^{\smile} ; b ; F^{\smile}) ; c^{\smile} & \text{back tabulation} \\
\sqsubseteq & f ; f^{\smile} \sqcap c ; (M ; M^{\smile} \sqcap F ; F^{\smile}) ; c^{\smile} & b \text{ univalent} \\
= & f ; f^{\smile} \sqcap c ; c^{\smile} & \text{assumption} \\
= & \mathbb{I}_{\mathcal{C}'} & \text{back tabulation} \\
2. & c ; c^{\smile} \sqcap m ; m^{\smile} ; f ; f^{\smile} & \\
= & c ; c^{\smile} \sqcap m ; m^{\smile} ; (f ; f^{\smile} \sqcap m ; m^{\smile} ; c ; c^{\smile}) & \text{modal rule} \\
= & c ; c^{\smile} \sqcap m ; m^{\smile} & \text{(i)} \\
= & \mathbb{I}_{\mathcal{C}'} & \text{left tabulation}
\end{array}$$

□

Injectivity of M makes $M^{\smile} ; F$ difunctional and also enforces injectivity of m and therewith difunctionality of $m^{\smile} ; f$.

In the general case, however, we have seen above that difunctionality of $m^{\smile} ; f$ requires not only difunctionality of $M^{\smile} ; F$, but also the front tabulation conditions.

This failure of difunctionality propagation can be understood as coming from the fact that in the difunctionality inclusion $M^{\smile} ; F ; F^{\smile} ; M ; M^{\smile} ; F \sqsubseteq M^{\smile} ; F$, the right-hand side passes through a "$\mathcal{C}$ element" that may be distinct from the three "$\mathcal{C}$ elements" of the left-hand side.

This distinct "$\mathcal{C}$ element" gives rise to a "$\mathcal{C}'$ element" that is, in the absence of the front tabulation conditions, determined only up to $c ; c^{\smile}$.

One way to avoid this unwanted factor is to specify that in any chain diagram documenting $M ; M^{\smile} ; F ; F^{\smile} ; M ; M^{\smile}$, the fourth (i.e., last) $\mathcal{C}$ element needs to be one of the previous three $\mathcal{C}$ elements. Referring to so many elements simultaneously in a relation-algebraic way requires direct products — we use π and ρ as the projections. The following is one formulation of this condition:

$$M ; M^{\smile} ; (\pi^{\smile} \sqcap F ; F^{\smile} ; M ; M^{\smile} ; \rho^{\smile}) \sqsubseteq M ; M^{\smile} ; (\pi^{\smile} \sqcap (F ; F^{\smile} \sqcup M ; M^{\smile}) ; \rho^{\smile})$$

However, it is not hard to see that this is equivalent to the following, much simpler condition:

$$F ; F^{\smile} ; M ; M^{\smile} \sqsubseteq F ; F^{\smile} \sqcup M ; M^{\smile}$$

This is obviously satisfied if one of M and F is injective. It can also be strengthened to an equality, since M and F are both total. This implies symmetry:

$$F ; F^{\smile} ; M ; M^{\smile} = F ; F^{\smile} \sqcup M ; M^{\smile} = M ; M^{\smile} ; F ; F^{\smile}$$

and, furthermore, difunctionality of $M^{\smile} ; F$:

$$M^{\smile} ; F ; F^{\smile} ; M ; M^{\smile} ; F = M^{\smile} ; M ; M^{\smile} ; F ; F^{\smile} ; F = M^{\smile} ; F \ .$$

Assuming also $M ; M^{\smile} \sqcap F ; F^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{C}}$, we obtain $f ; f^{\smile} ; m ; m^{\smile} = f ; f^{\smile} \sqcup m ; m^{\smile}$:

$$\begin{array}{lll}
& f ; f^{\smile} ; m ; m^{\smile} & \\
= & f ; f^{\smile} ; m ; m^{\smile} \sqcap c ; F ; F^{\smile} ; M ; M^{\smile} ; c^{\smile} & \\
\sqsubseteq & f ; f^{\smile} ; m ; m^{\smile} \sqcap c ; (F ; F^{\smile} \sqcup M ; M^{\smile}) ; c^{\smile} & \text{assumption} \\
= & f ; f^{\smile} ; m ; m^{\smile} \sqcap (c ; c^{\smile} ; f ; f^{\smile} \sqcup c ; c^{\smile} ; m ; m^{\smile}) & \\
= & (f ; f^{\smile} ; m ; m^{\smile} \sqcap c ; c^{\smile} ; f ; f^{\smile}) \sqcup (f ; f^{\smile} ; m ; m^{\smile} \sqcap c ; c^{\smile} ; m ; m^{\smile}) & \\
\sqsubseteq & f ; f^{\smile} \sqcup m ; m^{\smile} & \text{Lemma 5.3.1}
\end{array}$$

Therefore, $m^{\smile} ; f$ is difunctional, too, and together with Lemma 5.3.1 we obtain

$$m ; (m^{\smile} ; f)^{\circledast} ; f^{\smile} \sqcap c ; c^{\smile} = m ; m^{\smile} ; f ; f^{\smile} \sqcap c ; c^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{C}'} \ .$$

Altogether we have shown the following:

Theorem 5.3.2 In the category $\mathsf{Map}\,\mathbf{C}$ of maps over a bi-tabular collagory $\mathbf{C}$, pushouts satisfying also

$$F ; F^{\smile} \sqcap M ; M^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{C}}$$

and

$$F ; F^{\smile} ; M ; M^{\smile} \sqsubseteq F ; F^{\smile} \sqcup M ; M^{\smile}$$

are van Kampen squares. □

Both inclusions can be strengthened to equalities, and since the second condition implies difunctionality, both together imply that such pushouts are also pullbacks.

Chapter 6

Conclusion

We have streamlined the axiomatic basis of the relation-algebraic approach to graph structure transformation by introducing *collagories*, which, in comparison to earlier approaches, remove consideration of the zero-law and, to a certain extent, of difunctional closure defined via the Kleene star.

We showed that the concepts of tabulation and co-tabulation, which are essential for the relation-algebraic rewriting approach, can be formalised in collagories, and that the category of mappings in a bi-tabular collagory forms an *adhesive category*, thus establishing a powerful connection to the categorical approach to graph structure transformation. Our proof that co-tabulation are bicolimits appears to be somewhat related to Milius' results about colimit constructions in the 2-category of relations over a category [Mil03], and is a first step towards establishing connections between our characterisation of van Kampen squares in mapping categories in collagories with the characterisation as bicolimits in the bicategory of spans given by Heindel and Sobociński [HS09].

As shown in Chapter 3, all the important examples of adhesive categories can also be obtained as special cases of powerful collagory constructions; future work will investigate whether (respectively when) the category of relations [FS90, 1.412] in an adhesive category forms a collagory. Another interesting goal would be to identify a nicer collagory-level formulation of the van Kampen property, and establish connections with the characterisation as bicolimits in the bicategory of spans given by Heindel and Sobociński [HS09].

Further investigations will explore different variations of adhesive categories in a collagory setting, including the quasiadhesive categories of [LS05], and their applications.

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