Computation of Probabilistic Supervisory Controllers for Model Matching

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Abstract

Probabilistic discrete event systems (PDES) generalize discrete event systems (DES) by attaching an occurence probability to each event so that the underlying DES becomes a generator of a probabilistic language. In this paper PDES supervisors generalize DES supervisors by attaching a probability to the enablement of each controllable events that is updated after each event observation. When an event is disabled, its probability is redistributed via the probability distrubution conditioned on the remaining possible event outcomes. The control problem considered is to find, if possible, a probabilistic supervisory controller such that the probabilistic language generated by the closed loop system matches a given probabilistic specification language. In [6] necessary and sufficient conditions for the existence of a probabilistic supervisor were stated with partial proof. This paper completes the proof of the conditions and provides an algorithm that can be used to compute a solution to the model matching problem when it exists.

1 Introduction

Ramadge and Wonham first developed the Supervisory Control Problem (SCP) for Discrete Event Systems (DES) in [9]. The main concern of the supervisory control problem is to ensure that only acceptable strings or sequences of events occur. Although the deterministic language framework applied in [9] allows for nondeterminism in the sense that there may be more than one possible continuation of a string, there is no effort made to quantify this randomness. It is assumed that the choice of a possible continuation of a string is made by some internal structure unmodeled by the systems designer.

As pointed out in [2, 3], many DES have noise associated with them that we may be able to model by assigning probabilities to the possible one step continuations of a string. This results in the use of probabilistic languages over a set of events as the underlying model for these discrete event systems. In [2, 3] Garg *et al.* develop an algebra to model probabilistic languages.This treatment differs from the optimal control theory of Markov chains [1] and the application of supervisory control theory to Semi-Markov Decision

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problems [4], in its emphasis on sequences of events rather than states and state sequences. The work also allows for the possibility of termination of a system after completing a finite sequence of events. The algebraic theory is used to build complex models of probabilistic DES by combining subsystems using the defined algebraic operators but possible control mechanisms to alter the probabilistic behavior of the systems are not explicitly discussed.

A control mechanism for PDES was first introduced in [6] in the form of probabilistic supervisors which use the same event disablement control technology as employed in the control of DES [9]. The (deterministic) supervisors of [9] are generalized to probabilistic supervisors, allowing them to perform "probabilistic enablement" of controllable events. The standard deterministic supervisors later applied to PDES in [5] can be viewed as a special case of those used in [6].

These supervisors differ from the stochastic supervisors of [7] in that the probabilistic supervisors enable or disable a controllable event in the underlying deterministic automata with a certain probability according to the string observed thus far. The stochastic supervisors of [7] can disable a controllable event, or enable it and choose the probabilities with which the occurrence of the controllable event causes the system to move from the current state to any other state of the system in the underlying nondeterministic automata. In contrast, the standard deterministic controls for PDES of [5] and the probabilistic controls of [6] can simply enable or disable controllable transitions. They cannot directly change the underlying plant dynamics, but rather the effect of disablement is entirely determined by the plant.

We note that the main problem posed in [5] is to obtain a (deterministic) controller such that "the probabilistic language of the controlled system lies within a prespecified range, where the upper bound is a nonprobabilistic language representing a legality constraint." A condition for the existence of a (deterministic) supervisor solving the problem and an algorithm to test the existence condition when the probabilistic languages are regular are given. The probabilistic model matching problem describe in section 3 is a special case of this control problem where the upper bound languange is simply the support for the lower bound probabilistic language.

Probabilistic supervisors will be described in detail in the next section where we define the modeling framework. Section 3 defines the standard deterministic supervisory control model matching problem in terms of our notation and then extends it to the probabilistic case, restating the conditions given in [6]. A proof of the necessity and sufficiency of the conditions follows along with an probability matching algorithm that can be used in conjunction with existing supervisory control algorithms to compute a probabilistic supervisory controller.

2 Modeling Framework

A (deterministic) *system* S is a prefix-closed subset of Σ^* that contains the empty string ϵ. It can be equivalently represented by its *active event function*, which is a partial function $S_\Delta : \Sigma^* \to 2^\Sigma$. For $s \in \Sigma^*$ and $\sigma \in \Sigma$, if we write $S_\Delta(s)!$ to denote that $S_\Delta(s)$ is defined, then $S_\Delta(s)!$ iff $s \in S$ and $S_\Delta(s) = \{\sigma \in \Sigma | s\sigma \in S\}$. Since S is prefix closed $S(\epsilon)!$ and clearly $S_\Delta(s\sigma)!$ iff $S_\Delta(s)!$ and $\sigma \in S_\Delta(s)$. Using the latter representation, each element $s \in \operatorname{dom} S_\Delta$ is a state of the system, and ϵ is the initial state. In operation, the system is initialized in state ϵ and when in state s, it terminates if no events are active (i.e., $S_\Delta(s) = \emptyset$), or selects an event $\sigma \in S_\Delta(s)$ and moves to state $s\sigma$.

A *probabilistic system* is a pair (S, f), where $S \subseteq \Sigma^*$ is a system, and $f : \Sigma^* \to [0, 1]$

is an order-preserving function such that $f(\epsilon) = 1$, $supp(f) = S$, and $\sum_{\sigma\in\Sigma} f(s\sigma) \leq 1$. Here $supp(f)$ denotes the *support* of f, i.e., $\{s \in \Sigma^* | f(s) > 0\}$ and $f(s)$ is interpreted as the probability that string s occurs in the system. The system can be equivalently represented by the pair (S_Δ, S_ρ), where S_Δ is the active event function defined above, and $S_\rho : \mathrm{dom}\, S_\Delta \to [0,1]^\Sigma$ is a function such that $supp(S_\rho(s)) = S_\Delta(s)$, and $S_\rho(s)(\sigma) = f(s\sigma)/f(s)$ when $f(s) > 0$. When defined, $S_\rho(s)(\sigma)$ represents the probability that event σ occurs, given that the system is in state s. The extension of S_ρ is the function $S'_\rho : \mathrm{dom}\, S_\Delta \to [0,1]^{\Sigma\cup\perp}$ given by $S'_\rho(s)(\sigma) = S_\rho(s)(\sigma)$ when defined and $S'_\rho(s)(\perp) = 1 - \sum_\sigma S_\rho(s)(\sigma)$. Here $S'_\rho(s)(\perp)$ is the probability that the system terminates at state s. When in operation, the system is initialized in state ϵ. At each state s, the system terminates if $S_\Delta(s)$ is empty, or randomly selects an event $\sigma \in S_\Delta(s) \cup \{\perp\}$ such that σ has distribution $S'_\rho(s)$. If $\sigma = \perp$, the system terminates; otherwise the system moves to state $s\sigma$.

2.1 Control

A control structure on a system S is a set $\Sigma_c \subseteq \Sigma$ of controllable events. The set $\Sigma_u = \Sigma \backslash \Sigma_c$ is the set of uncontrollable events. A control input (enablement pattern) is an element $\theta \in 2^{\Sigma_c}$. The control system is the function $S^C_\Delta : \mathrm{dom}\, S_\Delta \to (2^{\Sigma_c} \to 2^\Sigma)$ such that $S^C_\Delta(s)(\theta) = S_\Delta(s) \cap (\Sigma_u \cup \theta)$. A control policy for S^C is a mapping $\theta : \Sigma^* \to 2^{\Sigma_c}$ such that $\theta(\epsilon)!$ and when $\theta(s)!$ and $S_\Delta(s)!$, for any $\sigma \in S^C(s)(\theta(s))$, $\theta(s\sigma)!$. The system under the supervision of deterministic control policy θ is the system S^θ, where $S^\theta_\Delta(\epsilon)!$ and $S^\theta_\Delta(s)! \Rightarrow S^\theta_\Delta(s) = S_\Delta(s) \cap (\Sigma_u \cup \theta(s))$.

When this control structure is applied to a probabilistic system (S_Δ, S_ρ), the probabilistic control system is the pair (S^C_Δ, S^C_ρ) such that $S^C_\rho : \mathrm{dom}\, S_\Delta \to (2^\Sigma_c \to [0,1]^{\Sigma\cup\{\perp\}})$ defined when $S^C_\Delta(s)(\theta)$ is nonempty, and given by $S^C_\rho(s)(\theta)(\sigma) = S'_\rho(\sigma | S^C_\Delta(s)(\theta))$. The probabilistic system under the supervision of deterministic control policy θ is $(S^\theta_\Delta, S^\theta_\rho)$, such that S^θ_Δ is the system under the control of θ, and $S^\theta_\rho : \mathrm{dom}\, S^\theta_\Delta \to [0,1]^{\Sigma\cup\{\perp\}}$ is such that when $S^\theta_\Delta(s)$ is nonempty, $S^\theta_\rho(s) = S^C_\rho(s)(\theta(s)) = S'_\rho(\sigma | S^\theta_\Delta(s))$.

In [6], Lawford *et al.* proposed a novel extension to this control paradigm in which control inputs are enablement probabilities on controllable events. Specifically, each active controllable event σ at state s is associated with a Bernoulli random variable X^s_σ with success probability $v^s(\sigma)$. The event σ is enabled at state s iff $X^s_\sigma = 1$. The probability that a particular enablement pattern θ occurs at state s is given by

$$P(v^s)(\theta) = \prod_{\sigma\in\theta} v^s(\sigma) \prod_{\sigma\in\Sigma_c\backslash\theta} (1 - v^s(\sigma)) \tag{1}$$

A control input is an element $v \in [0,1]^{\Sigma_c}$. Let $v_\Delta = supp(v)$. The probabilistic system with probabilistic enablement control structure is the pair $(S^{PC}_\Delta, S^{PC}_\rho)$, where $S^{PC}_\Delta : \mathrm{dom}\, S_\Delta \to [0,1]^{\Sigma_c} \to 2^{\Sigma\cup\{\perp\}}$, is given by $S^{PC}_\Delta(s)(v) = S^C_\Delta(v_\Delta)$, and $S^{PC}_\rho : \mathrm{dom}\, S^{PC}_\Delta \to [0,1]^{\Sigma_c} \to [0,1]^{\Sigma\cup\{\perp\}}$ is defined when $S^{PC}_\Delta(v)$ is nonempty and given by $S^{PC}_\rho(s)(v)(\sigma) = \sum_{\theta\in 2^{\Sigma_c}, S^C_\Delta(s)(\theta)\neq\emptyset} S^C_\rho(s)(\theta)(\sigma)P(v)(\theta)$. A control policy for S^{PC} is a mapping $\theta : \Sigma^* \to [0,1]^{\Sigma_c}$ such that $\theta(\epsilon)!$ and when $\theta(s)!$ and $S_\Delta(s)!$, for any $\sigma \in S^{PC}(s)(\theta(s))$, $\theta(s\sigma)!$. Let $\theta_\Delta : \mathrm{dom}\, \theta \to 2^{\Sigma_c}$ such that be the function such that $\theta_\Delta(s) = supp(\theta(s))$. The controlled probabilistic system is $S^{PC,\theta}$, where $S^{PC,\theta}_\Delta$ is the system under the control of θ_Δ, and $S^{PC,\theta}_\rho$ is such that when $S^{PC,\theta}_\Delta$ is nonempty, $S^{PC,\theta}_\rho(s) = S^{PC}_\rho(s)(\theta(s))$.

3 Matching Problem

The standard (deterministic) system matching problem is to construct a control policy θ such that $S^\theta = T$, where $T \subseteq \Sigma^*$ is a specified system. Following [9] we obtain:

Lemma 1 *There is a control policy θ such that $S^\theta = T$ iff when $T_\Delta(s)!$, $S_\Delta(s)!$ and $S_\Delta(s) \cap \Sigma_u = T_\Delta(s) \cap \Sigma_u$, and $S_\Delta(s) \cap \Sigma_c \supseteq T_\Delta(s) \cap \Sigma_c$.*

For probabilistic systems, the matching problem is identically stated, except now an additional condition (probability matching) is required. The primary result of [6] was the conjecture that a probability was matchable iff a linear constraint was feasible.

Conjecture 1 *There is a probabilistic control $\theta = (\theta_\Delta, \theta_\rho)$ such that $(S^\theta_\Delta, S^\theta_\rho) = (T_\Delta, T_\rho)$ iff when $T_\Delta(s)!$, $S_\Delta(s)!$, $S_\Delta(s) \cap \Sigma_u = T_\Delta(s) \cap \Sigma_u$, $S_\Delta(s) \cap \Sigma_c \supseteq T_\Delta(s) \cap \Sigma_c$ and*

1. *for all $i \in S_\Delta(s) \cap \Sigma_c$, $\frac{1 - \sum_{j \in \Sigma_c} S_\rho(s)(j)}{S_\rho(s)(i)} T_\rho(s)(i) + \sum_{j \in \Sigma_c} T_\rho(s)(j) \leq 1$*

2. *for all $i \in S_\Delta(s) \cap \Sigma_u$, $T_\rho(s)(i) = \frac{\sum_{j \in \Sigma_u} T_\rho(s)(j)}{\sum_{j \in \Sigma_u} S_\rho(s)(j)} S_\rho(s)(i)$*

The contribution of this paper are (i) the demonstration of the necessity and sufficiency of conditions (1) and (2) above for probability matching using the probabilistic enablement control technology, and (ii) a straightforward constructive method for computation of the enablement probability vector $\theta(s)$.

3.1 Probabilities

Let A be a finite set of outcomes. A probability function is a mapping $\rho \in [0,1]^A$ such that $\sum_{i \in A} \rho_i = 1$. Let $\mathcal{P}(A)$ denote the collection of probability functions on A. An event is an element $B \in 2^A$. The definition of ρ is extended to events by writing $\rho_B := \sum_{i \in B} \rho_i$. The probability that $i \in A$ occurs conditioned on the occurrence of event B where $\rho_B > 0$ is given by $\rho_{i|B} = \chi_B(i)\rho_i/\rho_B$, where $\chi_B(i)$ is the indicator function. Let $\mathcal{S}(A)$ denote the collection of functions $\rho \in [0,1]^A$ such that $\sum_{i \in A} \rho_i \leq 1$. Any $\rho \in \mathcal{S}(A)$, can be extended to form a probability function $\rho' \in \mathcal{P}(A \cup \perp)$ where $\rho'|A = \rho$ and $\rho'_\perp = 1 - \rho_A$.

The following result is used for chaining probabilities together via conditioning on an event function β that relates an outcome in the triggering space to an event in the accepting space. The chaining is valid if whenever a triggering outcome occurs, the event it triggers can actually occur in the accepting space.

Lemma 2 *Let $\rho_1 \in \mathcal{P}(A_1)$, $\rho_2 \in \mathcal{P}(A_2)$ and $\beta : A_1 \to 2^{A_2}$. If $\rho_2(\beta(i)) > 0$ when $\rho_1(i) > 0$, then $\gamma \in \mathcal{P}(A_2)$, where $\gamma(j) = \sum_{\substack{i \in A_1 \\ \rho_2(\beta(i))>0}} \rho_2(j|\beta(i))\rho_1(i)$*

Proof: The proof is immediate on observing that the additional constraint $\rho_1(i) > 0$ can be added without altering the sum:

$$\sum_{j \in A_2} \gamma(j) = \sum_{j \in A_2} \sum_{\substack{i \in A_1 \\ \rho_2(\beta(i))>0 \\ \rho_1(i)>0}} \rho_2(j|\beta(i))\rho_1(i) = \sum_{j \in A_2} \sum_{\substack{i \in A_1 \\ \rho_1(i)>0}} \chi_{\beta(i)}(j) \frac{\rho_2(j)}{\rho_2(\beta(i))} \rho_1(i) = \sum_{\substack{i \in A_1 \\ \rho_1(i)>0}} \rho_1(i) = 1 \quad (2)$$

$\square$

4 Probability Matching

Problem Statement 1 *(matching problem) Given $\rho, Q \in \mathcal{S}(\Sigma)$ and control structure Σ_c, find necessary and sufficient conditions for the existence of an enablement function $v \in [0,1]^{\Sigma_c}$ such that $P(v) = Q$, where $P \in [0,1]^{\Sigma_c} \to \mathcal{S}(\Sigma)$ given by*

$$P_i(v) := P(v)(i) = \sum_{\substack{\theta \in 2^{\Sigma_c} \\ \rho'(\theta \cup \Sigma_u \cup \perp) > 0}} \rho'(i|\theta \cup \Sigma_u \cup \perp) \cdot \prod_{j \in \theta} v_j \prod_{j \in \Sigma_c \setminus \theta} (1 - v_j) \tag{3}$$

Furthermore, provide a procedure to compute a $v \in [0,1]^{\Sigma_c}$ such that $P(v) = Q$.

Intuitively, $P_i(v)$ is interpreted as the probability that event $i \in \Sigma$ occurs, given enablement probability vector v is applied to the system. The probability that the system generates no event (terminates) is $P'(v)(\perp)$.

4.1 Assumptions

For existence of a solution, it will be necessary to assume that whenever $\rho_i = 0$, $Q_i = 0$, since the probability $P_i(v)$ is directly proportional to ρ_i. Without loss of generality, it is assumed that $\rho_i > 0$ when $i \in \Sigma_c$. If $\rho(\Sigma_c) = 1$, it follows that $\rho(\Sigma_u) = 0$, so $\rho_i = 0$ for all $i \in \Sigma_u$ and the matching problem effectively reduces to a matching problem for controllable events only. Hence, we will assume that $\rho(\Sigma_c) < 1$ (this condition is removed in [8]). Under these assumptions, $P'(v)$ is a probability function. For notational convenience it is assumed that $\Sigma_c = \{1, 2, \dots n\}$ and $\Sigma_u = \{n+1, \dots, n+m\}$.

Lemma 3 *Let $x \in \mathbb{R}^k$. Then $1 = \sum_{\mu \in 2^{\Sigma_c}} \prod_{j \in \mu} x_j \prod_{j \in \Sigma_c \setminus \mu} (1 - x_j)$*

Proof: We prove this by induction. The stated identity is valid when $k = 1$. Assume it is valid when $k < n$. Let $B = \{1, \dots, n-1\}$.

$$\sum_{\mu \in 2^{\Sigma_c}} \prod_{j \in \mu} x_j \prod_{j \in \Sigma_c \setminus \mu} (1 - x_j) = \sum_{\mu \in 2^B} (x_n + 1 - x_n) \prod_{j \in \mu} x_j \prod_{j \in B \setminus \mu} (1 - x_j) = \sum_{\mu \in 2^B} \prod_{j \in \mu} x_j \prod_{j \in B \setminus \mu} (1 - x_j) = 1$$

□

Lemma 4 *for all $v \in [0,1]^{\Sigma_c}$, $P'(v) \in \mathcal{P}(\Sigma \cup \perp)$.*

Proof: Let $v \in [0,1]^{\Sigma_c}$. Then $e : 2^{\Sigma_c} \to [0,1]$ given by $e(\theta) = \prod_{j \in \theta} v_j \prod_{j \in \Sigma_c \setminus \theta} (1 - v_j)$ is such that $e \in \mathcal{P}(2^{\Sigma_c})$, by lemma 3. Let $\beta : 2^{\Sigma_c} \to 2^{\Sigma}$ be given by $\beta(\theta) = \theta \cup \Sigma_u \cup \perp$. When $e(\theta) > 0$, $\rho'(\beta(\theta)) > \rho'(\Sigma_u \cup \perp) = 1 - \rho(\Sigma_c) > 0$. Hence, by lemma 2, $P'(v) \in \mathcal{P}(\Sigma \cup \perp)$. □

4.2 Characterization of the Set of Assignable Distributions

In this section, interpret set deletion as being by default left-associative result characterizes the probabilities of controllable and uncontrollable events.

Lemma 5 *Let $x \in [0,1]^n$. $P_i(x) = \rho_i x_i h_i(x)$ when $i \in \Sigma_c$ and $P_i(x) = \rho_i H(x)$ when $i \in \Sigma_u$, where $h_i, H : \mathbb{R}^n \to \mathbb{R}$ are given by*

$$h_i(x) = \sum_{\theta \in 2^{\Sigma_c \setminus i}} \frac{1}{1 - \sum_{j \in \theta} \rho_j} \prod_{j \in \theta} (1 - x_j) \prod_{j \in \Sigma_c \setminus i \setminus \theta} x_j \qquad H(x) = \sum_{\theta \in 2^{\Sigma_c}} \frac{1}{1 - \sum_{j \in \theta} \rho_j} \prod_{j \in \theta} (1 - x_j) \prod_{j \in \Sigma_c \setminus \theta} x_j \tag{4}$$

Proof: Let $x \in [0,1]^n$ and $i \in \{1,2,\ldots,n+m\}$. Under the assumption that $\rho(\Sigma_c) < 1$, for any $\theta \in 2^{\Sigma}$, $\rho'(\theta \cup \Sigma_u \cup \bot) \geq \rho'(\Sigma_u) + \rho'(\bot) = 1 - \rho(\Sigma_c) > 0$. Equation 3 may be equivalently expressed by

$$\begin{aligned} P_i(x) &= \sum_{\theta \in 2^{\Sigma_c}} \chi_{(\Sigma_c \backslash \theta) \cup \Sigma_u \cup \bot}(i) \frac{\rho_i}{\rho_{(\Sigma_c \backslash \theta) \cup \Sigma_u}} \prod_{i \in \theta}(1 - x_i) \prod_{i \in \Sigma_c \backslash \theta} x_i & (5) \\ &= \sum_{\theta \in 2^{\Sigma_c}} \chi_{\Sigma \backslash \theta}(i) \frac{\rho_i}{1 - \sum\limits_{j \in \theta} \rho_j} \prod_{i \in \theta}(1 - x_i) \prod_{i \in \Sigma_c \backslash \theta} x_i & (6) \end{aligned}$$

upon application of the isomorpshim β on 2^{Σ_c} given by $\beta(q) = \Sigma_c \backslash q$. If $i \in \Sigma_c$,

$$P_i(x) = \sum_{\substack{\theta \in 2^{\Sigma_c} \\ i \notin \theta}} \frac{\rho_i}{1 - \sum\limits_{j \in \theta} \rho_j} \prod_{j \in \theta}(1 - x_j) \prod_{j \in \Sigma_c \backslash \theta} x_j = \sum_{\theta \in 2^{\Sigma_c \backslash i}} \frac{\rho_i}{1 - \sum\limits_{j \in \theta} \rho_j} \prod_{j \in \theta}(1 - x_j) \cdot \left(x_i \prod_{j \in \Sigma_c \backslash i \backslash \theta} x_j \right) \quad (7)$$

from which it can be inferred that $P_i(x) = \rho_i x_i h_i(x)$. If $i \in \Sigma_u$, it is always true that $\chi_{\Sigma \backslash \theta}(i) = 1$ when $\theta \in 2^{\Sigma_c}$ and it can be inferred that $P_i(x) = \rho_i H(x)$. □

Equip $\mathbb{R}^n$ with the componentwise order induced from $\mathbb{R}$. For $a, b \in \mathbb{R}^n$ such that $a \leq b$, the order interval is given by $\langle a, b \rangle = \{x \in \mathbb{R}^n \mid 0 \leq x \leq 1\}$. Let $f : (X, \leq) \to (Y, \leq)$ be a mapping between posets. It is isotone if $f(x) \leq f(y)$ when $x \leq y$, and antitone when $f(y) \leq f(x)$ when $x \leq y$.

Lemma 6 *Let $R \in 2^{\Sigma_c}$ and $q : (2^{\Sigma_c}, \subseteq) \to \mathbb{R}$ be positive and isotone. $\Psi_R : \mathbb{R}^n \to \mathbb{R}$ given by $\Psi_R(x) = \sum\limits_{S \in 2^R} q(S) \prod\limits_{i \in S}(1 - x_i) \prod\limits_{i \in R \backslash S} x_i$ is positive and antitone on $\langle 0, 1 \rangle$.*

Proof: Assume $q : 2^R \to \mathbb{R}$ is positive and isotone. Let $R \subseteq \Sigma_c$ and $x \in \langle 0, 1 \rangle$. When $j \notin R$, Ψ_R does not depend on x_j and $\frac{\partial \Psi_R}{\partial x_j}(x) = 0$. When $j \in R$,

$$\begin{aligned} \Psi_R(x) &= \sum_{\substack{S \in 2^R \\ j \in S}} q(S) \prod_{i \in S}(1 - x_i) \prod_{i \in R \backslash S} x_i + \sum_{\substack{S \in 2^R \\ j \notin S}} q(S) \prod_{i \in S}(1 - x_i) \prod_{i \in R \backslash S} x_i & (8) \\ &= \sum_{S \in 2^{R \backslash j}} \left(q(S \cup j)(1 - x_j) + q(S) x_j\right) \prod_{i \in S}(1 - x_i) \prod_{i \in R \backslash j \backslash S} x_i & (9) \end{aligned}$$

and so

$$\frac{\partial \Psi_R}{\partial x_j}(x) = (-1) \cdot \sum_{S \in 2^{R \backslash j}} \left(q(S \cup j) - q(S)\right) \prod_{i \in S}(1 - x_i) \prod_{i \in R \backslash j \backslash S} x_i \leq 0 \quad (10)$$

Let $x, y \in \langle 0, 1 \rangle$ be such that $x \leq y$, and let $\phi : [0,1] \to \langle 0, 1 \rangle$ be the path $\phi(t) = (1-t)x + ty$. Then $\Psi_R(y) = \Psi_R(x) + \int_0^1 \frac{\partial \Psi_R}{\partial x}(\phi(t))(y - x)dt \leq \Psi_R(x)$ so Ψ_R is antitone. Since Ψ_R is antitone on $\langle 0, 1 \rangle$, $\Psi_R(x) \geq \Psi_R(1) = q(\emptyset) > 0$, so it is positive. □

Lemma 7 *Let $\rho \in \mathbb{R}^n$ be a positive vector such that $\sum_i \rho_i < 1$. Let $i \in \Sigma_c$. Then H and h_i are positive antitone, and such that $x_i h_i(x) \leq H(x)$ on $[0,1]^n$*

Proof: Let $R \subseteq \Sigma_c$ be a nonempty. Let $q : 2^R \to \mathbb{R}$ be given by $q(S) = \frac{1}{1 - \sum\limits_{i \in S} \rho_i}$. Let $A, B \in 2^R$ be such that $A \subseteq B$. Then $0 \leq \sum_{i \in A} \rho_i \leq \sum_{i \in B} \rho_i < 1$. Since $1/(1-x)$ is positive isotone on $[0, 1)$, it follows that $q(A) \le q(B)$, so q is isotone. From lemma 6, we can immediately conclude that H (choose $R = \Sigma_c$) and h_i (choose $R = \Sigma_c \backslash i$) are positive antitone. To establish $x_i h_i(x) \le H(x)$, let $R = \Sigma_c \backslash i$. Then

$$\begin{aligned} x_i h_i(x) &= x_i \sum_{\theta \in 2^{\Sigma_c \backslash i}} q(\theta) \prod_{j \in \theta} (1 - x_i) \prod_{j \in \Sigma_c \backslash i \backslash \theta} x_i = \sum_{\theta \in 2^{\Sigma_c \backslash i}} q(\theta) \prod_{j \in \theta} (1 - x_i) \prod_{j \in \Sigma_c \backslash \theta} x_i \quad (11) \\ &\le \sum_{\theta \in 2^{\Sigma_c}} q(\theta) \prod_{j \in \theta} (1 - x_i) \prod_{j \in \Sigma_c \backslash \theta} x_i = H(x) \quad (12) \end{aligned}$$

□

The following lemma provides sufficient conditions for the existence of a fixpoint of a mapping and is used in the main result. Note that a map $\phi : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is isotone if $\phi(x) \le \phi(y)$ when $x \le y$, and is lower continuous if $\lim\limits_{v_k \uparrow v} \phi(x) = \phi(v)$.

Lemma 8 *Let $\phi : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be isotone on D and suppose that $x_0 \le y_0$, $\langle x_0, y_0 \rangle \subseteq D$, $x_0 \le \phi(x_0)$. Suppose that $v \le y_0$ when $v \in \langle x_0, y_0 \rangle$ and $v \le \phi(v)$. Then the sequence $\{v^k\}$ given by*

$$v^0 = x_0, \quad v^{k+1} = \phi(v^k), \quad k = 0, 1, \ldots, \tag{13}$$

exists and is such that $v^k \uparrow v^*$ for some $v^* \in \langle x_0, y_0 \rangle$. If furthermore ϕ is lower continuous, then $v^* = \phi(v^*)$.

Proof: We first show by induction that the sequence $\{v^k\}$ exists and is a non-decreasing chain contained in $\langle x_0, y_0 \rangle$. Since $x_0 \le \phi(x_0) \le y_0$, the basis step is true. Assume that for $j \le k$, v^j exists and form a non-decreasing chain contained in $\langle x_0, y_0 \rangle$. Then $v^{k+1} = \phi(v^k)$ is defined since $v^k \in D$. Secondly, $v^k \le v^{k+1}$ since $v^{k-1} \le v^k$ and ϕ is isotone. Finally, $v^{k+1} \le y_0$. Hence for $j \le k + 1$, v^j exists and forms a non-decreasing chain contained in $\langle x_0, y_0 \rangle$, completing the induction proof.

Since $\{v^k\}$ is non-decreasing and has a finite upper bound y_0 it must converge to a point $v^* \le y_0$. Since ϕ is lower continuous, $v^k \uparrow v^*$ implies $v^{k+1} \to \phi(v^*)$, and so $v^* = \phi(v^*)$. □

The main result establishes the necessary and sufficient conditions for an controllable event to be assignable to a given probability, and can also be used to compute a probabilistic disablement policy via fixpoint iteration.

Theorem 9 *Let $\rho, Q \in \mathbb{R}^n$, where ρ is a positive vector such that $\sum\limits_i \rho_i < 1$ and Q is a non-negative vector such that*

$$\frac{1 - \sum\limits_{j \ne i} \rho_j}{\rho_i} Q_i + \sum_{j \ne i} Q_j \le 1 \tag{14}$$

Then the sequence $\{v^k\}$ given by

$$v^0 = 0, \quad v^{k+1} = \phi(v^k) \quad k = 0, 1, \ldots \qquad \textit{where} \quad \phi_i(x) = \frac{Q_i}{\rho_i h_i(x)}, i \in \Sigma_c \tag{15}$$

exists and is such that $v^k \uparrow v^*$ for some $v^* \in \langle 0, 1 \rangle$. Furthermore, $P_i(v^*) = Q_i$ for all $i \in \Sigma_c$. Conversely, for any $x \in \langle 0, 1 \rangle$, equation 14 is satisfied with $Q_i \triangleq P_i(x)$ for all $i \in \Sigma_c$.

Proof: We first show that ϕ is defined on $\langle 0,1\rangle$ and isotone. Since ρ is positive and $\sum_i \rho_i < 1$, by lemma 7 h_i is positive and non-increasing on on $\langle 0,1\rangle$. Hence ϕ_i is positive and isotone on $\langle 0,1\rangle$, and $0 \le \phi(0)$.

Next we show that $\phi(x) \le 1$ when $x \le \phi(x)$. Let $x \in \langle 0,1\rangle$ and suppose that $x \le \phi(x)$. Then $P_i(x) = \rho_i x_i h_i(x) \le Q_i$ for $i = 1,\dots,n$. Let $\Delta_i = \frac{1-\sum_{j\neq i}\rho_j}{\rho_i}$. Note that Δ_i is well-defined since $\rho_i > 0$.

$$\begin{aligned}
\Delta_i \rho_i h_i(x) &= \sum_{\mu\in 2^{\Sigma_c\setminus i}} \left(1 - \frac{\sum_{j\in\Sigma_c\setminus i\setminus\mu}\rho_j}{1-\sum_{j\in\mu}\rho_j}\right)\prod_{j\in\mu}(1-x_j)\prod_{j\in\Sigma_c\setminus i\setminus\mu} x_j && (16)\\
&=^A 1 - \sum_{\mu\in 2^{\Sigma_c\setminus i}}\sum_{j\in\Sigma_c\setminus i\setminus\mu}\frac{\rho_j}{1-\sum_{j\in\mu}\rho_j}\prod_{j\in\mu}(1-x_j)\prod_{j\in\Sigma_c\setminus i\setminus\mu} x_j && (17)
\end{aligned}$$

where (A) follows from lemma 3.

$$\begin{aligned}
&\ge^B 1 - \sum_{\mu\in 2^{\Sigma_c\setminus i}}\sum_{j\in\Sigma_c\setminus i\setminus\mu}\left(\frac{\rho_j}{1-\sum_{j\in\mu}\rho_j}x_i + \frac{\rho_j}{1-\rho_i-\sum_{j\in\mu}\rho_j}(1-x_i)\right)\prod_{j\in\mu}(1-x_j)\prod_{j\in\Sigma_c\setminus i\setminus\mu} x_j && (18)\\
&= 1 - \sum_{\mu\in 2_c^{\Sigma}}\sum_{j\in\Sigma_c\setminus i\setminus\mu}\frac{\rho_j}{1-\sum_{j\in\mu}\rho_j}\prod_{j\in\mu}(1-x_j)\prod_{j\in\Sigma_c\setminus\mu} x_j && (19)
\end{aligned}$$

$$\begin{aligned}
&\ge^C 1 - \sum_{j\in\Sigma_c\setminus i}\sum_{\mu\in 2^{\Sigma_c\setminus j}}\frac{\rho_j}{1-\sum_{j\in\mu}\rho_j}\prod_{j\in\mu}(1-x_j)\prod_{j\in\Sigma_c\setminus\mu} x_j && (20)\\
&=^D 1 - \sum_{j\in\Sigma_c\setminus i} P_j(x) && (21)\\
&\ge^E 1 - \sum_{j\in\Sigma_c\setminus i} Q_j && (22)
\end{aligned}$$

where (A) follows from lemma 3, (B) follows since on $\langle 0,1\rangle$,

$$\frac{p_j}{1-\sum_{j\in\mu}p_j}x_i + \frac{p_j}{1-p_i-\sum_{j\in\mu}p_j}(1-x_i) \ge \frac{p_j}{1-\sum_{j\in\mu}p_j} \tag{23}$$

(C) follows from the following logic: suppose $\mu \in 2^{\Sigma_c}$ and $j \in \Sigma_c\setminus i\setminus\mu$. Then $j \in \Sigma_c\setminus i$. Since $j \notin \mu, \mu\setminus j = \mu$, so $\mu \in 2^{\Sigma_c\setminus j}$. (D) follows from lemma 5, and (E) from an observation made earlier in this proof. Hence

$$\phi_i(x) = \frac{Q_i}{\rho_i h_i(x)} = \frac{\Delta_i Q_i}{\Delta_i\rho_i h_i(x)} \le \frac{\Delta_i Q_i}{1-\sum_{j\in\Sigma_c\setminus i} Q_j} \le 1 \tag{24}$$

Using lemma 8, the sequence $v^0 = 0$ and $v^{k+1} = \phi(v^k)$ for $k = 1,\dots,$ exists and is such that $v^k \uparrow v^*$ for some $v^* \in \langle 0,1\rangle$. Since ϕ is continuous on $\langle 0,1\rangle$, it follows that $\phi(v^*) = v^*$, or equivalently that $P^c(v^*) = Q$.

For necessity, suppose that there is an $x \in \langle 0,1\rangle$ such that $P_i(x) = Q_i$ for any $i \in \Sigma_c$. Since

$$1 = P_\perp(x) + \sum_{i\in\Sigma_c} P_i(x) = H(x)(1-\sum_i \rho_i) + \sum_{i\in\Sigma_c} Q_i \tag{25}$$

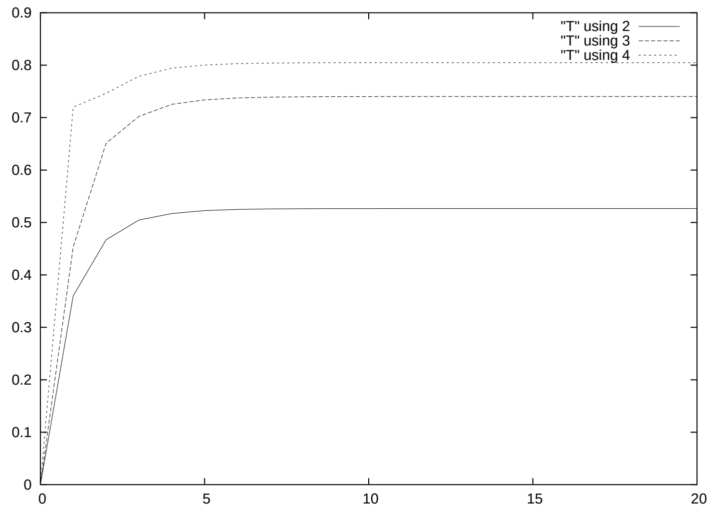

Figure 1: Fixed point computation for Example 1.

and $\rho_i H(x) \geq Q_i$ (lemma 7), it follows that

$$\sum_{i\in\Sigma_c} Q_i + \frac{1-\sum_{i\in\Sigma_c}\rho_i}{\rho_j} Q_j = \sum_{i\neq j} Q_i + \frac{1-\sum_{i\neq j}\rho_i}{\rho_j} Q_j \leq 1 \tag{26}$$

$\square$

Example 1 *Let* $\rho = (0.5, 0.3, 0.1), Q = (0.3, 0.34, 0.1)$. *Then* $\frac{6}{5}\rho_1 + \rho_2 + \rho_3 = 0.82$, $\rho_1 + \frac{4}{3}\rho_2 + \rho_3 = 0.87333$, $\rho_1 + \rho_2 + 2\rho_3 = 0.88$ *and so there is a* $v \in [0,1]^3$ *such that* $P(v) = Q$. *A value can be computed by fixpoint iteration. Let* $x^0 = (0,0,0)$ *and let* $x^i = \phi(x^{i-1})$*,where*

$$\begin{aligned}
\phi_1(x) &= \frac{Q_1}{\rho_1\left(x_2x_3 + \frac{1}{1-\rho_2}(1-x_2)x_3 + \frac{1}{1-\rho_3}x_2(1-x_3) + \frac{1}{1-\rho_2-\rho_3}(1-x_2)(1-x_3)\right)} \\
\phi_2(x) &= \frac{Q_2}{\rho_2\left(x_1x_3 + \frac{1}{1-\rho_1}(1-x_1)x_3 + \frac{1}{1-\rho_3}x_1(1-x_3) + \frac{1}{1-\rho_1-\rho_3}(1-x_1)(1-x_3)\right)} \\
\phi_3(x) &= \frac{Q_3}{\rho_3\left(x_2x_1 + \frac{1}{1-\rho_2}(1-x_2)x_1 + \frac{1}{1-\rho_1}x_2(1-x_1) + \frac{1}{1-\rho_2-\rho_1}(1-x_2)(1-x_1)\right)}
\end{aligned}$$

The progress of the iteration is recorded in the plot in Fig. 1 *and the limiting value is* $x^* = (0.526565, 0.740226, 0.804795)$.

The following lemma allows to handle the case when $S_\Delta(s) \cap \Sigma_u \neq \emptyset$ (i.e. some of events are uncontrollable). Let the partition $x = (x^c, x^u)$ be defined for vectors of length $n+m$, where $x_i^c = x_i$ for $i \in \Sigma_c$ and $x_{i-n}^u = x_i$ for $i \in \Sigma_u$. For $k \in \mathbb{N}$, let 1^k denote the row vector of length k such that $1_i^k = 1$ for all $i = 1, \ldots, k$.

Lemma 10 *Assume that* $\rho(\Sigma_u) > 0$. *Let* $Q \in \mathcal{S}(\Sigma)$. *There is a* $v \in [0,1]^{\Sigma_c}$ *such that* $P(v) = Q$ *iff* Q^c *satisfies* $\frac{1-\sum_{i\in\Sigma_c}\rho_i}{\rho_j} Q_i^c + \sum_{j\in\Sigma_c} Q_j^c \leq 1$ *from theorem* 9 *and* $Q^u = \rho^u \cdot \frac{Q(\Sigma_u)}{\rho(\Sigma_u)}$

Proof: Suppose there is a $v \in [0,1]^n$ such that $P(v) = Q$, or equivalently, $P^c(v) = Q^c$ and $P^u(v) = Q^u$. Since $1^{n+m}P(v) = 1^{n+m}Q$, $P^u(v) = \rho^u H(v)$, it follows that $1^{n+m}P(v) =$

$1^n P^c(v) + 1^m P^u(v) = 1^n Q^c + 1^m \rho^u H(v)$. Since $\rho(\Sigma_u) = 1^m \rho^u > 0$, we may solve for $H(v)$ in terms of Q^c to obtain an expression for $P^u(v)$ in terms of Q^u:

$$P^u(v) = \rho^u 1^m Q^u / 1^m \rho^u \tag{27}$$

Conversely, suppose the conditions are satisfied. Then there is a $v \in [0,1]^{\Sigma_c}$ such that $P^c(v) = Q^c$ by theorem 14 and $1^m Q^u = 1^m \rho^u H(v)$, it follows that $P^u(v) = Q^u$. □

5 Conclusions

In this paper we have proved the necessity and suffciency of conditions given in [6] for the existence of a probabilistic supervisory controller to solve the model maching problem for PDES. A constructive algorithm to solve the probability matching problem has been given that can be used as in conjunction with existing supervisory control algorithms to compute a solution to the probabilistic model matching problem when one exists.

Future work will generalize the control problem to obtain an "optimal" approximation to a probabilistic specification language when a solution to the model matching problem does not exist.

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