Parallel Program Design
Tasks, Critical Path

Ned Nedialkov

McMaster University

31 January 2023

Outline

Foster's Methodology

1. *Partitioning*. Divide the computation into small tasks that can be done in parallel
2. *Communication*. Determine the communications between the tasks
3. *Aggregation (or agglomeration)*. Combine tasks and communications into larger tasks
4. *Mapping*. Assign the combined tasks to processes (or threads)

From Ian Foster, Designing and Building Parallel Programs

Tasks

- We can decompose a computation into smaller parts or tasks.
- The goal is to determine which can be executed in parallel.
- Fine-grained decomposition: many small tasks.
- Coarse-grained decomposition: small number of larger tasks.
- Task-dependency graph
  - directed acyclic graph
  - nodes are tasks
  - there is an edge between task A and task B if B must be executed after A

Degrees of concurrency

- Maximum degree of concurrency: the maximum number of tasks that can be executed in parallel.
- Average degree of concurrency: the average number of tasks that can be executed in parallel.
- The average degree of concurrency is a more useful measure.

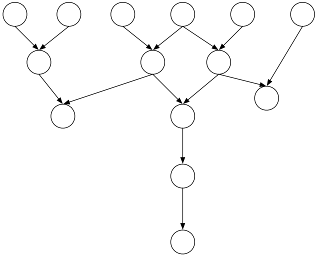
Critical path

- Start nodes: nodes with no incoming edges.
- Finish nodes: nodes with no outgoing edges.
- Critical path: the longest directed path between any pair of start and finish nodes.
- Critical path length: sum of the weights of the nodes on a critical path
- Average degree of concurrency

$$\text{ave degree of concurrency} = \frac{\text{total amount of work}}{\text{critical path length}}$$

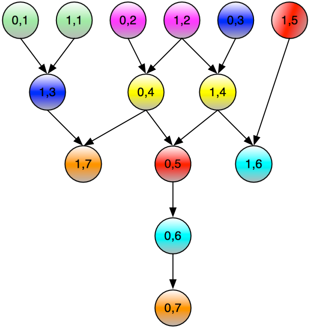
Example

Task-dependency graph

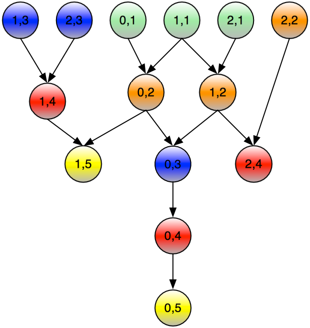


- maximum degree of concurrency is 6
- critical path length is 5
- total amount of work is 14 (assuming each task takes one unit of time)
- average degree of concurrency is $14/5 = 2.8$

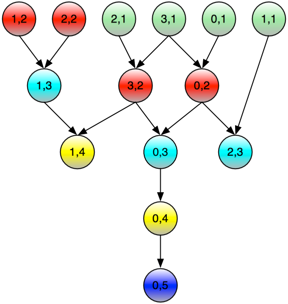
- An assignment to 2 processes and the order in which the tasks are executed
- First number is the process number
- The speedup is $14/7 = 2$



- An assignment to 3 processes
- The speedup is $14/5 = 2.8$



- An assignment to 4 processes
- The speedup is $14/5 = 2.8$



- What is the speedup on 8 processes?

Example: LU decomposition

- We want to compute the LU decomposition of a matrix A
- Assume A consists of 3×3 blocks
- We want

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}}_{U}$$

- We need to determine the L_{ij} and U_{ij}.
- Consider L times the first column of U.
  We have

$$A_{11} = L_{11}U_{11} \qquad \text{compute } L_{11} \text{ and } U_{11} \qquad (1)$$

$$A_{21} = L_{21}U_{11} \qquad L_{21} = A_{21}U_{11}^{-1} \qquad (2)$$

$$A_{31} = L_{31}U_{11} \qquad L_{31} = A_{31}U_{11}^{-1} \qquad (3)$$

- From multiplying the first row of L times U, we obtain

$$A_{12} = L_{11}U_{12} \qquad U_{12} = L_{11}^{-1}A_{21} \qquad (4)$$

$$A_{13} = L_{11}U_{13} \qquad U_{13} = L_{11}^{-1}A_{13} \qquad (5)$$

- Then we write

$$A_{22} = L_{21}U_{12} + L_{22}U_{22} \qquad A'_{22} = A_{22} - L_{21}U_{12} = L_{22}U_{22} \tag{6}$$

$$\text{compute } L_{22} \text{ and } U_{22} \tag{7}$$

$$A_{23} = L_{21}U_{13} + L_{22}U_{23} \qquad A'_{23} = A_{23} - L_{21}U_{13} = L_{22}U_{23} \tag{8}$$

$$U_{23} = L_{22}^{-1}A'_{23} \tag{9}$$

$$A_{32} = L_{31}U_{12} + L_{32}U_{22} \qquad A'_{32} = A_{32} - L_{31}U_{12} = L_{32}U_{22} \tag{10}$$

$$L_{32} = A'_{32}U_{22}^{-1} \tag{11}$$

$$A_{33} = L_{31}U_{13} + L_{32}U_{23} + L_{33}U_{33} \qquad A'_{33} = A_{33} - L_{31}U_{13} \tag{12}$$

$$A''_{33} = A'_{33} - L_{32}U_{23} = L_{33}U_{33} \tag{13}$$

$$\text{compute } L_{33} \text{ and } U_{22} \tag{14}$$

- Using the right column in (1–14), we can decompose the computation in the following tasks

1. $A_{11} = L_{11}U_{11}$ (compute LU factorization)
2. $L_{21} = A_{21}U_{11}^{-1}$
3. $L_{31} = A_{31}U_{11}^{-1}$
4. $U_{12} = L_{11}^{-1}A_{21}$
5. $U_{13} = L_{11}^{-1}A_{13}$

---

6. $A'_{22} = A_{22} - L_{21}U_{12}$
7. $A'_{32} = A_{32} - L_{31}U_{12}$
8. $A'_{23} = A_{23} - L_{21}U_{13}$
9. $A'_{33} = A_{33} - L_{31}U_{13}$
10. $A'_{22} = L_{22}U_{22}$ (compute LU factorization)

---

11. $L_{32} = A'_{32}U_{22}^{-1}$
12. $U_{23} = L_{22}^{-1}A'_{23}$

---

13. $A''_{33} = A'_{33} - L_{32}U_{23}$
14. $A''_{33} = L_{33}U_{33}$ (compute LU factorization)

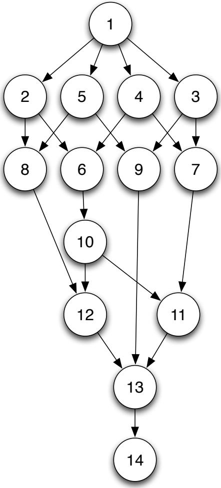


Figure: Task-dependency graph

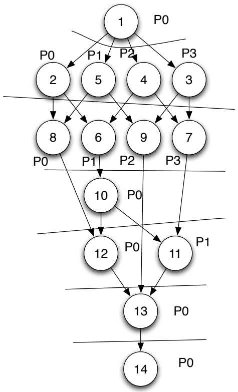


Figure: Task distribution on 4 processes. Maximum speedup is 2