CS 3DA3 2007 Problem Set I

Solutions

1.

$$
\begin{aligned}
f(t) &= at^2 + bt + c \\
&= a(t^2 + \tfrac{b}{at} + \tfrac{c}{a}) \\
&= a[(t + \tfrac{b}{2a})^2 - \tfrac{b^2}{4a^2} + \tfrac{4ac}{4a^2}] \\
&= a[(t + \tfrac{b}{2a})^2 - \tfrac{D}{4a^2}] \qquad .....(\alpha)
\end{aligned}
$$

(1) if $D < 0$ then $-\frac{D}{4a^2} > 0$

$a > 0$

$(t + \frac{b}{2a})^2 \leq 0$

$\Rightarrow f(t) > 0$ for all t

(2) if $D = 0$ then from (α) we have $f(t) = a(t + \frac{b}{2a})^2$

so $f(t) = 0 \Rightarrow t = -\frac{b}{2a}$

and $t = -\frac{b}{2a} \Rightarrow f(t) = 0$

(3) if $D > 0$ and $f(t) = 0$ then from (α) we have

$(t + \frac{b}{2a})^2 - \frac{D}{4a^2} = 0$

$\Rightarrow t = -\frac{b}{2a} \pm \sqrt{\frac{D}{4a^2}}$

$= \frac{-b \pm \sqrt{D}}{2a}$

2.

$|f(n)| = |13n^4 - 722n^2 + \frac{47}{n^2+31} + 6|$

$\leq |13n^4| + |722n^2| + |\frac{47}{n^2+31}| + |6|$

$\leq 13n^4 + 722n^4 + \frac{47}{n^2} + 6$

when $n \geq 1$

$|f(n)| \leq 13n^4 + 722n^4 + \frac{47}{n^2} \times n^6 + 6 \times n^4$

$= 788n^4$

$\therefore$ for any $n \geq 1, |f(n)| \leq 788|n^4|$

3.

(1)

$g \in O(\alpha f)$

$\Rightarrow \exists k, n_0 > 0, s.t.\ \forall n \geq n_0, |g(n)| \leq k|\alpha f(n)|$

$\Rightarrow \exists k, n_0 > 0, s.t.\ \forall n \geq n_0, |g(n)| \leq k|\alpha||f(n)|$

$\Rightarrow \exists k, n_0 > 0, s.t.\ \forall n \geq n_0, \frac{|g(n)|}{k|\alpha|} \leq |f(n)|$

$\Rightarrow \exists k, n_0 > 0, s.t.\ \forall n \geq n_0, \frac{|\alpha||g(n)|}{k|\alpha|^2} \leq |f(n)|$

$\Rightarrow \exists k, n_0 > 0, s.t.\ \forall n \geq n_0, \frac{1}{k|\alpha|^2} \cdot |\alpha g(n)| \geq |f(n)|$

$\therefore \exists k' = \frac{1}{k|a|^2} > 0$ and $n_0 > 0$ $s.t.\ \forall n \geq n_0\ |f(n)| \leq k'|\alpha g(n)|$

$\Rightarrow f \in \Omega(\alpha g)$

similarly,

$f \in O(\alpha g)$

$\Rightarrow \exists k, n_0 > 0, s.t.\ \forall n \geq n_0, |f(n)| \geq k|\alpha g(n)|$

$\Rightarrow \exists k, n_0 > 0, s.t.\ \forall n \geq n_0, |f(n)| \geq k|\alpha||g(n)|$

$\Rightarrow \exists k, n_0 > 0, s.t.\ \forall n \geq n_0, \frac{|f(n)|}{k|\alpha|} \geq |g(n)|$

$\Rightarrow \exists k, n_0 > 0, s.t.\ \forall n \geq n_0, \frac{|\alpha||f(n)|}{k|\alpha|^2} \geq |g(n)|$

$\Rightarrow \exists k, n_0 > 0, s.t.\ \forall n \geq n_0, \frac{1}{k|\alpha|^2} \cdot |\alpha f(n)| \geq |g(n)|$

$\therefore \exists k' = \frac{1}{k|a|^2} > 0$ and $n_0 > 0$ $s.t.\ \forall n \geq n_0,\ |g(n)| \leq k'|\alpha f(n)|$

$\Rightarrow g \in \Omega(\alpha f)$

(2)

Let $g \in O(f)$, then $\exists k, n_0 > 0,\ s.t.\ \forall n \geq n_0,\ |g(n)| \leq k|f(n)|$.

$\Rightarrow \exists n_1 = \max\{n_0, 1\},\ s.t.\ |g(n)| \leq k|n||f(n)|$, for all $n \geq n_1$

$\Rightarrow |g(n)| \leq k|nf(n)|$

$\Rightarrow g \in O(nf)$

4.

Note that $100^{\log n} = n^{\log 100}$, the idea is that f is polynomially bounded, g is not.

Specifically, $e > 2 \Rightarrow e^7 > 128 > 100$ so $\log 100 < 7$ and $n^{\log 100} < n^7$. Then $|f(n)| \leq 2n^3 + n^7 + n^2 \log n \leq 4n^7$ since $\log n < n$ for all $n \geq 1$.

(One could probably prove in another ways but this is simple and

elementary).

We already showed that $n^m \in O(a^n)$ for all $a > 1, m \geq 1$. So $f \in O(1.5^n)$

However, suppose $g \in O(f)$, namely there exists $k > 0, n_0 > 0,\ s.t.$

$g \leq k|f(n)|$, then $|g(n)| \leq 4k|n^7|$.

$\Rightarrow 1.5^n \leq (4k)n^7$

$\Rightarrow n \log 1.5 \leq \log 4k + 7 \log n$

$\Rightarrow n \leq \frac{\log 4k}{\log 1.5} + \frac{7}{\log 1.5} \cdot \log n$

We already proved $\log n \in O(n)$ and it is easy to prove $a \log n \in O(bn)$, so we can choose n_0 s.t. $n \geq n_0 \Rightarrow \frac{7}{\log 1.5} \cdot \log n \leq 1/2n$ then

$1/2n \leq \frac{\log 4k}{\log 1.5}$ for all $n \geq n_0$, which is a contradiction.

5.

Since when $n \geq 4,\ n(1/4n - 1) \geq 0$

$\Rightarrow \frac{n^2}{4} \geq n$

$\Rightarrow \frac{n}{2} \geq \sqrt{n}$

$\Rightarrow \frac{1}{2} n \log n \geq \sqrt{n} \log n$

$\Rightarrow \log n^{n/2} \geq \log n^{\sqrt{n}}$

$\Rightarrow \log \sqrt{n}^n \geq \log n^{\sqrt{n}}$

$\therefore$ for $n_0 = 4$ and $k = 1$

$\forall n \geq n_0, n^{\sqrt{n}} \leq k \cdot \sqrt{n}^n$

$\therefore g(n) \in O(f(n))$

However, $f \notin O(g)$, to see this, suppose $f(n) \leq kg(n)$ for all $n \geq n_0$, then $\log f(n) \leq \log k + \log g(n)$ for all $n \geq n_0$, i.e.

$n/2 \log n \leq \log k + \sqrt{n} \cdot \log n$

$(n/2 - \sqrt{n}) \log n \leq \log k$

In fact, if $n \geq 8$, then

$n/2 - \sqrt{n} \geq 1$, so $\log k \geq \log n \Rightarrow k \geq n$,

This is a contradiction.