Proceedings of the 2^{nd} International Conference of Control, Dynamic Systems, and Robotics
Ottawa, Ontario, Canada, May 7-8 2015
Paper No. 183

Structured Condition Numbers of Symmetric Algebraic Riccati Equations

Huai-An Diao
Northeast Normal University
Key Laboratory for Applied Statistics of MOE and
School of Mathematics and Statistics
Chang Chun, P.R. China 130024
hadiao@nenu.edu.cn

Sanzheng Qiao
McMaster University, Department of Computing and Software
1280 Main Street W., Hamilton, Ontario, Canada L8S 4K1
qiao@mcmaster.ca

Abstract

This paper presents a structured perturbation analysis of the symmetric algebraic Riccati equations by exploiting the symmetry structure. Based on the analysis, structured normwise, componentwise, and mixed condition numbers are defined and their explicit expressions are derived. Due to the exploitation of the symmetry structure, our results are improvements of previous work on the perturbation analysis and condition numbers of the symmetric algebraic Riccati equations. Our preliminary numerical experiments demonstrate that our condition numbers provide accurate estimates for the errors in the solution caused by the perturbations on the data.

Keywords: Symmetric algebraic Riccati equation, perturbation analysis, condition number.

1 Introduction

Algebraic Riccati equations arise in optimal control problems in continuous-time or discrete-time. The theory, applications, and numerical methods for solving the equations can be found in [1, 5] and references therein. The continuous-time algebraic Riccati equation (CARE) is given in the form:

$$Q + A^H X + XA - XBR^{-1}B^H X = 0, \tag{1}$$

where X is the unknown matrix, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$ and Q, R are $n \times n$ Hermitian matrices with Q being positive semi-definite (p.s.d.) and R being positive definite. The discrete-time algebraic Riccati equation (DARE) is given in the form:

$$Y - A^H YA + A^H YB(R + B^H YB)^{-1}B^H YA - C^H C = 0, \tag{2}$$

where Y is the unknown matrix, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{r\times n}$, and $R \in \mathbb{C}^{m\times m}$ with R being Hermitian and positive definite.

For the complex CARE (1), let $G = BR^{-1}B^H$, a Hermitian and p.s.d. matrix, then we obtain its simplified form

$$Q + A^H X + XA - XGX = 0. \tag{3}$$

For the complex DARE (2), let $Q = C^H C$ and $G = BR^{-1}B^H$, both Hermitian and p.s.d., then we obtain its simplified form

$$Y - A^H Y(I + GY)^{-1}A - Q = 0. \tag{4}$$

The existence and uniqueness of the solution is essential for perturbation analysis. It is known that if (A, G) in the CARE (3) is a c-stabilizable pair and (A, Q) is a c-detectable pair, then there exists a unique Hermitian and p.s.d. solution X for the CARE. Similarly, if (A, B) in the DARE (2) is a d-stabilizable pair and (A, C) is a d-detectable pair, then there exists a unique Hermitian and p.s.d. solution Y for the DARE (4) [1]. In this paper, we assume that for the CARE or DARE the conditions are satisfied, thus the solution exists and is unique.

Perturbation analysis concerns the sensitivity of the solution to the perturbations in the data of a problem. A condition number is a measurement of the sensitivity. Combining with the backward error analysis, it can be used to estimate the error in a computed solution. In this paper, we consider structured perturbation. Specifically, because Q, G in (3) and (4) are Hermitian, it is reasonable to require the perturbation matrices on Q and G be Hermitian. Sun [9] defined the structured normwise condition numbers for the CARE and DARE and showed that the expressions of structured normwise condition numbers are the same as their unstructured counterparts for both real and complex cases. Later, Zhou etc. [10] performed componentwise perturbation analyses of the real CARE and real DARE and obtained the exact expressions for mixed and componentwise condition numbers defined in [3]. However, in their paper, the perturbation matrices on Q and G are general (unstructured). In this paper, we perform structured perturbation analyses and define the structured normwise, mixed and componentwise condition numbers for the complex CARE and DARE and derive their expressions using the Kronecker product.

Throughout this paper we adopt the following notations:

- $\mathbb{C}^{m\times n}$ ($\mathbb{R}^{m\times n}$) denotes the set of complex (real) $m \times n$ matrices; $\mathbb{H}^{n\times n}$ the set of $n \times n$ Hermitian matrices; $\mathbb{S}^{n\times n}$ the set of $n \times n$ real symmetric matrices; $\mathbb{SK}^{n\times n}$ the set of $n \times n$ real skew-symmetric matrices.

- A^T denotes the transpose of A; A^H the complex conjugate and transpose of A; I the identity matrix; 0 the zero matrix; $\mathsf{Re}(A)$ ($\mathsf{Im}(A)$) the real (imaginary) part of a complex matrix A.

- The mapping $\mathsf{sym}(\cdot)$: $\mathbb{S}^{n\times n} \to \mathbb{R}^{n(n+1)/2}$ maps a symmetric matrix $A = [a_{ij}] \in \mathbb{S}^{n\times n}$ to a $(n(n+1)/2)$-vector:

$$[a_{11}, ..., a_{1n}, a_{22}, ..., a_{2n}, ..., a_{n-1,n-1}, a_{n-1,n}, a_{nn}]^T.$$

- The mapping $\mathsf{skew}(\cdot)$: $\mathbb{SK}^{n\times n} \to \mathbb{R}^{n(n-1)/2}$ maps a skew-symmetric matrix $A = [a_{ij}] \in \mathbb{SK}^{n\times n}$ to the $(n(n-1)/2)$-vector:

$$[a_{12}, ..., a_{1n}, a_{23}, ..., a_{2n}, ..., a_{n-2,n-1}, a_{n-2,n}a_{n-1,n}]^T.$$

- $A \succ 0$ ($A \succeq 0$) means that A is positive definite (positive semi-definite).
- $\| \ \|_F, \| \ \|_2$ and $\| \ \|_\infty$ are the Frobenius norm, the spectral norm and the infinity norm respectively. For $A \in \mathbb{C}^{m\times n}$, $\|A\|_{\max} = \max_{ij} |a_{ij}|$.
- $A \otimes B = [a_{ij}B]$ is the Kronecker product of matrices $A = [a_{ij}]$ and B; $\mathsf{vec}(A)$ is the vector defined by $\mathsf{vec}(A) = [a_1^T, ..., a_n^T]^T \in \mathbb{C}^{mn}$, where a_j is the jth column of A; Π is an $n^2 \times n^2$ permutation matrix, such that, for an $n \times n$ real matrix A, $\mathsf{vec}(A^T) = \Pi\mathsf{vec}(A)$.
- $|A| \le |B|$ means $|a_{ij}| \le |b_{ij}|$ for A, $B \in \mathbb{C}^{m\times n}$; $A./B$ is the componentwise division of matrices A and B of the same dimensions, where if $b_{ij} = 0$ we assume $a_{ij} = 0$ and define $a_{i,j}/b_{i,j} = 0$.

The rest of the paper is organized as follows. In Section 2, we present our structured perturbation analyses and the definitions and expressions of the structured normwise condition numbers of the CARE and the DARE. The definitions and expressions of the structured mixed and componentwise condition numbers are given in Section 3. Our preliminary numerical experiment results are demonstrated in Section 4. Finally, Section 5 concludes this paper.

2 Structured normwise condition numbers

In this section, using the Kronecker product, we first present a structured perturbation analysis of the CARE (3) and derive an expression of the corresponding structured normwise condition number. Then, analogously, we give a structured perturbation analysis of the DARE (4) and the corresponding structured normwise condition number.

2.1 CARE

Let $\Delta A \in \mathbb{C}^{n\times n}$, $\Delta Q \in \mathbb{H}^{n\times n}$, and $\Delta G \in \mathbb{H}^{n\times n}$ be the perturbations to the data A, Q, and G respectively. From [8], for Q, $G \succeq 0$ and sufficiently small $\|[\Delta A, \Delta Q, \Delta G]\|_F$, there exists a unique Hermitian and p.s.d. matrix $\tilde{X}$ such satisfying the perturbed equation:

$$\tilde{X}\tilde{G}\tilde{X} - \tilde{X}\tilde{A} - \tilde{A}^H\tilde{X} - \tilde{Q} = 0, \tag{5}$$

where $\tilde{A} = A + \Delta A$, $\tilde{G} = G + \Delta G$, and $\tilde{Q} = Q + \Delta Q$. Let $\Delta X = \tilde{X} - X$ be the change in the solution due to the perturbation, then for small $\|[\Delta A, \Delta Q, \Delta G]\|_F$, the first order approximation of (5) is the continuous Lyapunov equation:

$$(A - GX)^H\Delta X + \Delta X(A - GX) = -\Delta Q - X\Delta A - \Delta A^H X + X\Delta GX,$$

where ΔX is the unknown matrix. Applying the vec operator to the above equation and using the identity $\mathsf{vec}(UVW) = (W^T \otimes U)\mathsf{vec}(V)$, we obtain

$$\mathsf{vec}(\Delta X) = -Z^{-1}[-(X^T\otimes X)\mathsf{vec}(\Delta G)+(I_n\otimes X)\mathsf{vec}(\Delta A)+(X^T\otimes I_n)\mathsf{vec}(\Delta A^H)+\mathsf{vec}(\Delta Q)], \tag{6}$$

where

$$Z = I_n \otimes (A - GX)^H + (A - GX)^T \otimes I_n. \tag{7}$$

To exploit the symmetry structure of the perturbation matrices ΔG and ΔQ, we define an n^2-by-$n(n+1)/2$ matrix $\mathcal{S}_1$ such that $\mathcal{S}_1\, \mathsf{sym}(\mathsf{Re}(\Delta G)) = \mathsf{vec}(\mathsf{Re}(\Delta G))$. Basically, $\mathcal{S}_1$ expands the $n(n+1)/2$-vector $\mathsf{sym}(\mathsf{Re}(\Delta G))$ to the n^2-vector $\mathsf{vec}(\mathsf{Re}(\Delta G))$ by copying its entries. Similarly, we define an n^2-by-$n(n-1)/2$ matrix $\mathcal{S}_2$ such that $\mathcal{S}_2\, \mathsf{skew}(\mathsf{Im}(\Delta G)) = \mathsf{vec}(\mathsf{Im}(\Delta G))$. Then (6) becomes

$$\begin{aligned}\mathsf{vec}(\Delta X) \;=\; & -Z^{-1}[I_n \otimes X + (X^T \otimes I_n)\Pi,\ \mathbf{i}\,(I_n \otimes X - (X^T \otimes I_n)\Pi), \\ & -(X^T \otimes X)\mathcal{S}_1,\ -\mathbf{i}\,(X^T \otimes X)\mathcal{S}_2,\ \mathcal{S}_1,\ \mathbf{i}\,\mathcal{S}_2] \cdot \Delta,\end{aligned}$$

recalling that Π is the permutation matrix such that $\Pi\, \mathsf{vec}(\mathsf{Re}(\Delta A)) = \mathsf{vec}(\mathsf{Re}(\Delta A^H))$ and

$$\begin{aligned}\Delta \;=\; & [\mathsf{vec}(\mathsf{Re}(\Delta A))^T,\ \mathsf{vec}(\mathsf{Im}(\Delta A))^T,\ \ \mathsf{sym}(\mathsf{Re}(\Delta G))^T,\ \mathsf{skew}(\mathsf{Im}(\Delta G))^T, \\ & \mathsf{sym}(\mathsf{Re}(\Delta Q))^T,\ \mathsf{skew}(\mathsf{Im}(\Delta Q))^T]^T\end{aligned} \tag{8}$$

is the augmented structured perturbation vector. Denoting $\mathcal{M} = [M_A\ M_G\ M_Q]$, where

$$M_A = [I_n \otimes X + (X^T \otimes I_n)\Pi,\ \ \mathbf{i}(I_n \otimes X - (X^T \otimes I_n)\Pi)]$$

corresponds to $[\mathsf{vec}(\mathsf{Re}(\Delta A))^T,\ \mathsf{vec}(\mathsf{Im}(\Delta A))^T]^T$,

$$M_G = [-(X^T \otimes X)\mathcal{S}_1,\ -\mathbf{i}(X^T \otimes X)\mathcal{S}_2]$$

corresponds to $[\mathsf{sym}(\mathsf{Re}(\Delta G))^T,\ \mathsf{skew}(\mathsf{Im}(\Delta G))^T]^T$, and

$$M_Q = [\mathcal{S}_1,\ \mathbf{i}\,\mathcal{S}_2]$$

corresponds to $[\mathsf{sym}(\mathsf{Re}(\Delta Q))^T,\ \mathsf{skew}(\mathsf{Im}(\Delta Q))^T]^T$, we get

$$\mathsf{vec}(\Delta X) = -Z^{-1}\mathcal{M}\Delta. \tag{9}$$

Now, following the condition number theory [6], we define the structured normwise condition numbers:

$$\kappa_n = \lim_{\epsilon \to 0} \sup_{\substack{\eta \le \epsilon \\ \Delta A \in \mathbb{C}^{n\times n},\ \Delta G, \Delta Q \in \mathbb{H}^{n\times n} \\ G+\Delta G,\ Q+\Delta Q \succeq 0}} \frac{\|\Delta X\|_F}{\epsilon \|X\|_F}, \tag{10}$$

where

$$\begin{aligned}\eta \;=\; & \left\| \left[\frac{\|\mathsf{Re}(\Delta A)\|_F}{\delta_1},\ \frac{\|\mathsf{Im}(\Delta A)\|_F}{\delta_2},\ \frac{\|\mathsf{sym}(\mathsf{Re}(\Delta G))\|_2}{\delta_3},\ \frac{\|\mathsf{skew}(\mathsf{Im}(\Delta G))\|_2}{\delta_4}, \right.\right. \\ & \left.\left. \frac{\|\mathsf{sym}(\mathsf{Re}(\Delta Q))\|_2}{\delta_5},\ \frac{\|\mathsf{skew}(\mathsf{Im}(\Delta Q))\|_2}{\delta_6} \right] \right\|_2\end{aligned}$$

is a scaled augmented structured perturbation vector and the scaling factors $\delta_i > 0$, $i = 1, ..., 6$, are generally chosen to be the functions of $\|\mathsf{Re}(A)\|_F$, $\|\mathsf{Im}(A)\|_F$, $\|\mathsf{sym}(\mathsf{Re}(G))\|_2$, $\|\mathsf{skew}(\mathsf{Im}(G))\|_2$, $\|\mathsf{sym}(\mathsf{Re}(Q))\|_2$, and $\|\mathsf{skew}(\mathsf{Im}(Q))\|_2$. Here, we choose $\delta_1 = \|\mathsf{Re}(A)\|_F$, $\delta_2 = \|\mathsf{Im}(A)\|_F$, $\delta_3 = \|\mathsf{sym}(\mathsf{Re}(G))\|_2$, $\delta_4 = \|\mathsf{skew}(\mathsf{Im}(G))\|_2$, $\delta_5 = \|\mathsf{sym}(\mathsf{Re}(Q))\|_2$ and $\delta_6 = \|\mathsf{skew}(\mathsf{Im}(Q))\|_2$.

The following theorem gives an explicit expression of the structured normwise condition number of the CARE.

Theorem 1. *Using the above notations, an expression of the normwise condition number of the complex CARE is*

$$\kappa_n = \frac{\|Z^{-1}\mathcal{M}D\|_2}{\|X\|_F}, \tag{11}$$

where

$$D = \mathsf{Diag}\left([\delta_1 I_{n^2},\ \delta_2 I_{n^2},\ \delta_3 I_{n(n+1)/2},\ \delta_4 I_{n(n-1)/2},\ \delta_5 I_{n(n+1)/2},\ \delta_6 I_{n(n-1)/2}]\right) \tag{12}$$

is a diagonal scaling matrix.

Proof. From (9), we have

$$\mathsf{vec}(\Delta X) = -Z^{-1}\mathcal{M}DD^{-1}\Delta.$$

Then, from the definition (10), we have

$$\kappa_n = \lim_{\epsilon \to 0} \sup_{\eta \le \epsilon} \frac{\|Z^{-1}\mathcal{M}DD^{-1}\Delta\|_2}{\epsilon\|X\|_F} = \lim_{\epsilon \to 0} \sup_{\|\epsilon^{-1}D^{-1}\Delta\|_2 \le 1} \frac{\| - Z^{-1}\mathcal{M}D(\epsilon^{-1}D^{-1}\Delta)\|_2)}{\|X\|_F},$$

noting that $\|D^{-1}\Delta\|_2 = \eta$. The upper bound (11) can be attained, because Δ can vary freely.

Different from the perturbation analysis in [9], in our analysis, we treat the perturbations on the real and imaginary parts of a complex matrix separately. It is more practical, since in computation, the real and imaginary parts are stored and computed separately. It can be shown that our condition number κ_n is smaller than its counterpart in [9].

2.2 DARE

Following the perturbation analysis of the CARE, for the DARE (4), we define $\mathcal{N} = [N_A\ N_G\ N_Q]$, where

$$\begin{aligned}
N_A &= [I_n \otimes (A^H Y W) + ((YWA)^T \otimes I_n)\Pi,\ \mathbf{i}(I_n \otimes (A^H Y W) - ((YWA)^T \otimes I_n)\Pi)], \\
M_G &= [-((YWA)^T \otimes (A^H Y W)\mathcal{S}_1,\ -\mathbf{i}((YWA)^T \otimes (A^H Y W))\mathcal{S}_2], \\
M_Q &= [\mathcal{S}_1,\ \mathbf{i}\,\mathcal{S}_2],
\end{aligned}$$

then

$$\mathsf{vec}(\Delta Y) = -T^{-1}\mathcal{N}\Delta,$$

where Δ is defined in (8) and

$$T = I_n - (A^T W^T) \otimes (A^H W^H). \tag{13}$$

Replacing ΔX and X in (10) with ΔY and Y respectively, we can define the structured normwise condition number for the DARE. An expression of the condition number similar to (11) can be obtained. Also, it can be shown that our condition number is an improvement of its counterpart in [9].

3 Structured mixed and componentwise condition numbers

Componentwise analysis [2, 4, 7] is more informative than its normwise counterpart when the data are imbalanced or sparse. Here, we consider the two kinds of condition numbers introduced by Gohberg and Koltracht [3]. The first kind, called the mixed condition number, measures the output errors in norm while the input perturbations componentwise. The second kind, called the componentwise condition number, which measures both the output error and the input perturbations componentwise.

Following [3], we define the structured mixed and componentwise condition numbers for the CARE (3):

$$\begin{aligned}
\kappa_m &= \lim_{\epsilon\to 0} \sup_{\substack{|\Delta|./|c|\le\epsilon \\ \Delta A\in\mathbb{C}^{n\times n},\ \Delta G,\Delta Q\in\mathbb{H}^{n\times n} \\ G+\Delta G,\, Q+\Delta Q\succeq 0}} \frac{\|\Delta X\|_{\max}}{\epsilon\|X\|_{\max}}, \\
\kappa_c &= \lim_{\epsilon\to 0} \sup_{\substack{|\Delta|./|c|\le\epsilon \\ \Delta A\in\mathbb{C}^{n\times n},\ \Delta G,\Delta Q\in\mathbb{H}^{n\times n} \\ G+\Delta G,\, Q+\Delta Q\succeq 0}} \epsilon^{-1}\|\Delta X./X\|_{\max},
\end{aligned}$$

where Δ is the augmented structured perturbation vector defined in (8) and

$$\begin{aligned}
c \;=\; & [\mathsf{vec}(\mathsf{Re}(A))^T,\ \mathsf{vec}(\mathsf{Im}(A))^T,\ \ \mathsf{sym}(\mathsf{Re}(G))^T,\ \mathsf{skew}(\mathsf{Im}(G))^T, \\
& \mathsf{sym}(\mathsf{Re}(Q))^T,\ \mathsf{skew}(\mathsf{Im}(Q))^T]^T \qquad (14)
\end{aligned}$$

is the augmented structured data vector. Thus $|\Delta|./|c| \le \epsilon$ means that the relative perturbation is componentwise less than or equal to ϵ.

The following theorem presents the structured mixed and componentwise condition numbers of the CARE.

Theorem 2. *For the structured mixed and componentwise condition numbers of the complex CARE (3), we have*

$$\kappa_m = \frac{\big\||Z^{-1}\mathcal{M}|\,|C|\big\|_\infty}{\|X\|_{\max}}, \tag{15}$$

and

$$\kappa_c = \left\|\mathrm{diag}(\mathsf{vec}(X))^\dagger \left(|Z^{-1}\mathcal{M}|\,|C|\right)\right\|_\infty, \tag{16}$$

where $\mathrm{diag}(\mathsf{vec}(X))^\dagger$ *is the Moore-Penrose inverse of* $\mathrm{diag}(\mathsf{vec}(X))$.

Proof. From (9), we have

$$\|\Delta X\|_{\max} = \|\mathsf{vec}(\Delta X)\|_\infty = \|Z^{-1}\mathcal{M}\Delta\|_\infty \le \|Z^{-1}\mathcal{M}D_c\|_\infty \|D_c^\dagger\Delta\|_\infty,$$

where $D_c^\dagger$ is the Moore-Penrose inverse of $D_c = \mathrm{diag}(c)$. Note that $|\Delta|./|c| \le \epsilon$ implies $\|D_c^\dagger\Delta\|_\infty \le \epsilon$. Let $\mathbf{e}$ be the vector consisting all 1's, then

$$\left\|Z^{-1}\mathcal{M}D_c\right\|_\infty = \left\||Z^{-1}\mathcal{M}|\,|D_c\mathbf{e}|\right\|_\infty = \left\||Z^{-1}\mathcal{M}|\,|c|\right\|_\infty,$$

leading to the expression (15) of κ_m. The expression (16) of the componentwise condition number κ_c can be obtained similarly.

Analogously, we can define the mixed and componentwise condition numbers for the DARE and derive their expressions.

4 Numerical examples

In this section, we adopt the examples in [9] to illustrate the effectiveness of our methods. All the experiments were performed using MATLAB 7.0.

Given $A \in \mathbb{R}^{n\times n}$ and G, $Q \in \mathbb{S}^{n\times n}$, we generated the perturbations on A, G and Q as follows: $\Delta A = \epsilon(M_1 \boxdot A)$, $\Delta G = \epsilon(M_2 \boxdot G)$, and $\Delta Q = \epsilon(M_3 \boxdot Q)$, where $\epsilon = 10^{-j}$, $\boxdot$ denotes the componentwise matrix multiplication, and $M_1 \in \mathbb{R}^{n\times n}$, and $M_2, M_3 \in \mathbb{S}^{n\times n}$ are matrices whose entries are random variables uniformly distributed in the open interval $(-1, 1)$.

Example 1. Consider the CARE (3), where

$$A = \text{diag}([-0.1, -0.02]), \quad Q = C^T C, \quad G = BR^{-1}B,$$

where

$$B = \begin{bmatrix} 0.1 & 0 \\ 0.001 & 0.01 \end{bmatrix}, \; R = \begin{bmatrix} 1.001 & 1 \\ 1 & 1 \end{bmatrix}, \; C = [10, 100].$$

The pair (A, G) is c-stabilizable and the pair (A, Q) is c-detectable. Thus there exists a unique symmetric and p.s.d. solution X

Let $\tilde{Q} = Q + \Delta Q$, $\tilde{A} = A + \Delta A$, $\tilde{G} = G + \Delta G$ be the coefficient matrices of the perturbed CARE (5). We used the MATLAB function `are` to compute the solution X to (3) and the solution $\tilde{X}$ to (5). Thus $\|\Delta X\|_F / \|X\|_F$, where $\Delta X = \tilde{X} - X$, gives the relative change in the solution due to the perturbation. From the definitions (8) of Δ and (14) of c, $\|\Delta\|_2/\|c\|_2$ gives the size of relative perturbation on the data A, G, and Q. In our experiments, we set $j = -12$, that is, $\epsilon = 10^{-12}$. The following table compares the estimates using our condition numbers with the change in the solution computed by MATLAB.

$\|\Delta X\|_F/\|X\|_F$	$\kappa_n\|\Delta\|_2/\|c\|_2$	$\kappa_c\|\Delta\|_2/\|c\|_2$
4.1375×10^{-10}	8.3197×10^{-8}	3.0826×10^{-9}

Example 2. Consider the DARE (4), where

$$A = B\,\text{diag}([0, 0.001, 1.0])\,B, \quad Q = B\,\text{diag}([1000, 1.0, 0.001])\,B,$$

and

$$G = B\,\text{diag}([0.001, 0.001, 0.001])\,B,$$

where

$$B = I_3 - \frac{2}{3}[1, 1, 1]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is a Householder matrix. The pair (A, B) is d-stabilizable and the pair (A, Q) is d-detectable. Thus there exists a unique symmetric and p.s.d. solution Y

Similar to Example 1, let $\tilde{Q} = Q + \Delta Q$, $\tilde{A} = A + \Delta A$, $\tilde{G} = G + \Delta G$ be the coefficient matrices of the perturbed DARE. We used the MATLAB function `dre` to compute the solution Y to (4) and the solution $\tilde{Y}$ to the perturbed equation. Thus $\|\Delta Y\|_F/\|Y\|_F$, where $\Delta Y = \tilde{Y} - Y$, gives the relative change in the solution due to the perturbation. From the definitions (8) of Δ and (14) of c, $\|\Delta\|_2/\|c\|_2$ gives the size of relative perturbation on the data A, G, and Q. In our experiments, we set $j = -12$, that is, $\epsilon = 10^{-12}$. The following table compares the estimates using our condition numbers with the change in the solution computed by MATLAB.

$\|\Delta Y\|_F/\|Y\|_F$	$\kappa_n\|\Delta\|_2/\|c\|_2$	$\kappa_c\|\Delta\|_2/\|c\|_2$
1.3307×10^{-10}	5.5545×10^{-10}	2.6434×10^{-9}

As shown above, our condition number estimates are accurate.

5 Conclusion

In this paper, by exploiting the symmetry structure, we present structured perturbation analyses of both the continuous-time and the discrete-time symmetric algebraic Riccati equations. From the analyses, we define the structured normwise, mixed and componentwise condition numbers and derive their explicit expressions. Our condition numbers are improvements of the results in previous work [9, 10]. Our preliminary experiments show that the three kinds of condition numbers, especially the componentwise condition number, provide accurate estimate for the change in the solution for the perturbed equation. The expressions (11), (15), and (16) show that the condition number for Z (7) can be used as an indicator for the condition of solving the CARE. Similarly, the condition number of T (13) can be used as an indicator for the condition of solving the DARE.

References

[1] B. Anderson and J. Moore. *Optimal Filtering.* Prentice-Hall, Englewood Cliffs, NJ, 1979.

[2] F. Cucker, H. Diao, and Y. Wei. On mixed and componentwise condition numbers for Moore-Penrose inverse and linear least squares problems. *Math. Comput.*, 76:947–963, 2007.

[3] I. Gohberg and I. Koltracht. Mixed, componentwise, and structured condition numbers. *SIAM J. Matrix Anal. Appl.*, 14:688–704, 1993.

[4] N. Higham. *Accuracy and Stability of Numerical Algorithms.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 2002.

[5] P. Lancaster and L. Rodman. *Algebraic Riccati Equations.* Oxford University Press, New York, 1995.

[6] J.R. Rice. A theory of condition. *SIAM J. Numer. Anal.*, 3:287–310, 1966.

[7] J. Rohn. New condition numbers for matrices and linear systems. *Computing*, 41:167–169, 1989.

[8] J. Sun. Perturbation theory for algebraic Riccati equations. *SIAM J. Matrix Anal. Appl.*, 19:39–65, 1998.

[9] J. Sun. Condition numbers of algebraic Riccati equations in the Frobenius norm. *Linear Algebra Appl.*, 350:237–261, 2002.

[10] L. Zhou, Y. Lin, Y. Wei and S. Qiao. Perturbation Analysis and Condition Numbers of Symmetric Algebraic Riccati Equations. *Automatica*, 45:1005–1011, 2009.